

Commission on Higher Education
in collaboration with the Philippine Normal University



TEACHING GUIDE FOR SENIOR HIGH SCHOOL
General Mathematics
CORE SUBJECT

This Teaching Guide was collaboratively developed and reviewed by educators from public and private schools, colleges, and universities. We encourage teachers and other education stakeholders to email their feedback, comments, and recommendations to the Commission on Higher Education, K to 12 Transition Program Management Unit - Senior High School Support Team at k12@ched.gov.ph. We value your feedback and recommendations.

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Introduction

As the Commission supports DepEd’s implementation of Senior High School (SHS), it upholds the vision and mission of the K to 12 program, stated in Section 2 of Republic Act 10533, or the Enhanced Basic Education Act of 2013, that “every graduate of basic education be an empowered individual, through a program rooted on...the competence to engage in work and be productive, the ability to coexist in fruitful harmony with local and global communities, the capability to engage in creative and critical thinking, and the capacity and willingness to transform others and oneself.”

To accomplish this, the Commission partnered with the Philippine Normal University (PNU), the National Center for Teacher Education, to develop Teaching Guides for Courses of SHS. Together with PNU, this Teaching Guide was studied and reviewed by education and pedagogy experts, and was enhanced with appropriate methodologies and strategies.

Furthermore, the Commission believes that teachers are the most important partners in attaining this goal. Incorporated in this Teaching Guide is a framework that will guide them in creating lessons and assessment tools, support them in facilitating activities and questions, and assist them towards deeper content areas and competencies. Thus, the introduction of the **SHS for SHS Framework**.

The SHS for SHS Framework

The SHS for SHS Framework, which stands for “Saysay-Husay-Sarili for Senior High School,” is at the core of this book. The lessons, which combine high-quality content with flexible elements to accommodate diversity of teachers and environments, promote these three fundamental concepts:

SAYSAY: MEANING

Why is this important?

Through this Teaching Guide, teachers will be able to facilitate an understanding of the value of the lessons, for each learner to fully engage in the content on both the cognitive and affective levels.

HUSAY: MASTERY

How will I deeply understand this?

Given that developing mastery goes beyond memorization, teachers should also aim for deep understanding of the subject matter where they lead learners to analyze and synthesize knowledge.

SARILI: OWNERSHIP

What can I do with this?

When teachers empower learners to take ownership of their learning, they develop independence and self-direction, learning about both the subject matter and themselves.

The Parts of the Teaching Guide

This Teaching Guide is mapped and aligned to the DepEd SHS Curriculum, designed to be highly usable for teachers. It contains classroom activities and pedagogical notes, and integrated with innovative pedagogies. All of these elements are presented in the following parts:

1. INTRODUCTION

- Highlight key concepts and identify the essential questions
- Show the big picture
- Connect and/or review prerequisite knowledge
- Clearly communicate learning competencies and objectives
- Motivate through applications and connections to real-life

2. MOTIVATION

- Give local examples and applications
- Engage in a game or movement activity
- Provide a hands-on/laboratory activity
- Connect to a real-life problem

3. INSTRUCTION/DELIVERY

- Give a demonstration/lecture/simulation/hands-on activity
- Show step-by-step solutions to sample problems
- Give applications of the theory
- Connect to a real-life problem if applicable

4. PRACTICE

- Provide easy-medium-hard questions
- Give time for hands-on unguided classroom work and discovery
- Use formative assessment to give feedback

5. ENRICHMENT

- Provide additional examples and applications
- Introduce extensions or generalisations of concepts
- Engage in reflection questions
- Encourage analysis through higher order thinking prompts
- Allow pair/small group discussions
- Summarize and synthesize the learnings

6. EVALUATION

- Supply a diverse question bank for written work and exercises
- Provide alternative formats for student work: written homework, journal, portfolio, group/individual projects, student-directed research project

On DepEd Functional Skills and CHED’s College Readiness Standards

As Higher Education Institutions (HEIs) welcome the graduates of the Senior High School program, it is of paramount importance to align Functional Skills set by DepEd with the College Readiness Standards stated by CHED.

The DepEd articulated a set of 21st century skills that should be embedded in the SHS curriculum across various subjects and tracks. These skills are desired outcomes that K to 12 graduates should possess in order to proceed to either higher education, employment, entrepreneurship, or middle-level skills development.

On the other hand, the Commission declared the College Readiness Standards that consist of the combination of knowledge, skills, and reflective thinking necessary to participate and succeed - without remediation - in entry-level undergraduate courses in college.

The alignment of both standards, shown below, is also presented in this Teaching Guide - prepares Senior High School graduates to the revised college curriculum which will initially be implemented by AY 2018-2019.

College Readiness Standards Foundational Skills	DepEd Functional Skills
<p>Produce all forms of texts (written, oral, visual, digital) based on:</p> <ol style="list-style-type: none"> 1. Solid grounding on Philippine experience and culture; 2. An understanding of the self, community, and nation; 3. Application of critical and creative thinking and doing processes; 4. Competency in formulating ideas/arguments logically, scientifically, and creatively; and 5. Clear appreciation of one’s responsibility as a citizen of a multicultural Philippines and a diverse world; 	<p>Visual and information literacies Media literacy Critical thinking and problem solving skills Creativity Initiative and self-direction</p>
<p>Systematically apply knowledge, understanding, theory, and skills for the development of the self, local, and global communities using prior learning, inquiry, and experimentation</p>	<p>Global awareness Scientific and economic literacy Curiosity Critical thinking and problem solving skills Risk taking Flexibility and adaptability Initiative and self-direction</p>
<p>Work comfortably with relevant technologies and develop adaptations and innovations for significant use in local and global communities;</p>	<p>Global awareness Media literacy Technological literacy Creativity Flexibility and adaptability Productivity and accountability</p>
<p>Communicate with local and global communities with proficiency, orally, in writing, and through new technologies of communication;</p>	<p>Global awareness Multicultural literacy Collaboration and interpersonal skills Social and cross-cultural skills Leadership and responsibility</p>
<p>Interact meaningfully in a social setting and contribute to the fulfilment of individual and shared goals, respecting the fundamental humanity of all persons and the diversity of groups and communities</p>	<p>Media literacy Multicultural literacy Global awareness Collaboration and interpersonal skills Social and cross-cultural skills Leadership and responsibility Ethical, moral, and spiritual values</p>

The General Mathematics Teaching Guide

Implementing a new curriculum is always subject to a new set of challenges. References are not always available, and training may be too short to cover all the required topics. Under these circumstances, providing teachers with quality resource materials aligned with the curricular competencies may be the best strategy for delivering the expected learning outcomes. Such is the rationale for creating a series of teaching guides for several Grade 11 and 12 subjects. The intention is to provide teachers a complete resource that addresses all expected learning competencies, as stated in the Department of Education's official curriculum guide.

This resource is a teaching guide for General Mathematics. The structure is quite unique, reflective of the wide scope of General Mathematics: functions, business mathematics, and logic. Each lesson begins with an introductory or motivational activity. The main part of the lesson presents important ideas and provides several solved examples. Explanations to basic properties, the rationale for mathematical procedures, and the derivation of important formulas are also provided. The goal is to enable teachers to move learners away from regurgitating information and towards an authentic understanding of, and appreciation for, the subject matter.

The chapters on functions are an extension of the functions learned in Junior High School, where the focus was primarily on linear, quadratic, and polynomial functions. In Grade 11, learners will be exposed to other types of functions such as piecewise, rational, exponential, and logarithmic functions. Related topics such as solving equations and inequalities, as well as identifying the domain, range, intercepts, and asymptotes are also included.

The chapters on business mathematics in Grade 11 may be learners' first opportunity to be exposed to topics related to financial literacy. Here, they learn about simple and compound interest, annuities, loans, stocks, and bonds. These lessons can hopefully prepare learners to analyze business-related problems and make sound financial decisions.

The final chapter on logic exposes learners to symbolic forms of propositions (or statements) and arguments. Through the use of symbolic logic, learners should be able to recognize equivalent propositions, identify fallacies, and judge the validity of arguments. The culminating lesson is an application of the rules of symbolic logic, as learners are taught to write their own justifications to mathematical and real-life statements.

This Teaching Guide is intended to be a practical resource for teachers. It includes activities, explanations, and assessment tools. While the beginning teacher may use this Teaching Guide as a “script,” more experienced teachers can use this resource as a starting point for writing their own lesson plans. In any case, it is hoped that this resource, together with the Teaching Guide for other subjects, can support teachers in achieving the vision of the K to 12 Program.

First Quarter

	Hour 1	Hour 2	Hour 3	Hour 4
Week a	Lesson 1	Lesson 1, 2	Lesson 3	Lesson 3
Week b	Lesson 4	Lesson 5, 6	Lesson 6	Lesson 7
Week c	Lesson 7	Lesson 8	Lesson 8	Review/Exam
Week d	Lesson 9	Lesson 10	Lesson 10	Lesson 11
Week e	Lesson 11	Review/Exam	Lesson 12	Lesson 12, 13
Week f	Lesson 14	Lesson 14	Lesson 15	Lesson 15
Week g	Lesson 16	Lesson 16	Review/Exam	Review/Exam
Week h	Lesson 17	Lesson 17	Lesson 18, 19	Lesson 19, 20
Week i	Lesson 20	Lesson 21	Lesson 21	Lesson 21
Week j	Lesson 22	Lesson 22	Review/Exam	Review/Exam

Second Quarter

	Hour 1	Hour 2	Hour 3	Hour 4
Week a	Lesson 23	Lesson 24	Lesson 25	Lesson 25, 26
Week b	Lesson 26	Lesson 27	Lesson 27 / Review	Review/Exam
Week c	Lesson 28	Lesson 28	Lesson 29	Lesson 29
Week d	Lesson 29	Lesson 30	Lesson 30	Review/Exam
Week e	Lesson 31	Lesson 31	Lesson 32	Lesson 33
Week f	Lesson 34	Lesson 35	Lesson 35	Review/Exam
Week g	Lesson 36	Lesson 36	Lesson 37	Lesson 37
Week h	Lesson 38	Lesson 39	Lesson 39	Lesson 39
Week i	Lesson 40	Lesson 40	Lesson 40	Lesson 41
Week j	Lesson 41	Lesson 41	Review/Exam	Review/Exam

**K to 12 BASIC EDUCATION CURRICULUM
SENIOR HIGH SCHOOL – CORE SUBJECT**

Grade: 11
Core Subject Title: General Mathematics

Semester: First Semester
No. of Hours/Semester: 80 hours/semester
Prerequisite (if needed):

Core Subject Description: At the end of the course, the students must know how to solve problems involving rational, exponential and logarithmic functions; to solve business-related problems; and to apply logic to real-life situations.

CONTENT	CONTENT STANDARDS	PERFORMANCE STANDARDS	LEARNING COMPETENCIES	CODE
Functions and Their Graphs	<i>The learner demonstrates understanding of...</i> 2. key concepts of rational functions.	<i>The learner is able to...</i> 2. accurately formulate and solve real-life problems involving rational functions.	<i>The learner...</i> 1. represents real-life situations using functions, including piece-wise functions.	M11GM-Ia-1
			2. evaluates a function.	M11GM-Ia-2
			3. performs addition, subtraction, multiplication, division, and composition of functions	M11GM-Ia-3
			4. solves problems involving functions.	M11GM-Ia-4
			5. represents real-life situations using rational functions.	M11GM-Ib-1
			6. distinguishes rational function, rational equation, and rational inequality.	M11GM-Ib-2
			7. solves rational equations and inequalities.	M11GM-Ib-3
			8. represents a rational function through its: (a) table of values, (b) graph, and (c) equation.	M11GM-Ib-4
			9. finds the domain and range of a rational function.	M11GM-Ib-5
			10. determines the: (a) intercepts (b) zeroes; and (c) asymptotes of rational functions	M11GM-Ic-1
			11. graphs rational functions.	M11GM-Ic-2
			12. solves problems involving rational functions, equations, and inequalities.	M11GM-Ic-3

**K to 12 BASIC EDUCATION CURRICULUM
SENIOR HIGH SCHOOL – CORE SUBJECT**

CONTENT	CONTENT STANDARDS	PERFORMANCE STANDARDS	LEARNING COMPETENCIES	CODE
	<p>3. key concepts of inverse functions, exponential functions, and logarithmic functions.</p>	<p>3. apply the concepts of inverse functions, exponential functions, and logarithmic functions to formulate and solve real-life problems with precision and accuracy.</p>	<ol style="list-style-type: none"> 1. represents real-life situations using one-to-one functions. 2. determines the inverse of a one-to-one function. 3. represents an inverse function through its: (a) table of values, and (b) graph. 4. finds the domain and range of an inverse function. 5. graphs inverse functions. 6. solves problems involving inverse functions. 7. represents real-life situations using exponential functions. 8. distinguishes between exponential function, exponential equation, and exponential inequality. 9. solves exponential equations and inequalities. 10. represents an exponential function through its: (a) table of values, (b) graph, and (c) equation. 11. finds the domain and range of an exponential function. 12. determines the intercepts, zeroes, and asymptotes of an exponential function. 13. graphs exponential functions. 14. solves problems involving exponential functions, equations, and inequalities. 15. represents real-life situations using logarithmic functions. 16. distinguishes logarithmic function, logarithmic equation, and logarithmic inequality. 17. illustrates the laws of logarithms. 18. solves logarithmic equations and inequalities. 19. represents a logarithmic function through its: (a) table of values, (b) graph, and (c) equation. 20. finds the domain and range of a logarithmic function. 21. determines the intercepts, zeroes, and asymptotes of logarithmic functions. 22. graphs logarithmic functions. 23. solves problems involving logarithmic functions, equations, and inequalities. 	<p>M11GM-Id-1 M11GM-Id-2 M11GM-Id-3 M11GM-Id-4 M11GM-Ie-1 M11GM-Ie-2 M11GM-Ie-3 M11GM-Ie-4 M11GM-Ie-f-1 M11GM-Ie-f-2 M11GM-Ie-f-3 M11GM-Ie-f-4 M11GM-Ig-1 M11GM-Ig-2 M11GM-Ih-1 M11GM-Ih-2 M11GM-Ih-3 M11GM-Ih-i-1 M11GM-Ii-1 M11GM-Ii-2 M11GM-Ii-3 M11GM-Ii-4 M11GM-Ij-1 M11GM-Ij-2</p>

**K to 12 BASIC EDUCATION CURRICULUM
SENIOR HIGH SCHOOL – CORE SUBJECT**

CONTENT	CONTENT STANDARDS	PERFORMANCE STANDARDS	LEARNING COMPETENCIES	CODE
Basic Business Mathematics	<i>The learner demonstrates understanding of...</i> 1. key concepts of simple and compound interests, and simple and general annuities. 2. basic concepts of stocks and bonds. 3. basic concepts of business and consumer loans.	<i>The learner is able to...</i> 1. investigate, analyze and solve problems involving simple and compound interests and simple and general annuities using appropriate business and financial instruments. 2. use appropriate financial instruments involving stocks and bonds in formulating conclusions and making decisions. 3. decide wisely on the appropriateness of business or consumer loan and its proper utilization.	24. illustrates simple and compound interests.	M11GM-IIa-1
			25. distinguishes between simple and compound interests.	M11GM-IIa-2
			26. computes interest, maturity value, future value, and present value in simple interest and compound interest environment.	M11GM-IIa-b-1
			27. solves problems involving simple and compound interests.	M11GM-IIb-2
			28. illustrates simple and general annuities.	M11GM-IIc-1
			29. distinguishes between simple and general annuities.	M11GM-IIc-2
			30. finds the future value and present value of both simple annuities and general annuities.	M11GM-IIc-d-1
			31. calculates the fair market value of a cash flow stream that includes an annuity.	M11GM-IId-2
			32. calculates the present value and period of deferral of a deferred annuity.	M11GM-IId-3
			33. illustrate stocks and bonds.	M11GM-IIe-1
			34. distinguishes between stocks and bonds.	M11GM-IIe-2
			35. describes the different markets for stocks and bonds.	M11GM-IIe-3
			36. analyzes the different market indices for stocks and bonds.	M11GM-IIe-4
Logic	<i>The learner demonstrates understanding of...</i> 1. key concepts of propositional logic; syllogisms and fallacies.	<i>The learner is able to...</i> 1. judiciously apply logic in real-life arguments.	37. interprets the theory of efficient markets.	M11GM-IIe-5
			38. illustrates business and consumer loans.	M11GM-IIf-1
			39. distinguishes between business and consumer loans.	M11GM-IIf-2
			40. solves problems involving business and consumer loans (amortization, mortgage).	M11GM-IIf-3
			41. illustrates a proposition.	M11GM-IIg-1
			42. symbolizes propositions.	M11GM-IIg-2
			43. distinguishes between simple and compound propositions.	M11GM-IIg-3
			44. performs the different types of operations on propositions.	M11GM-IIg-4
			45. determines the truth values of propositions.	M11GM-IIh-1
			46. illustrates the different forms of conditional propositions.	M11GM-IIh-2
47. illustrates different types of tautologies and fallacies.	M11GM-IIi-1			

**K to 12 BASIC EDUCATION CURRICULUM
SENIOR HIGH SCHOOL – CORE SUBJECT**

CONTENT	CONTENT STANDARDS	PERFORMANCE STANDARDS	LEARNING COMPETENCIES	CODE
	2. key methods of proof and disproof.	2. appropriately apply a method of proof and disproof in real-life situations.	48. determines the validity of categorical syllogisms. 49. establishes the validity and falsity of real-life arguments using logical propositions, syllogisms, and fallacies. 50. illustrates the different methods of proof (direct and indirect) and disproof (indirect and by counterexample).	M11GM-IIj-2 M11GM-III-3
			51. justifies mathematical and real-life statements using the different methods of proof and disproof.	M11GM-IIj-2

Chapter 1

Functions

LESSON 1: Functions as Models

TIME FRAME: 90 minutes

CODE: M11GM-Ia-1 and M11GM-Ia-4

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to represent real life situations using functions, including piecewise functions, and solve problems involving functions.

LESSON OUTLINE:

1. Review: of relations and functions
 2. Review: The function as a machine
 3. Review: Functions and relations as a table of values
 4. Review: Functions as a graph in the Cartesian plane
 5. Review: Vertical line test
 6. Functions as representations of real life situations
 7. Piecewise functions
-

DEVELOPMENT OF THE LESSON

(A) Review of Functions from Junior High School


Note to the Teacher:

The concept of functions were introduced during Grade 8. The review provided in this teaching guide is extensive. Discuss only as much as necessary.

The following table shows how relations and functions are the same and how they are different.

Relations

A **relation** is a rule that relates values from a set of values (called the domain) to a second set of values (called the range).

The elements of the domain can be imagined as input to a machine that applies a rule to these inputs to generate one or more outputs.

A relation is also a set of ordered pairs (x, y) .

Functions

A **function** is a relation where each element in the domain is related to only one value in the range by some rule.

The elements of the domain can be imagined as input to a machine that applies a rule so that each input corresponds to only one output.

A function is a set of ordered pairs (x, y) such that no two ordered pairs have the same x -value but different y -values.

The function as a machine

Mention that we will represent mathematical relations as machines with an input and an output, and that the output is related to the input by some rule.

Draw the following six ‘machines’ on the board similar to the figures below. Write the following rules inside the figures to indicate how the machine functions. Then provide the guide questions given below.

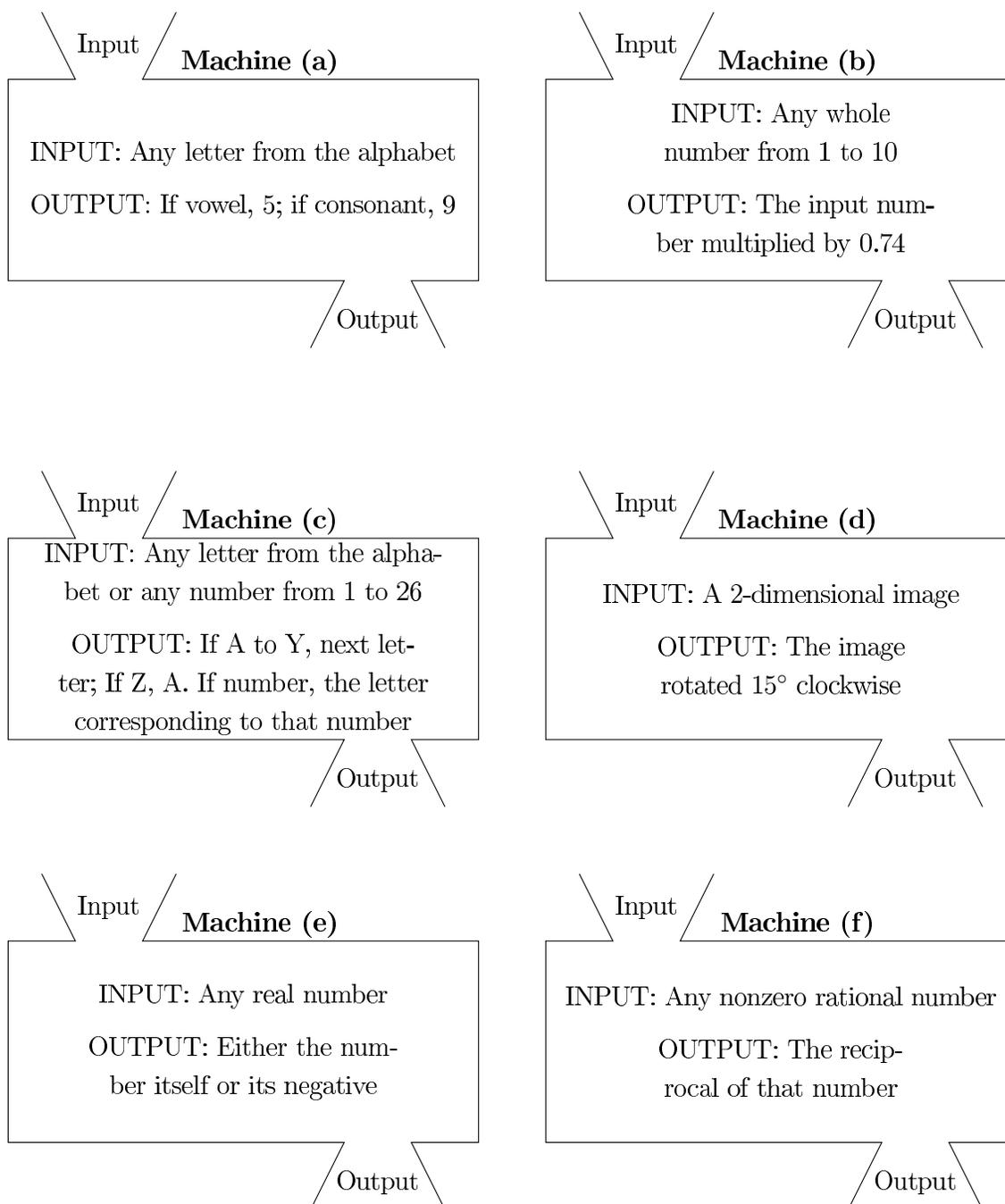


Figure 1.1: Function machines activity

Ask the class to give descriptive names to these five machines. Raise the following guide questions to the class about these machines:

- (a) Which of these machines, if you know the input, can you determine a *single* or unique output? The answers should be (a), (c), (d), and (f). Have the class explain why. Ask why (e) is not part

of this list. Introduce the term function to describe these machines.

- (b) Which of these machines, if the output is known, can you determine a single or unique input? The answers should be (d) and (f). Have the class explain why. This will be used in the discussion of one-to-one functions later.
- (c) Suppose we connect machine (a) to machine (c) such that the output of (a) becomes the input of (c). Give the outputs for the following inputs: A, R, S, U.

Functions and relations as a table of values

For the following illustrations of functions and relations, we draw from the definitions for **relations** and **functions** given at the start of the chapter.

Give the following examples to illustrate these concepts:

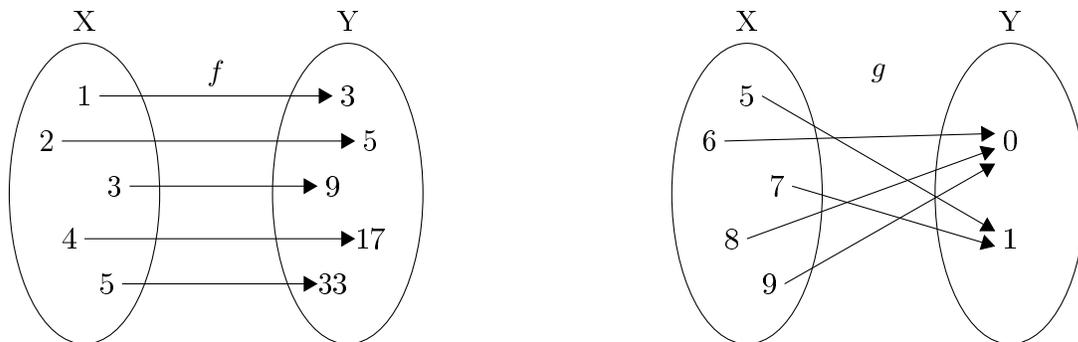
EXAMPLE 1. Which of the following relations are functions?

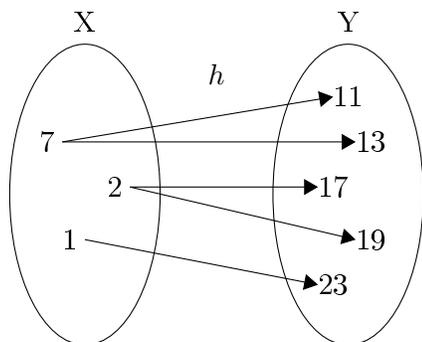
- (a) $f = \{(1, 2), (2, 2), (3, 5), (4, 5)\}$
- (b) $g = \{(1, 3), (1, 4), (2, 5), (2, 6), (3, 7)\}$
- (c) $h = \{(1, 3), (2, 6), (3, 9), \dots, (n, 3n), \dots\}$

Solution. The relations f and h are functions because no two ordered pairs have the same x -value but different y -values, while g is not a function because $(1, 3)$ and $(1, 4)$ are ordered pairs with the same x -value but different y -values. \square

Relations and functions can be represented by mapping diagrams where the elements of the domain are mapped to the elements of the range using arrows. In this case, the relation or function is represented by the set of all the connections represented by the arrows.

EXAMPLE 2. Which of the following mapping diagrams represent functions?





Solution. The relations f and g are functions because each $x \in X$ corresponds to a unique $y \in Y$. The relation h is not a function because there is at least one element in X for which there is more than one corresponding y -value. For example, $x = 7$ corresponds to $y = 11$ or 13 . Similarly, $x = 2$ corresponds to both $y = 17$ or 19 . \square

Functions as a graph in the Cartesian plane

Recall from Grade 8 that a relation between two sets of numbers can be illustrated by a graph in the Cartesian plane, and that a function passes the vertical line test (i.e., a vertical line can be drawn anywhere and intersect the graph in at most one point).

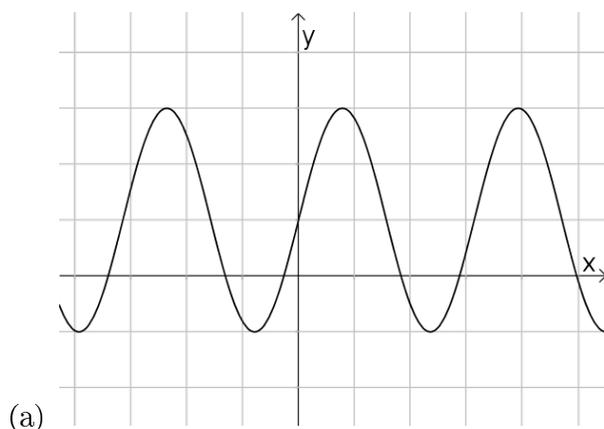
The Vertical Line Test

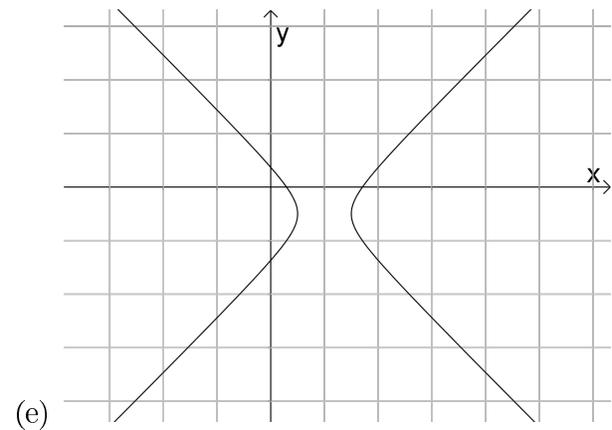
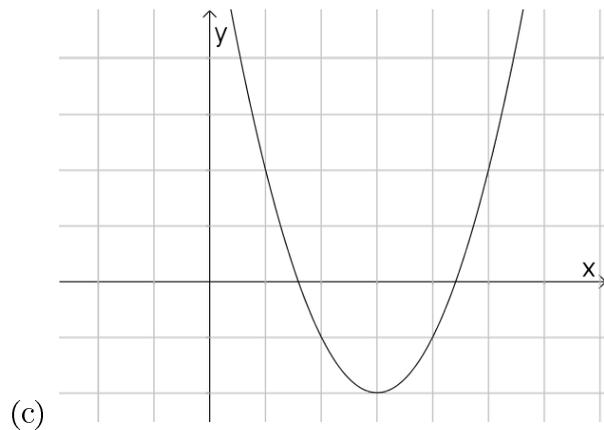
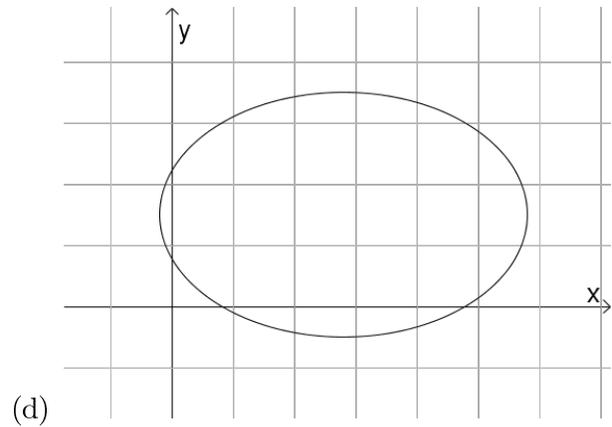
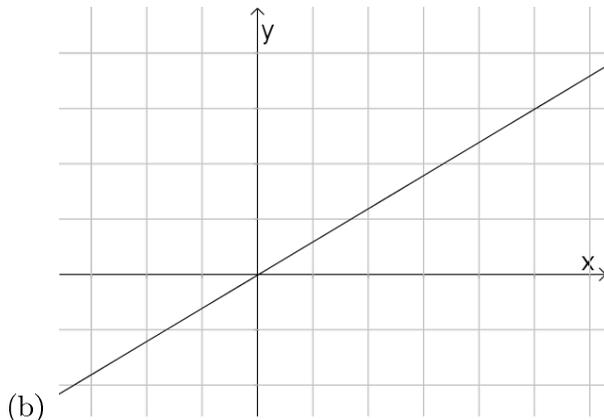
A graph represents a function if and only if each vertical line intersects the graph at most once.

If a vertical line $x = a$ intersects a graph twice, say at (a, b) and at (a, c) , then the graph cannot represent a function because two different y -values correspond to $x = a$.

Give the following examples and ask the students to identify which graphs represent functions and which do not.

EXAMPLE 3. Which of the following graphs can be graphs of functions?





Solution. Graphs (a), (b), (c) are graphs of functions while (d) and (e) are not because they do not pass the vertical line test. □

Teaching Tip

Explain that by convention, the x -variable is the input variable and that the value of the y -variable is computed based on the value of the x -variable. A relation is a function if for each x -value there corresponds only one y -value.

EXAMPLE 4. Which of the following represents a function?

- (a) $y = 2x + 1$
- (b) $y = x^2 - 2x + 2$
- (c) $x^2 + y^2 = 1$
- (d) $y = \sqrt{x + 1}$
- (e) $y = \frac{2x + 1}{x - 1}$

(f) $y = \lfloor x \rfloor + 1$ where $\lfloor x \rfloor$ is the greatest integer function. (Optional example)

Solution. All are relations. All are functions except (c). Equation (c) is not a function because we can find an x -value that corresponds to more than one y -value (ex. if $x = 0$, then y can be $+1$ or -1 .) \square

Definition.

The **domain** of a relation is the set of all possible values that the variable x can take.

EXAMPLE 5. Identify the domain for each relation above using set builder notation.

Solution. The domains for the relations above are as follows:

- | | |
|------------------|--------------------------------------|
| (a) \mathbb{R} | (d) $[-1, +\infty)$ |
| (b) \mathbb{R} | (e) $(-\infty, 1) \cup (1, +\infty)$ |
| (c) $[-1, 1]$ | (f) \mathbb{R} |

\square

If a relation is a function, then y can be replaced with $f(x)$ to denote that the value of y depends on the value of x . Do this for the functions in the examples above, replacing f with other letters to distinguish the functions from each other:

- (a) $f(x) = 2x + 1$
 (b) $q(x) = x^2 - 2x + 2$
 (d.) $g(x) = \sqrt{x + 1}$
 (e.) $r(x) = \frac{2x + 1}{x - 1}$
 (f.) $F(x) = \lfloor x \rfloor + 1$ where $\lfloor x \rfloor$ is the greatest integer function.

(B) Lesson Proper

Functions as representations of real-life situations

Functions can often be used to model real situations. Identifying an appropriate functional model will lead to a better understanding of various phenomena.

EXAMPLE 6. Give a function C that can represent the cost of buying x meals, if one meal costs ₱40.

Solution. Since each meal costs ₱40, then the cost function is $C(x) = 40x$. \square

EXAMPLE 7. One hundred meters of fencing is available to enclose a rectangular area next to a river (see figure). Give a function A that can represent the area that can be enclosed, in terms of x .

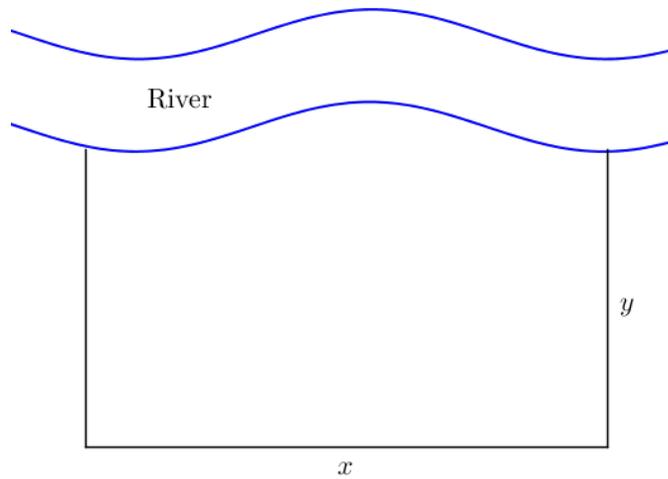


Figure 1.2

Solution. The area of the rectangular enclosure is $A = xy$. We will write this as a function of x . Since only 100 m of fencing is available, then $x + 2y = 100$ or $y = (100 - x)/2 = 50 - 0.5x$. Thus, $A(x) = x(50 - 0.5x) = 50x - 0.5x^2$. \square

Piecewise Functions

Some situations can only be described by more than one formula, depending on the value of the independent variable.

EXAMPLE 8. A user is charged ₱300 monthly for a particular mobile plan, which includes 100 free text messages. Messages in excess of 100 are charged ₱1 each. Represent the amount a consumer pays each month as a function of the number of messages m sent in a month.

Solution. Let $t(m)$ represent the amount paid by the consumer each month. It can be expressed by the piecewise function

$$t(m) = \begin{cases} 300 & \text{if } 0 < m \leq 100 \\ 300 + m & \text{if } m > 100 \end{cases}$$

\square

EXAMPLE 9. A jeepney ride costs ₱8.00 for the first 4 kilometers, and each additional integer kilometer adds ₱1.50 to the fare. Use a piecewise function to represent the jeepney fare in terms of the distance d in kilometers.

Solution. The input value is distance and the output is the cost of the jeepney fare. If $F(d)$ represents the fare as a function of distance, the function can be represented as follows:

$$F(d) = \begin{cases} 8 & \text{if } 0 < d \leq 4 \\ 8 + 1.5[d] & \text{if } d > 4 \end{cases}$$

The symbol $[d]$ denotes the **greatest integer** or **floor** function applied to d . The floor function gives the largest integer less than or equal to d , e.g. $[4.1] = [4.9] = 4$. □

EXAMPLE 10. Water can exist in three states: solid ice, liquid water, and gaseous water vapor. It takes a certain amount of heat before water can change state. Use the following information to sketch the graph of the function representing the temperature of water as a function of the amount of total heat added in Joules (J). Explain why this is a piecewise function.

- A solid block of ice is at -25°C and heat is added until it completely turns into water vapor.
- The temperature of the ice increases steadily and reaches 0° after 940 J of heat is applied.
- The temperature stays constant at 0°C until all the ice melts, which happens once a total of 6,950 J of heat is applied.
- The temperature increases steadily and reaches 100°C after a total of 14,470 J is applied.
- The temperature stays constant at 100° until all the water evaporates, which happens after a total of 55,260 J is applied.
- When all the water is in a gaseous state, its temperature can rise above 100°C (This is why steam can cause third-degree burns!)

Solution. Let $T(x)$ represent the temperature of the water in degrees Celsius as a function of cumulative or total heat added in Joules. The function $T(x)$ can be graphed as follows:

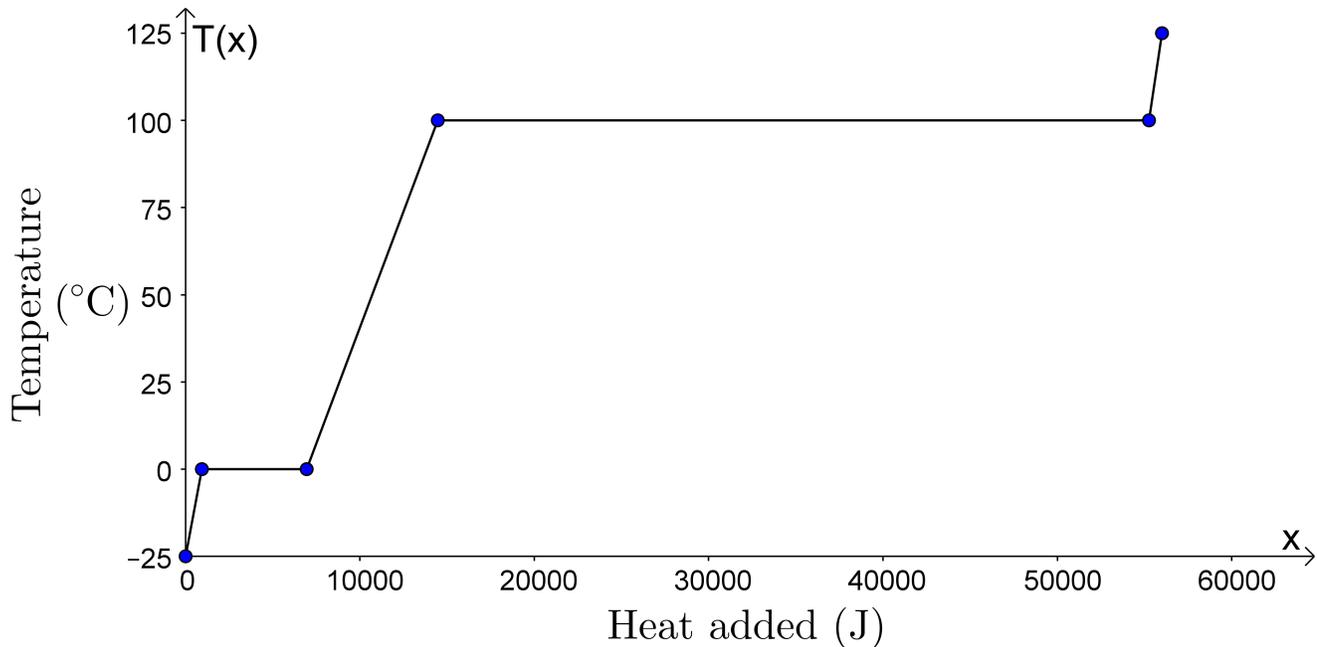


Figure 1.3

This is a piecewise function because it behaves differently across a sequence of intervals. □

(C) Seatwork

Choose several of the activities below to assign to the class as homework or seatwork.

Seatwork 1. Provide a real-world example or scenario that can be expressed as a relation that is not a function. **Sample Answer:** Destination versus tricycle fare: For ₱15 you can go anywhere within 3 kilometers.

Seatwork 2. Provide a real-world scenario of a scenario that can be modeled by a linear function. **Sample Answer:** Distance versus time if traveling at a constant speed.

Seatwork 3. A videoke machine can be rented for ₱1,000 for three days, but for the fourth day onwards, an additional cost of ₱400 per day is added. Represent the cost of renting a videoke machine as a piecewise function of the number of days it is rented and plot its graph. **Answer:**

$$f(x) = \begin{cases} 1000 & \text{if } 0 \leq x \leq 3 \\ 1000 + 400[x - 3] & \text{if } x > 3 \end{cases}$$

Seatwork 4. The fee for hiring a guide to explore a cave is ₱700. A guide can only take care of a maximum of 4 persons, and additional guides can be hired as needed. Represent the cost of hiring guides as a function of the number of tourists who wish to explore the cave and plot its graph. **An-**

swer: $f(x) = 700\lceil\frac{x}{4}\rceil$, $x \in \mathbb{N}$

Seatwork 5. The cost of hiring a catering service to serve food for a party is ₱150 per head for 20 persons or less, ₱130 per head for 21 to 50 persons, and ₱110 per head for 51 to 100 persons. For 100 or more persons, the cost is at ₱100 per head. Represent the total cost as a piecewise function of the

number of attendees of the party.

$$\mathbf{Answer:} \quad f(x) = \begin{cases} 150x & \text{if } 0 \leq x \leq 20 \\ 130x & \text{if } 21 \leq x \leq 50 \\ 110x & \text{if } 51 \leq x \leq 100 \\ 100x & \text{if } x > 100 \end{cases}, x \in \mathbb{N}$$

Performance Task

Ask students to research about **tax rates in the Philippines** (e.g., if you earn a certain income level, then your tax rate is such and such). If there are no references available, you can just give them the tax structure (this can be seen at the back of your Income Tax Return or ITR).

Based on the existing tax structure, ask students to define a **piecewise function** that models this situation by giving(a) the function's equation, (b) a table showing some function values, and (c) a graph.

Students will use the piecewise function representing the **tax structure in the Philippines** as a basis for discussing whether they think the tax rate is fair. For example, they can compare the amount of tax deducted from taxpayers with various levels of income. Students may present their work or make a report.

LESSON 2: Evaluating Functions

TIME FRAME: 30 minutes

CODE: M11GM-Ia-2 and M11GM-Ia-4

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to evaluate functions and solve problems involving functions.

LESSON OUTLINE:

1. Evaluating functions

DEVELOPMENT OF THE LESSON

(A) Lesson Proper

Evaluating a function means replacing the variable in the function, in this case x , with a value from the function's domain and computing for the result. To denote that we are evaluating f at a for some a in the domain of f , we write $f(a)$.

If one thinks of functions as a function machine, evaluating a function is analogous to providing our machines with a valid input.

EXAMPLE 1. Evaluate the following functions at $x = 1.5$:

(a) $f(x) = 2x + 1$

(b) $q(x) = x^2 - 2x + 2$

(c) $g(x) = \sqrt{x + 1}$

(d) $r(x) = \frac{2x + 1}{x - 1}$

(e) $F(x) = [x] + 1$ where $[x]$ is the greatest integer function.

Solution. Substituting 1.5 for x in the functions above, we have

(a) $f(1.5) = 2(1.5) + 1 = 4$

(b) $q(1.5) = (1.5)^2 - 2(1.5) + 2 = 2.25 - 3 + 2 = 1.25$

(c) $g(1.5) = \sqrt{1.5 + 1} = \sqrt{2.5}$

(d) $r(1.5) = \frac{2x + 1}{x - 1} = \frac{2(1.5) + 1}{(1.5) - 1} = \frac{3 + 1}{0.5} = 8$

(e) $F(1.5) = [x] + 1 = [1.5] + 1 = 1 + 1 = 2$

□

EXAMPLE 2. Find $g(-4)$ and $r(1)$ where g and r are as defined in the previous example.

Solution. This is not possible because -4 is not in the domain of $g(x)$ and 1 is not in the domain of $r(x)$. □

EXAMPLE 3. Evaluate the following functions, where f and q are as defined in Example 1:

(a) $f(3x - 1)$

(b) $q(2x + 3)$

Solution. To evaluate $f(3x - 1)$, replace all the x in $f(x) = 2x + 1$ by $(3x - 1)$:

$$f(3x - 1) = 2(3x - 1) + 1 = 6x - 2 + 1 = 6x - 1$$

To evaluate $q(3x + 3)$, replace all the x in $q(x) = x^2 - 2x + 2$ with $(2x + 3)$:

$$q(2x + 3) = (2x + 3)^2 - 2(2x + 3) + 2 = (4x^2 + 12x + 9) - 4x - 6 + 2 = 4x^2 + 8x + 5$$

□

(B) Seatwork

Choose several of the activities below to give to assign to the class as homework or seatwork.

Seatwork 1. Given $f(x) = x - 2$, find the following values:

(a) $f(0)$	Answer: -2	(d) $f(\pi)$	Answer: $\pi - 2$
(b) $f(3)$	Answer: 1	(e) $f(x + 1)$	Answer: $x - 1$
(c) $f(-1)$	Answer: -3	(f) $f(3x)$	Answer: $3x - 2$

Seatwork 2. Given $f(x) = \frac{4}{x}$, find the following values:

(a) $f(1)$	Answer: 4
(b) $f(2)$	Answer: 2
(c) $f(-1)$	Answer: -4
(d) $f(\sqrt{2})$	Answer: $2\sqrt{2}$
(e) $f(1/x)$	Answer: $4x$
(f) $f(2x)$	Answer: $2/x$

Seatwork 3. Given $f(x) = \sqrt{x - 3}$, find the following values:

(a) $f(3)$	Answer: 0
(b) $f(4)$	Answer: 1
(c) $f(12)$	Answer: 3
(d) $f(x - 3)$	Answer: $\sqrt{x - 6}$
(e) $f\left(\frac{1}{1 - x}\right)$	Answer: $\frac{3x - 2}{1 - x}$
(f) $f(x^2 + 4x + 7)$	Answer: $\sqrt{x^2 + 4x + 4}$ or $ x + 2 $

Seatwork 4. Mark started selling snacks in the nearby school. In one day he spends ₱200 for rent and ₱25 for each snack item he prepares. His expenses in a single day can be expressed as the function $C(x) = 25x + 200$, where x is the number of items and $C(x)$ is his daily expenses in pesos. How much are his expenses if he prepares 100 snack items? 150 snack items? **Answer:** ₱2700 and ₱3950, respectively

Seatwork 5. The function for the height of an object dropped from a 100-meter tall platform at time t seconds is approximated by $s(t) = -5t^2 + 100$ (This formula is based on an approximated value for the acceleration $g = 10m/s^2$ due to gravity). What is the height of the object after 2 seconds? After 4 seconds? **Answer:** 80 and 20 meters, respectively

LESSON 3: Operations on Functions

TIME FRAME: 120 minutes

CODE: M11GM-Ia-3

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to perform addition, subtraction, multiplication, division, and composition of functions.

LESSON OUTLINE:

1. Review: Addition and subtraction of algebraic expressions
2. Review: Multiplication and division of algebraic expressions
3. Addition, subtraction, multiplication, and division of functions
4. Composition of functions

DEVELOPMENT OF THE LESSON

(A) Introduction, motivation.

Do a quick review of the rules for adding, subtracting, multiplying and dividing fractions and algebraic expressions. You may skip this review if the class is proficient with performing arithmetic operations on algebraic expressions.

Addition and Subtraction:

- (a) Find the least common denominator (LCD) of both fractions.
- (b) Rewrite the fractions as equivalent fractions with the same LCD.
- (c) The LCD is the denominator of the resulting fraction.
- (d) The sum or difference of the numerators is the numerator of the resulting fraction.

EXAMPLE 1. Find the sum of $\frac{1}{3}$ and $\frac{2}{5}$.

Solution. The LCD of the two fractions is 15.

$$\begin{aligned}\frac{1}{3} + \frac{2}{5} &= \frac{5}{15} + \frac{6}{15} = \frac{5+6}{15} \\ &= \frac{11}{15}\end{aligned}$$

□

EXAMPLE 2. Find the sum of $\frac{1}{x-3}$ and $\frac{2}{x-5}$.

Solution. The LCD of the two rational expressions is $(x-3)(x-5)$ or $(x^2 - 8x + 15)$.

$$\begin{aligned}\frac{1}{x-3} + \frac{2}{x-5} &= \frac{x-5}{x^2-8x+15} + \frac{2(x-3)}{x^2-8x+15} = \frac{x-5+2x-6}{x^2-8x+15} \\ &= \frac{3x-11}{x^2-8x+15}\end{aligned}$$

□

Multiplication:

- (a) Rewrite the numerator and denominator in terms of its prime factors.
- (b) Common factors in the numerator and denominator can be simplified as (this is often called canceling).
- (c) Multiply the numerators together to get the new numerator.
- (d) Multiply the denominators together to get the new denominator.

EXAMPLE 3. Find the product of $\frac{10}{21}$ and $\frac{15}{8}$. Use cancellation of factors when convenient.

Solution. Express the numerators and denominators of the two fractions into their prime factors. Multiply and cancel out common factors in the numerator and the denominator to reduce the final answer to lowest terms.

$$\frac{10}{21} \cdot \frac{15}{8} = \frac{2 \cdot 5}{3 \cdot 7} \cdot \frac{3 \cdot 5}{2 \cdot 2 \cdot 2} = \frac{\cancel{2} \cdot 5 \cdot \cancel{3} \cdot 5}{\cancel{3} \cdot 7 \cdot \cancel{2} \cdot 2 \cdot 2} = \frac{25}{28}$$

□

EXAMPLE 4. Find the product of $\frac{x^2 - 4x - 5}{x^2 - 3x + 2}$ and $\frac{x^2 - 5x + 6}{x^2 - 3x - 10}$.

Solution. Express the numerators and denominators of the two rational expressions into their prime factors. Multiply and cancel out common factors in the numerator and the denominator to reduce the final answer to lowest terms. Note the similarity in the process between this example and the previous one on fractions.

$$\begin{aligned} \frac{x^2 - 4x - 5}{x^2 - 3x + 2} \cdot \frac{x^2 - 5x + 6}{x^2 - 3x - 10} &= \frac{(x+1)(x-5)}{(x-2)(x-1)} \cdot \frac{(x-2)(x-3)}{(x-5)(x+2)} \\ &= \frac{(x+1)\cancel{(x-5)}\cancel{(x-2)}(x-3)}{\cancel{(x-2)}(x-1)\cancel{(x-5)}(x+2)} \\ &= \frac{(x+1)(x-3)}{(x-1)(x+2)} = \frac{x^2 - 2x - 3}{x^2 + x - 2} \end{aligned}$$

□

Division:

To divide two fractions or rational expressions, multiply the dividend with the reciprocal of the divisor.

(B) Lesson Proper

The important concept that the students need to understand in this lesson is that *adding, subtracting and multiplying two or more functions together will result in another function*. Dividing two functions together will also result in another function if the denominator or divisor is not the zero function.

Introduce examples to build up to the following notations for function operations:

Definition.

Let f and g be functions.

- (a) Their **sum**, denoted by $f + g$, is the function defined by $(f + g)(x) = f(x) + g(x)$.
- (b) Their **difference**, denoted by $f - g$, is the function defined by $(f - g)(x) = f(x) - g(x)$.
- (c) Their **product**, denoted by $f \cdot g$, is the function defined by $(f \cdot g)(x) = f(x) \cdot g(x)$.
- (d) Their **quotient**, denoted by f/g , is the function defined by $(f/g)(x) = f(x)/g(x)$, excluding the values of x where $g(x) = 0$.

Write down the following functions on the board. Use these functions to develop the examples below.

Solution. Again, the solution can involve trial and error. Check if $q(x) - f(x)$ gives $x^2 - 4x + 1$.

$$\begin{aligned} q(x) - f(x) &= (x^2 - 2x + 2) - (2x + 1) \\ &= x^2 - 4x + 1 \\ &= f_2(x) \end{aligned}$$

Thus $f_2(x) = q(x) - f(x)$. □

EXAMPLE 8. Express the function $f_3(x) = \frac{2x^2 + x}{x - 1}$ as a sum or difference of the functions above.

Solution. Because $\frac{2x^2 + x}{x - 1}$ has $x - 1$ as a denominator, then $r(x) = \frac{2x + 1}{x - 1}$ must be one of the functions in our solution. Let us try to add $f(x)$ and $r(x)$:

$$\begin{aligned} f(x) + r(x) &= 2x + 1 + \frac{2x + 1}{x - 1} \\ &= \frac{(2x + 1)(x - 1)}{x - 1} + \frac{2x + 1}{x - 1} \\ &= \frac{(2x + 1)(x - 1) + (2x + 1)}{x - 1} \\ &= \frac{(2x^2 - x - 1) + (2x + 1)}{x - 1} \\ &= \frac{2x^2 + x}{x - 1} \\ &= f_3(x) \end{aligned}$$

□

Recall the definition for adding functions, i.e., $(f + g)(x) = f(x) + g(x)$. Emphasize that this is done to indicate that there is single new function from the sum of two other functions.

EXAMPLE 9. Write down the answers from the previous item in this new notation.

Solution.

- (a) $f_1(x) = q(x) + f(x) = (q + f)(x)$
- (b) $f_2(x) = q(x) - f(x) = (q - f)(x)$
- (c) $f_3(x) = f(x) + r(x) = (f + r)(x)$

□

Continue the discussion to products and quotients of functions by giving the following examples.

EXAMPLE 10. Express the function $g_1(x) = 2x^3 - 3x^2 + 2x + 2$ as a product or quotient of the functions above.

Solution. Since $2x^3 - 3x^2 + 2x + 2$ is cubic, then it is possibly the product of $f(x)$ and $q(x)$.

$$\begin{aligned} f(x) \cdot q(x) &= (2x + 1)(x^2 - 2x + 2) \\ &= (2x)(x^2 - 2x + 2) + (x^2 - 2x + 2) \\ &= (2x^3 - 4x^2 + 4x) + (x^2 - 2x + 2) \\ &= 2x^3 - 3x^2 + 2x + 2 \\ &= g_1(x) \end{aligned}$$

Thus, $g_1(x) = f(x) \cdot q(x)$. □

EXAMPLE 11. (*Can be used for seatwork*) Express the function $g_2(x) = x - 1$ as a product or quotient of the functions above.

Solution. The function $r(x) = \frac{2x + 1}{x - 1}$ involves $x - 1$. The goal is to get rid of $2x + 1$. This can be done by dividing $f(x)$ by $r(x)$:

$$\begin{aligned} \frac{f(x)}{r(x)} &= (2x + 1) \div \frac{2x + 1}{x - 1} \\ &= (2x + 1) \cdot \frac{x - 1}{2x + 1} \\ &= \frac{2x + 1}{2x + 1} \cdot (x - 1) \\ &= x - 1 \\ &= g_2(x) \end{aligned}$$

Thus, $g_2(x) = \frac{f(x)}{r(x)}$. □

EXAMPLE 12. (*Can be used for seatwork*) Express the function $g_3(x) = \frac{1}{x - 1}$ as a product or quotient of the functions above.

Solution. The function $g_3(x) = \frac{1}{x - 1}$ is very similar to $r(x) = \frac{2x + 1}{x - 1}$. The goal is to “get rid” of $2x + 1$. This can be done by dividing $r(x)$ by $f(x) = 2x + 1$:

$$\begin{aligned} \frac{r(x)}{f(x)} &= \frac{2x + 1}{x - 1} \div (2x + 1) \\ &= \frac{2x + 1}{x - 1} \cdot \frac{1}{2x + 1} \\ &= \frac{1}{x - 1} \\ &= g_3(x) \end{aligned}$$

Thus, $g_3(x) = \frac{r(x)}{f(x)}$. □


Teaching Tip

You can present similar problems through a classroom game: Write three functions on the board. Get the sum, difference, product, or quotient of any pair of these functions and give the answer to the class. Students will try to determine which pair of functions gave rise to your answer.

Composition of Functions

Recall in the discussion of the five machines that the output of one machine was connected to the input of another machine. Ask the class how a similar scenario can be applied to the functions discussed above. Ask what would happen if the “output” of the function $f(x) = 2x + 1$ is used as the input for the function $g(x) = x^2 - 2x + 2$. The result is $(2x + 1)^2 - 2(2x + 1) + 2$. Using a function as an input leads us to function composition, as defined below.

Definition.

Let f and g be functions. The **composite function**, denoted by $(f \circ g)$, is defined by

$$(f \circ g)(x) = f(g(x)).$$

The process of obtaining a composite function is called **function composition**.

Write the following functions on the board.

- $f(x) = 2x + 1$
- $g(x) = \sqrt{x + 1}$
- $q(x) = x^2 - 2x + 2$
- $F(x) = \lfloor x \rfloor + 1$
- $r(x) = \frac{2x + 1}{x - 1}$

EXAMPLE 13. Find and simplify $(g \circ f)(x)$.

Solution.

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= \sqrt{f(x) + 1} \\ &= \sqrt{(2x + 1) + 1} \\ &= \sqrt{2x + 2} \end{aligned}$$

□

EXAMPLE 14. Find and simplify $(q \circ f)(x)$. Is it the same with $(f \circ q)(x)$?

Solution.

$$\begin{aligned}
 (q \circ f)(x) &= q(f(x)) \\
 &= [f(x)]^2 - 2[f(x)] + 2 \\
 &= (2x + 1)^2 - 2(2x + 1) + 2 \\
 &= (4x^2 + 4x + 1) - (4x + 2) + 2 \\
 &= 4x^2 + 1
 \end{aligned}$$

$$\begin{aligned}
 (f \circ q)(x) &= f(q(x)) \\
 &= 2(x^2 - 2x + 2) + 1 \\
 &= 2x^2 - 4x + 5
 \end{aligned}$$

The functions $(q \circ f)(x)$ and $(f \circ q)(x)$ are not the same. □

EXAMPLE 15. Find and simplify $(f \circ r)(x)$.

Solution.

$$\begin{aligned}
 (f \circ r)(x) &= f(r(x)) \\
 &= 2r(x) + 1 \\
 &= 2 \left[\frac{2x + 1}{x - 1} \right] + 1 \\
 &= \frac{4x + 2}{x - 1} + 1 \\
 &= \frac{(4x + 2) + (x - 1)}{x - 1} \\
 &= \frac{5x + 1}{x - 1}
 \end{aligned}$$
□

EXAMPLE 16. Find and simplify $(F \circ r)(5)$.

Solution.

$$\begin{aligned}
 (F \circ r)(5) &= F(r(5)) \\
 &= [r(5)] + 1 \\
 &= \left[\frac{2(5) + 1}{5 - 1} \right] + 1 \\
 &= \left[\frac{11}{4} \right] + 1 = 2 + 1 = 3
 \end{aligned}$$
□

(C) Seatwork

Choose several of the activities below to give to assign to the class as homework or seatwork.

Seatwork 1. Given functions f and g below, find and simplify the following functions: $f + g$; $f - g$; $f \cdot g$; f/g ; g/f .

(a) $f(x) = x + 2$; $g(x) = x^2 - 4$

Answer: $x^2 + x - 2$; $-x^2 + x + 6$; $x^3 + 2x^2 - 4x - 8$; $\frac{1}{x-2}$; $x - 2$

(b) $f(x) = \sqrt{x-1}$; $g(x) = x^2 + 4$

Answer: $\sqrt{x-1} + x^2 + 4$; $\sqrt{x-1} - x^2 - 4$; $\sqrt{x-1}(x^2 + 4)$; $\frac{\sqrt{x-1}}{x^2 + 4}$; $x - 2$

(c) $f(x) = \frac{x-2}{x+2}$; $g(x) = \frac{1}{x}$

Answer: $\frac{x-2}{x+2} + \frac{1}{x}$; $\frac{x-2}{x+2} - \frac{1}{x}$; $\frac{x-2}{x(x+2)}$; $\frac{x(x-2)}{x+2}$; $\frac{x+2}{x(x-2)}$

(d) $f(x) = \frac{1}{x+2}$; $g(x) = \frac{x-2}{x}$

Answer: $\frac{1}{x+2} + \frac{x-2}{x}$; $\frac{1}{x+2} - \frac{x-2}{x}$; $\frac{x-2}{x(x+2)}$; $\frac{x}{(x+2)(x-2)}$; $\frac{(x+2)(x-2)}{x}$

(e) $f(x) = \frac{1}{x^2}$; $g(x) = \sqrt{x}$

Answer: $\frac{1}{x^2} + \sqrt{x}$; $\frac{1}{x^2} - \sqrt{x}$; $\frac{\sqrt{x}}{x^2}$; $\frac{1}{x^2\sqrt{x}}$; $x^2\sqrt{x}$

Seatwork 2. If $f(x) = x^2 + 3x$ and $g(x) = x - 2$, find and simplify the following composite functions: $f \circ g$; $g \circ f$; $f \circ f$; $g \circ g$.

Answer: $x^2 - x - 2$; $x^2 + 3x - 2$; $x^4 + 6x^3 + 12x^2 + 9x$; $x - 4$

Seatwork 3. Each pair of functions below have a special relationship because $(f \circ g) = x$ and $(g \circ f) = x$. Verify these.

(a) $f(x) = 3x - 2$; $g(x) = \frac{1}{3}(x + 2)$

(b) $f(x) = \frac{x}{2-x}$; $g(x) = \frac{2x}{x-1}$

(c) $f(x) = (x-1)^3 + 2$; $g(x) = \sqrt[3]{x-2} + 1$



Chapter 2

Rational Functions

LESSON 4: Representing Real-life Situations Using Rational Functions

TIME FRAME: 60 minutes

CODE: M11GM-Ib-1

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to represent real-life situations using rational functions.

LESSON OUTLINE:

1. Review: Polynomial functions
 2. Real-life situations that can be modeled by rational functions
 3. Definition of a polynomial function
-

DEVELOPMENT OF THE LESSON

(A) Concept Review

The students have covered the concept of **polynomial functions** during the second quarter of their Grade 10. Have the students recall the following concepts about polynomial functions:

Definition.

A **polynomial function p of degree n** is a function that can be written in the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \dots, a_n \in \mathbb{R}$, $a_n \neq 0$, and n is a positive integer. Each summand is a **term** of the polynomial function. The constants $a_0, a_1, a_2, \dots, a_n$ are the **coefficients**. The **leading coefficient** is a_n . The **leading term** is $a_n x^n$ and the **constant term** is a_0 .

(B) Introduction

Present the following scenario to demonstrate real-world applications of rational functions:

- (a) The local barangay received a budget of ₱100,000 to provide medical checkups for the children in the barangay. The amount is to be allotted equally among all the children in the barangay. Write an equation representing the relationship of the allotted amount per child (y -variable) versus the total number of children (x -variable). **Answer:** $y = \frac{100,000}{x}$

Fill up the table below with the different allotment amounts for different values for the number of children:

No. of children, x	10	20	50	100	200	300	500	1000
Allocated amount, y								

- (b) A philanthropist wants to supplement the budget allotted for each child by providing an additional ₱750 for each child in the barangay. If $g(x)$ represents this new amount allotted for each child, construct a function representing this relationship. **Answer:** $g(x) = \frac{100,000}{x} + 750$

Fill up the table below with the new allotment amounts under this scenario:

No. of children, x	10	20	50	100	200	300	500	1000
Allocated amount, y								

(C) Lesson proper

**Teaching Tip**

This lesson makes use of graphs as examples. You can use a web-based graphing software such as Desmos (<https://www.desmos.com/calculator>) or download a graphing software such as Geogebra (<https://www.geogebra.org/download>) to generate the graphs for your class.

Definition.

A **rational function** is a function of the form $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions, and $q(x)$ is not the zero function (i.e., $q(x) \not\equiv 0$). The domain of $f(x)$ is all values of x where $q(x) \neq 0$.

Present the following scenarios to the class. These are real-world relationships that can be modeled by rational functions. Indicate that unlike polynomial functions, rational functions may contain a variable in the denominator.

EXAMPLE 1. An object is to travel a distance of 10 meters. Express velocity v as a function $v(t)$ of travel time t , in seconds.

Solution. The following table of values show v for various values of t .

t (seconds)	1	2	4	5	10
v (meters per second)	10	5	2.5	2	1

The function $v(t) = \frac{10}{t}$ can represent v as a function of t . □

EXAMPLE 2. Suppose that $c(t) = \frac{5t}{t^2 + 1}$ (in mg/mL) represents the concentration of a drug in a patient's bloodstream t hours after the drug was administered. Construct a table of values for $c(t)$ for $t = 1, 2, 5, 10$. Round off answers to three decimal places. Use the table to sketch a graph and interpret the results.

Solution.

t	0	1	2	5	10
$c(t)$	0	2.5	2	0.962	0.495

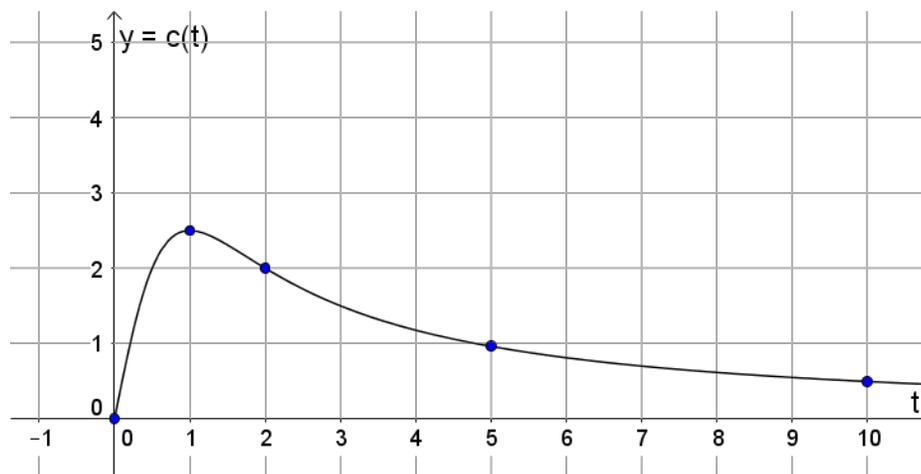


Figure 2.1

The graph indicates that the maximum drug concentration occurs around 1 hour after the drug was administered (calculus can be used to determine the exact value at which the maximum occurs). After 1 hour, the graph suggests that drug concentration decreases until it is almost zero.

□

Ask the class what these examples have in common, and the class should observe that in all cases, there is a variable in the denominator.

(D) Experiment (could be done outside class)

This experiment is adapted from the *Light it up* activity from the National Council of Teachers of Mathematics (NCTM) Illuminations project. <http://illuminations.nctm.org/Lesson.aspx?id=1968>

Have the class perform this experiment involving light reflecting off a small mirror into a wall. The class can be divided into groups of 3 or 4 students with their own materials.

Each group should bring or be provided with the following materials:

- Flashlight with narrow beam or laser pointer. (LED flashlights may not be suitable for this activity)
- Flat mirror
- Measuring tape or meterstick
- A thick book

Instructions.

- (a) Place the thick book a short distance from the wall and place the mirror on top of the book.
- (b) Stand on the side of the mirror opposite the wall. Aim the flashlight toward the center of the mirror so that the light reflects off the mirror and onto the wall.
- (c) Take the following measurements in centimeters and note down the values:
 - a = the height of the flashlight from the floor
 - b = the distance between the center of mirror and the wall
 - c = the distance between the floor and the mirror (or the thickness of the book).
- (d) The student holding the flashlight goes to various distances away from the mirror, while making sure that the flashlight is held at the same height throughout the experiment. At each location, the student measures the following distances, in centimeters.
 - x = the distance from the wall to the flashlight
 - y = the height of the reflection on the wall
- (e) Collect at least 10 data points and enter them into the table below:

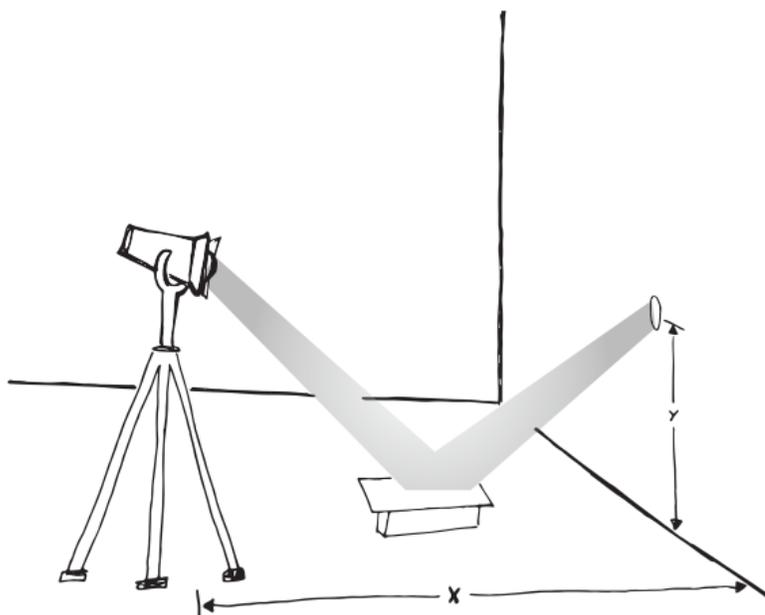


Figure 2.2

Distance from flashlight to wall (x)									
Distance from reflection to floor (y)									

- (f) On a Cartesian plane or graphing paper, plot a graph of your data.
- (g) Plot the graph of the equation $y = \frac{a}{x-b} + c$ on the same grid as your data. You can use a graphing tool like Geogebra or Desmos (see links above). Are your experiment results consistent with the results predicted by the graph?
- (h) Using the equation above, determine the distance from the floor of the reflection if the person holding the flashlight is:
- 3 meters from the wall
 - 4 meters from the wall

Note that each group may have different results due to height differences between students.

(E) Seatwork

Seatwork 1. A fence is to enclose a rectangular vegetable farm with an area of 400 square meters. If x is the length of one side of this fence, find a function $P(x)$ representing the perimeter of the fencing

material required.

$$\text{Answer: } P(x) = \frac{2x^2 + 800}{x}$$

Find the perimeter of the fence if one side has length 10 meters, 50 meters. **Answer:** 100 m, 116 m

Seatwork 2. Suppose the amount of bacteria growing in a petri dish is represented by the function

$$b(t) = \frac{50t}{t+1} \quad 0 \leq t \leq 20$$

where t is in hours and $b(t)$ is in millions.

Evaluate the function at $t = 1, 2, 5, 10, 15, 20$. Interpret the obtained values. **Answer:** 25; 33.33, 41.67; 45.45; 46.88; 47.62; The population increases but does not exceed 48 million.

LESSON 5: Rational Functions, Equations and Inequalities

TIME FRAME: 30 minutes

CODE: M11GM-Ib-2

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to distinguish rational functions, rational equations and rational inequalities

LESSON OUTLINE:

1. Review: Rational expressions
2. Definition and examples of rational functions, rational equations, and rational inequalities

DEVELOPMENT OF THE LESSON

(A) Review of Rational Expressions

Definition.

A **rational expression** is an expression that can be written as a ratio of two polynomials.

A rational expression can be described as a function where either the numerator, denominator, or both have a variable on it.

Provide these examples of algebraic expressions and have the class identify if they are rational or not. Have them explain their reasoning.

$\frac{x^2 + 3x + 2}{x + 4}$	Rational expression because it is a ratio of two polynomials.
$\frac{1}{3x^2}$	Rational expression because the numerator 1 is a polynomial (of degree 0).
$\frac{x^2 + 4x - 3}{2}$	Rational expression which is also a polynomial.
$\frac{\sqrt{x+1}}{x^3 - 1}$	Not a rational expression since the numerator is not a polynomial.
$\frac{1}{\frac{x+2}{x-2}}$	The expression is equal to $\frac{1}{(x+2)(x-2)}$ so it is a rational expression.

(B) Lesson Proper

Use the table below to show how to distinguish among rational equations, rational inequalities and rational functions:

	Rational Equation	Rational Inequality	Rational Function
Definition	An equation involving rational expressions	An inequality involving rational expressions	A function of the form $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials, and $q(x)$ is not the zero function
Example	$\frac{2}{x} - \frac{3}{2x} = \frac{1}{5}$	$\frac{5}{x-3} \leq \frac{2}{x}$	$f(x) = \frac{x^2 + 2x + 3}{x+1}$ or $y = \frac{x^2 + 2x + 3}{x+1}$

A rational equation or inequality can be solved for all x values that satisfy the equation or inequality. Whereas we solve an equation or inequality, we do not "solve" functions. Rather, a function (and in particular, a rational function) expresses a relationship between two variables (such as x and y), and can be represented by a table of values or a graph.

(C) Seatwork

Seatwork 1. Determine which of the following are rational expressions.

(a) $\frac{15}{\sqrt{x-1}}$

(b) $5x^4 - 6x^7 + 1$

(c) $\frac{5-x^3}{x}$

Answer: Only (b) and (c) are rational functions.

Seatwork 2. Determine whether the given is a rational function, a rational equation, a rational inequality, or none of these.

(a) $y = 5x^3 - 2x + 1$

Answer: rational function

(b) $\frac{8}{x} - 8 = \frac{x}{2x-1}$

Answer: rational equation

(c) $\sqrt{x-2} = 4$

Answer: none of these

(d) $\frac{x-1}{x+1} = x^3$

Answer: rational equation

(e) $y = \frac{7x^3 - 4\sqrt{x} + 1}{x^2 + 3}$

Answer: none of these

(f) $6x - \frac{5}{x+3} \geq 0$

Answer: rational inequality

LESSON 6: Solving Rational Equations and Inequalities

TIME FRAME: 90 minutes

CODE: M11GM-Ib-3

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to solve rational equations and inequalities, and solve problems involving rational equations and inequalities.

LESSON OUTLINE:

1. Solving rational equations
 2. Interval and set notation
 3. Solving rational inequalities
 4. Solving word problems involving rational equations or inequalities.
-

DEVELOPMENT OF THE LESSON

(A) Motivation

Give some rational equations such as $\frac{x+1}{2x} = 10$ and ask students some strategies they could try for solving it.

Next, turn the equation to an inequality, e.g., $\frac{x+1}{2x} \leq 10$. Challenge the class to provide as many solutions and non-solutions as they can.

Inform the class that this lesson provides systematic techniques to solve rational equations and inequalities.

(B) Lesson Proper

Procedure for Solving Rational Equations

To solve rational equations:

- (a) Eliminate denominators by multiplying each term of the equation by the least common denominator.
- (b) Note that eliminating denominators may introduce extraneous solutions. Check the solutions of the transformed equations with the original equation.

EXAMPLE 1. Solve for x : $\frac{2}{x} - \frac{3}{2x} = \frac{1}{5}$.

Solution. The LCD of all denominators is $10x$. Multiply both sides of the equation by $10x$ and solve

the resulting equation.

$$\begin{aligned} 10x \left(\frac{2}{x} \right) - 10x \left(\frac{3}{2x} \right) &= 10x \left(\frac{1}{5} \right) \\ 20 - 15 &= 2x \\ 5 &= 2x \\ x &= \frac{5}{2} \end{aligned}$$

□

EXAMPLE 2. Solve for x : $\frac{x}{x+2} - \frac{1}{x-2} = \frac{8}{x^2-4}$.

Solution. Factor each denominator in the fraction.

$$\frac{x}{x+2} - \frac{1}{x-2} = \frac{8}{(x-2)(x+2)}$$

The LCD is $(x-2)(x+2)$. Multiply the LCD to both sides of the equation to remove the denominators.

$$\begin{aligned} (x-2)(x+2) \cdot \frac{x}{x+2} - (x-2)(x+2) \cdot \frac{1}{x-2} &= [(x-2)(x+2)] \left(\frac{8}{(x-2)(x+2)} \right) \\ (x-2)x - (x+2) &= 8 \\ x^2 - 3x - 10 &= 0 \end{aligned}$$

Upon reaching this step, we can use strategies for solving polynomial equations.

$$\begin{aligned} x^2 - 3x - 10 &= 0 \\ (x+2)(x-5) &= 0 \\ x+2 = 0 \text{ or } x-5 &= 0 \\ x = -2 \text{ or } x &= 5 \end{aligned}$$

Check for extraneous solutions by substituting the answers back into the original equation. Since $x = 2$ will make the original equation undefined, it is an extraneous solution. Since $x = 5$ satisfies the original equation, it is the only solution. □

Let us apply the concepts we learned in solving rational equations to solving word problems. Rational equations can be applied to problems with variables in the denominator. Discuss the following examples in class.

EXAMPLE 3. In an inter-barangay basketball league, the team from Barangay Culiati has won 12 out of 25 games, a winning percentage of 48%. How many games should they win in a row to improve

their win percentage to 60%?

Solution. Let x represent the number of games that they need to win to raise their percentage to 60%. The team has already won 12 out of their 25 games. If they win x games in a row to increase their percentage to 60%, then they would have played $12 + x$ games out of $25 + x$ games. The equation is

$$\frac{12 + x}{25 + x} = 0.6$$

The equation is a rational equation. Solve the equation using the techniques previously discussed:

Since $25 + x$ is the only denominator, we multiply it to both sides of the equation. We then solve the resulting equation:

$$\begin{aligned}\frac{12 + x}{25 + x} &= 0.6 \\ 12 + x &= 0.6(25 + x) \\ 12 + x &= 0.6(25) + 0.6x \\ x - 0.6x &= 15 - 12 \\ 0.4x &= 3 \\ x &= 7.5\end{aligned}$$

Since x represents the number of games, this number should be an integer. Therefore Barangay Culiati needs to win 8 games in a row to raise their winning percentage to 60%.

□

EXAMPLE 4. Jens walks 5 kilometers from his house to Quiapo to buy a new bike which he uses to return home. He averaged 10 kilometers faster on his bike than on on foot. If his total trip took 1 hour and 20 minutes, what is his walking speed in kph? Use the formula $v = \frac{d}{t}$.

Solution. Using the formula $v = \frac{d}{t}$, we have the formula for time $t = \frac{d}{v}$.

Let v be Jens' walking speed. Then $v + 10$ is his speed on his new bike. Jens' walking time is $\frac{5}{v}$ and his biking time is $\frac{5}{v+10}$.

Rewriting the time 1 hour and 20 minutes as the fraction $\frac{4}{3}$ hours, the equation becomes:

$$\frac{5}{v} + \frac{5}{v + 10} = \frac{4}{3}$$

The LCD is $3v(v + 10)$. Multiply both sides by the LCD the remove the denominators and solve the resulting equation:

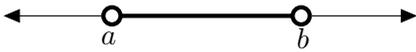
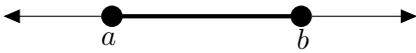
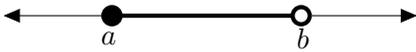
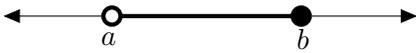
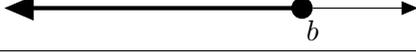
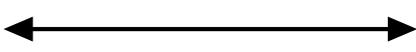
$$\frac{5}{v} + \frac{5}{v + 10} = \frac{4}{3}$$

$$\begin{aligned}
3v(v+10) \cdot \frac{5}{v} + 3v(v+10) \cdot \frac{5}{v+10} &= 3v(v+10) \cdot \frac{4}{3} \\
15(v+10) + 15v &= 4v(v+10) \\
30v + 150 &= 4v^2 + 40v \\
4v^2 + 10v - 150 &= 0 \\
2v^2 + 5v - 75 &= 0 \\
(2v+15)(v-5) &= 0 \\
v = -\frac{15}{2} \text{ or } v &= 5
\end{aligned}$$

Since v is the walking speed, it cannot be negative. We conclude that Jens' walking speed is 5 kph. \square

Interval and Set Notation

An inequality may have infinitely many solutions. The set of all solutions can be expressed using *set notation* or *interval notation*. These notations are presented in the table below:

Interval	Set Notation	Graph
(a, b)	$\{x a < x < b\}$	
$[a, b]$	$\{x a \leq x \leq b\}$	
$[a, b)$	$\{x a \leq x < b\}$	
$(a, b]$	$\{x a < x \leq b\}$	
(a, ∞)	$\{x a < x\}$	
$[a, \infty)$	$\{x a \leq x\}$	
$(-\infty, b)$	$\{x x < b\}$	
$(-\infty, b]$	$\{x x \leq b\}$	
$(-\infty, \infty)$	\mathbb{R} (set of all real numbers)	

Solutions to inequalities in this text will be represented using set notation.

Procedure for Solving Rational Inequalities

To solve rational inequalities:

- (a) Rewrite the inequality as a single fraction on one side of the inequality symbol and 0 on the other side.
- (b) Determine over what intervals the fraction takes on positive and negative values.
- Locate the x -values for which the rational expression is zero or undefined (factoring the numerator and denominator is a useful strategy).
 - Mark the numbers found in (i) on a number line. Use a shaded circle to indicate that the value is included in the solution set, and a hollow circle to indicate that the value is excluded. These numbers partition the number line into intervals.
 - Select a test point within the interior of each interval in (ii). The sign of the rational expression at this test point is also the sign of the rational expression at each interior point in the aforementioned interval.
 - Summarize the intervals containing the solutions.

 **Warning!**

It is not valid to multiply both sides of an inequality by a variable. Recall that

- multiplying both sides of an inequality by a positive number *retains* the direction of the inequality, and
- multiplying both sides of an inequality by a negative number *reverses* the direction of the inequality.

Since the sign of a variable is unknown, then it is not valid to multiply both sides of an inequality by a variable.

EXAMPLE 5. Solve the inequality $\frac{2x}{x+1} \geq 1$

Solution. (a) Rewrite the inequality as a single fraction on one side, and 0 on the other side.

$$\begin{aligned}\frac{2x}{x+1} - 1 &\geq 0 \\ \frac{2x - (x+1)}{x+1} &\geq 0 \\ \frac{x-1}{x+1} &\geq 0\end{aligned}$$

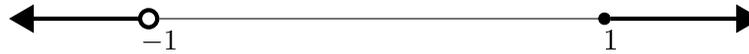
- (b) The value $x = 1$ is included in the solution since it makes the fraction equal to zero, while $x = -1$ makes the fraction undefined. Mark these on a number line. Use a shaded circle for $x = 1$ (a solution) and an unshaded circle for $x = -1$ (not a solution).



- (c) Choose convenient test points in the intervals determined by -1 and 1 to determine the sign of $\frac{x-1}{x+1}$ in these intervals. Construct a table of signs as shown below.

Interval	$x < -1$	$-1 < x < 1$	$x > 1$
Test point	$x = -2$	$x = 0$	$x = 2$
$x - 1$	-	-	+
$x + 1$	-	+	+
$\frac{x-1}{x+1}$	(+)	(-)	(+)

- (d) Since we are looking for the intervals where the fraction is positive or zero, we determine the solution intervals to be $x < -1$ and $x \geq 1$. Plot these intervals on the number line.



The solution set is $\{x \in \mathbb{R} | x < -1 \text{ or } x \geq 1\}$. It can also be written using interval notation: $(-\infty, -1) \cup [1, \infty)$.

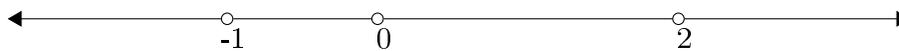
□

EXAMPLE 6. Solve: $\frac{3}{x-2} < \frac{1}{x}$.

Solution. (a) Rewrite as an inequality with zero on one side.

$$\begin{aligned} \frac{3}{x-2} - \frac{1}{x} &< 0 \\ \frac{3x - (x-2)}{x(x-2)} &< 0 \\ \frac{2x+2}{x(x-2)} &< 0 \\ \frac{2(x+1)}{x(x-2)} &< 0 \end{aligned}$$

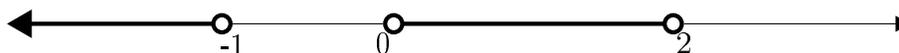
- (b) The fraction will be zero for $x = -1$ and undefined for 0 and 2 . Plot on a number line. Use hollow circles since these values are not part of the solutions.



- (c) Construct a table of signs to determine the sign of the function in each interval determined by -1 , 0 , and 2 .

Interval	$x < -1$	$-1 < x < 0$	$0 < x < 2$	$x > 2$
Test point	$x = -2$	$x = -\frac{1}{2}$	$x = 1$	$x = 3$
$2(x + 1)$	-	+	+	+
x	-	-	+	+
$x - 2$	-	-	-	+
$\frac{2(x + 1)}{x(x - 2)}$	(-)	(+)	(-)	(+)

(d) Summarize the intervals satisfying the inequality. Plot these intervals on the number line.



The solution set of the inequality is the set $\{x \in \mathbb{R} | x < -1 \text{ or } 0 < x < 2\}$.

□

We now look at word problems that can be modeled using rational inequalities.

EXAMPLE 7. A box with a square base is to have a volume of 8 cubic meters. Let x be the length of the side of the square base and h be the height of the box. What are the possible measurements of a side of the square base if the height should be longer than a side of the square base?

Solution. The volume of a rectangular box is the product of its width, length, and height. Since the base of the box is square, its width and length are equal.

The variable x is the length of a side of the box, while h is its height. The equation relating h and x is

$$8 = x^2 h$$

Express h in terms of x :

$$h = \frac{8}{x^2}$$

Since the height is greater than the width, $h > x$ and our inequality is

$$\frac{8}{x^2} > x$$

(a) To solve this inequality, we rewrite with zero on one side:

$$\begin{aligned} \frac{8}{x^2} &> x \\ \frac{8}{x^2} - x &> 0 \\ \frac{8 - x^3}{x^2} &> 0 \\ \frac{(2 - x)(x^2 + 2x + 4)}{x^2} &> 0 \end{aligned}$$

(b) The rational expression will be zero for $x = 16$ and $x = 28$ and undefined for $x = 0$ and $x = 4$. Plot on a number line and use hollow circles since these values are not part of the solution set.



(c) Construct a table of signs to determine the sign of the function in each interval determined by 0 and 2.

Interval	$x < 0$	$0 < x < 2$	$x > 2$
Test point	$x = -1$	$x = 1$	$x = 3$
$2 - x$	+	+	-
$x^2 + 2x + 4$	+	+	+
x^2	+	+	+
$\frac{(2-x)(x^2+2x+4)}{x^2}$	(+)	(+)	(-)

(d) Since the rational expression is positive in the interval $0 < x < 2$, this is the solution set of the inequality. We reject the interval $x < 0$ even if the expression is positive here since we only consider positive values of x for this problem.

Therefore the height of the box should be less than 2 meters.

□

EXAMPLE 8. A dressmaker ordered several meters of red cloth from a vendor, but the vendor only had 4 meters of red cloth in stock. The vendor bought the remaining lengths of red cloth from a wholesaler for ₱ 1120. He then sold those lengths of red cloth to the dressmaker along with the original 4 meters of cloth for a total of ₱1600. If the vendor's price per meter is at least ₱10 more than the wholesaler's price per meter, what possible lengths of cloth did the vendor purchase from the wholesaler?

Solution. Let the variable x be the length of the additional cloth purchased by the vendor from the wholesaler.

The wholesaler's price of red cloth per meter can be expressed as $\frac{1120}{x}$. The vendor's price of red cloth per meter can be expressed as $\frac{1600}{x+4}$.

Since the vendor's price per meter is at least P10 more than the wholesaler's price per meter, the inequality can be written as

$$\frac{1600}{x+4} - \frac{1120}{x} \geq 10$$

(a) To solve this inequality, rewrite it with zero on one side and simplify:

$$\begin{aligned} \frac{1600}{x+4} - \frac{1120}{x} &\geq 10 \\ \frac{160}{x+4} - \frac{112}{x} &\geq 1 \\ \frac{160}{x+4} - \frac{112}{x} - 1 &\geq 0 \\ \frac{160x - 112(x+4) - (x^2 + 4x)}{x(x+4)} &\geq 0 \\ \frac{160x - 112x - 448 - x^2 - 4x}{x(x+4)} &\geq 0 \end{aligned}$$

$$\begin{aligned} \frac{x^2 - 44x + 448}{x(x+4)} &\leq 0 \\ \frac{(x-16)(x-28)}{x(x+4)} &\leq 0 \end{aligned}$$

(b) The rational expression will be zero for $x = 16$ and $x = 28$ and undefined for $x = 0$ and $x = -4$. Plot on a number line and use hollow circles since these values are not part of the solution set. The figure below is not drawn to scale.



(c) Construct a table of signs to determine the sign of the function in each interval determined by the values above.

Interval	$x < -4$	$-4 < x < 0$	$0 < x < 16$	$16 < x < 28$	$x > 28$
Test point	$x = -5$	$x = -1$	$x = 10$	$x = 20$	$x = 30$
$x - 16$	-	-	-	+	+
$x - 28$	-	-	-	-	+
x	-	-	+	+	+
$x + 4$	-	+	+	+	+
$\frac{(x-16)(x-28)}{x(x+4)}$	(+)	(-)	(+)	(-)	(+)

(d) The expression is negative in the interval $-4 < x < 0$ and in the interval $16 < x < 28$. However, since we are dealing with lengths of cloth, we discard the interval where the length is negative. Also, as shown above $x = 16$ and $x = 28$ are part of the solution. **Therefore the vendor bought and sold an additional length of red cloth between 16 to 28 meters (inclusive) to the dressmaker.**

□

(C) Seatwork

Seatwork 1. Find the solutions for each rational equation below. Make sure to check for extraneous solutions.

$$(a) \frac{3}{x+1} = \frac{2}{x-3} \qquad \text{Answer: } 11$$

$$(b) \frac{2x}{x+1} + \frac{5}{2x} = 2 \qquad \text{Answer: } -5$$

$$(c) \frac{x^2 - 10}{x - 1} = \frac{-14 - 5x}{x - 1} \qquad \text{Answer: } -4, -1$$

$$(d) \frac{x^2 - 4x}{x - 2} = \frac{14 - 9x}{x - 2} \qquad \text{Answer: } -7$$

Seatwork 2. Find the solution set for each rational inequality below. Graph the solution set on a number line.

$$(a) \frac{(x+3)(x-2)}{(x+2)(x-1)} \geq 0 \qquad \text{Answer: } (-\infty, -3] \cup [2, \infty)$$

$$(b) \frac{(x+4)(x-3)}{(x-2)(x^2+2)} \geq 0 \qquad \text{Answer: } [-4, 2) \cup [3, \infty)$$

$$(c) \frac{x+1}{x+3} \leq 2 \qquad \text{Answer: } (-\infty, -5] \cup (-3, \infty)$$

$$(d) \frac{x-2}{x^2-3x-10} < 0 \qquad \text{Answer: } (-\infty, -2) \cup (2, 5)$$

Seatwork 3. You have 6 liters of a pineapple juice blend that is 50% pineapple juice. How many liters of pure pineapple juice needs to be added to make a juice blend that is 75% pineapple juice?

Hint: If x is the amount of pure pineapple juice to be added, the percentage can be written as $\frac{3+x}{6+x}$.

Answer: 1.5 L

Seatwork 4. Two ships travelling from Dumaguete to Cagayan de Oro differ in average speed by 10 kph. The slower ship takes 3 hours longer to travel a 240 kilometer route than for the faster ship to

travel a 200 kilometer route. Find the speed of the slower ship. Hint: Construct an equation with time t as the variable. **Answer:** 30 kph

LESSON 7: Representations of Rational Functions

TIME FRAME: 120 minutes

CODE: M11GM-Ib-4

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to represent a rational function through its table of values, graphs and equation, and solve problems involving rational functions.

LESSON OUTLINE:

1. Review: Definition of rational functions
 2. Representing a rational function through a table of values, a graph or an equation
-

DEVELOPMENT OF THE LESSON

(A) Recall of important concepts

Begin the session by recalling the definition of a rational function.

Definition

A **rational function** is a function of the form $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions, and $q(x)$ is not the zero function, i.e, $q(x) \neq 0$. The domain of $f(x)$ is all values of x where $q(x) \neq 0$.

In this lesson we demonstrate how we can represent rational functions using tables of values, graphs and equation.

(B) Lesson proper

Present a simple scenario involving rational functions to the class:

Average speed (or velocity) can be computed by the formula $s = \frac{d}{t}$. Consider a 100-meter track used for foot races. The speed of a runner can be computed by taking the time it will take him to run the track and applying it to the formula $s = \frac{100}{t}$, since the distance is fixed at 100 meters.

EXAMPLE 1. Represent the speed of a runner as a function of the time it takes to run 100 meters in the track.

Solution. Since the speed of a runner depends on the time it takes the runner to run 100 meters, we can represent speed as a function of time.

Let x represent the time it takes the runner to run 100 meters. Then the speed can be represented as a function $s(x)$ as follows:

$$s(x) = \frac{100}{x}$$

Observe that it is similar in structure to the formula $s = \frac{d}{t}$ relating speed, distance and time. \square

EXAMPLE 2. Continuing the scenario above, construct a table of values for the speed of a runner against different run times.

Solution. A table of values can help us determine the behavior of a function as the variable x changes.



Software tools

You can use a spreadsheet program such as Microsoft Excel, LibreOffice Calc (<https://www.libreoffice.org/download/libreoffice-fresh/>) or the Google Docs spreadsheet (<http://docs.google.com>) to generate a table of values.

The current world record (as of October 2015) for the 100-meter dash is 9.58 seconds set by the Jamaican Usain Bolt in 2009. We start our table of values at 10 seconds.

Let x be the run time and $s(x)$ be the speed of the runner in meters per second, where $s(x) = \frac{100}{x}$. The table of values for run times from 10 seconds to 20 seconds is as follows:

x	10	12	14	16	18	20
$s(x)$	10	8.33	7.14	6.25	5.56	5

From the table we can observe that the speed decreases with time. We can use a graph to determine if the points on the function follow a smooth curve or a straight line. \square

EXAMPLE 3. Plot the points on the table of values on a Cartesian plane. Determine if the points on the function $s(x) = \frac{100}{x}$ follow a smooth curve or a straight line.

Solution. Assign points on the Cartesian plane for each entry on the table of values above:

$$A(10,10) \quad B(12,8.33) \quad C(14,7.14) \quad D(16, 6.25) \quad E(18,5.56) \quad F(20,5)$$

Plot these points on the Cartesian plane.

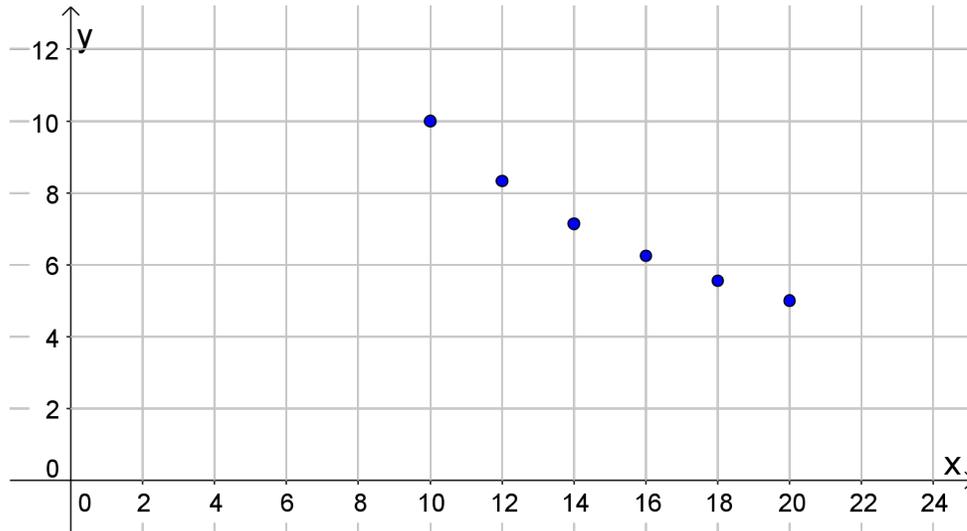


Figure 2.3

By connecting the points, we can see that they are not collinear but rather follow a smooth curve.

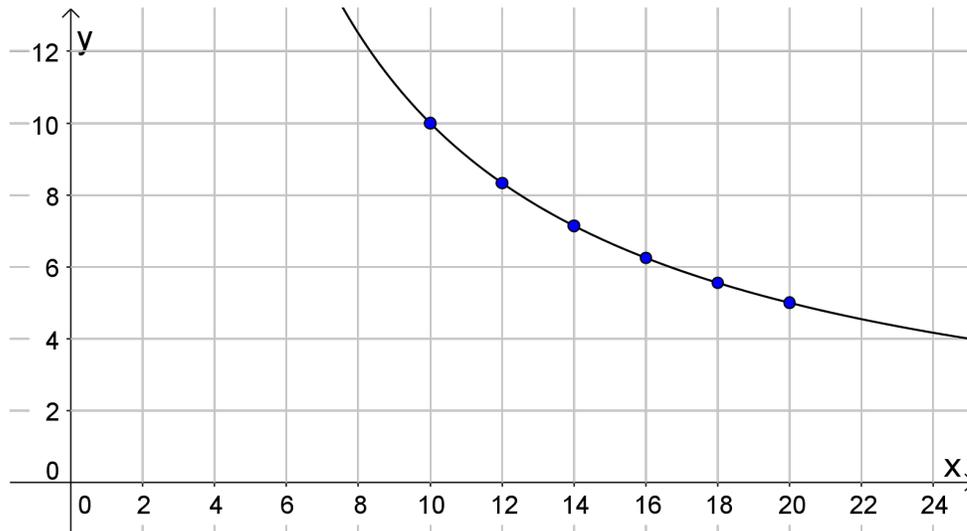


Figure 2.4

For the 100-meter dash scenario, we have constructed a function of speed against time, and represented our function with a table of values and a graph.

□

The previous example is based on a real world scenario and has limitations on the values of the x -variable. For example, a runner cannot have negative time (which would mean he is running backwards in time!), nor can a exceed the limits of human physiology (can a person run 100-meters in 5 seconds?). However, we can apply the skills of constructing tables of values and plotting graphs to observe the

behavior of rational functions.

EXAMPLE 4. Represent the rational function given by $f(x) = \frac{x-1}{x+1}$ using a table of values and plot a graph of the function by connecting points.

Solution. Since we are now considering functions in general, we can find function values across more values of x . Let us construct a table of values for some x -values from -10 to 10 :

x	-10	-8	-6	-4	-2	0	2	4	6	8	10
$f(x)$	1.22	1.29	1.4	1.67	3	-1	0.33	0.6	0.71	0.78	0.82

Make the observation that the function values appear to be increasing, but suddenly jumps from 3 to -1 , and then starts to increase again. Why would this be the case?

Let us attempt to get a better picture by plotting the points on a Cartesian plane and connecting the points.

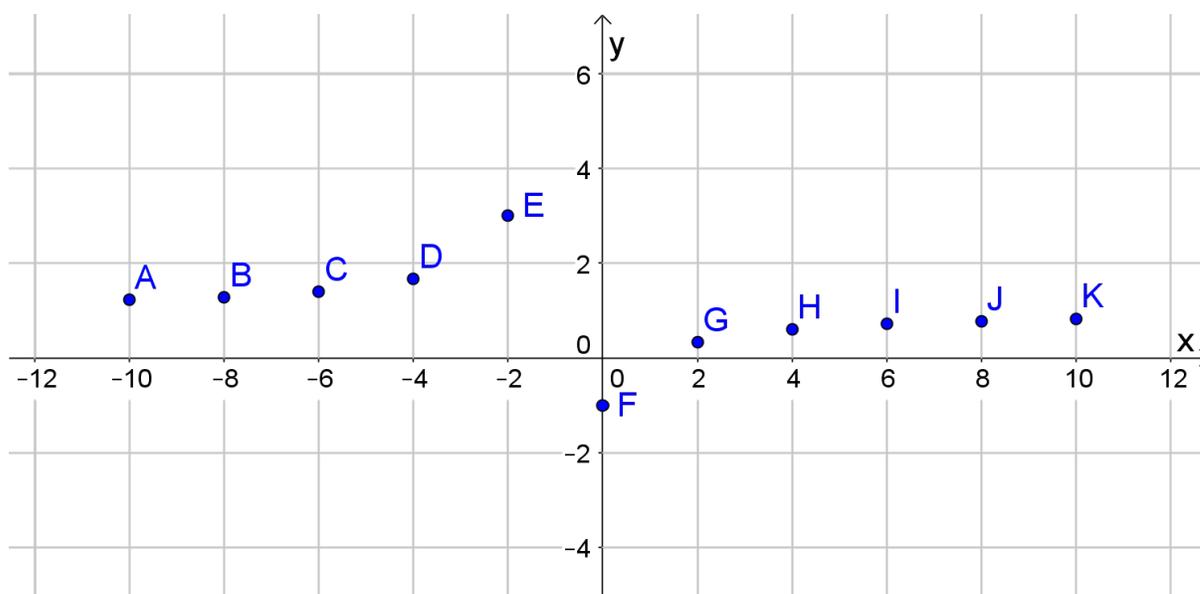


Figure 2.5

Ask a student to connect the points on this graph. At this point we can expect that the student will make the mistake and simply connect all points, specifically the mistake of connecting E to F . Allow this to happen but make sure to process the mistake right away.

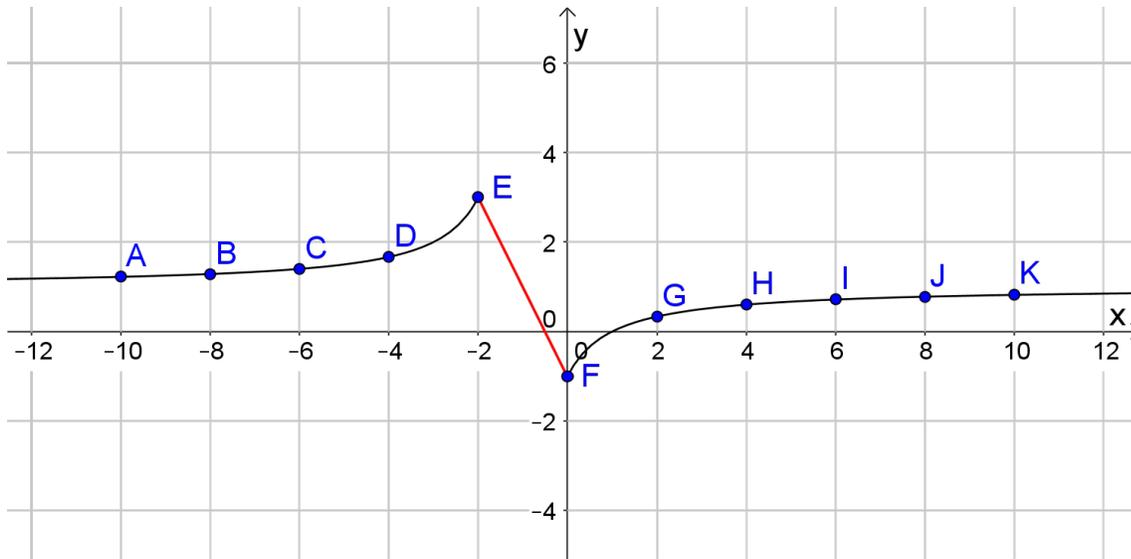


Figure 2.6: An expected mistake in graphing the function $f(x) = \frac{x-1}{x+1}$

Why would the graph unexpectedly break the smooth curve and jump from point E to point F ? The answer is that it doesn't! Let us take a look at the function again:

$$f(x) = \frac{x-1}{x+1}$$

Observe that the function will be undefined at $x = -1$. This means that there cannot be a line connecting point E and point F as this implies that there is a point in the graph of the function where $x = -1$. We will cover this aspect of graphs of rational functions in a future lesson, so for now we just present a partial graph for the function above as follows:

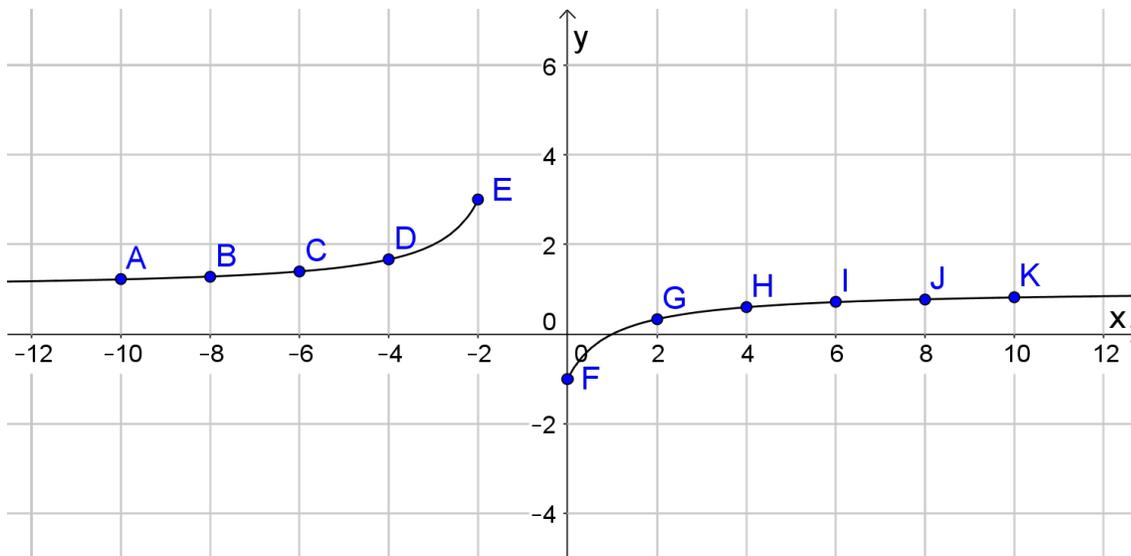


Figure 2.7: Remove the segment connecting E and F as the graph does not pass through points with an x -value of -1 .

□

EXAMPLE 5. Represent the rational function $f(x) = \frac{x^2 - 3x - 10}{x}$ using a table of values. Plot the points given in the table of values and sketch a graph by connecting the points.

Solution. As we have seen in the previous example, we will need to look at the x -values which will make the denominator zero. Since $x = 0$ will make the denominator zero, then f is not defined at 0. Taking function values integers in $-6 \leq x \leq 10$, $x \neq 0$ we get the following table of values:

x	-5	-4	-3	-2	-1	1	2
$f(x)$	-6	-4.5	-2.67	0	6	-12	-6

x	3	4	5	6	7	8	9	10
$f(x)$	-3.33	-1.5	0	1.33	2.57	3.75	4.89	6

Plotting the values above as points in the Cartesian plane:

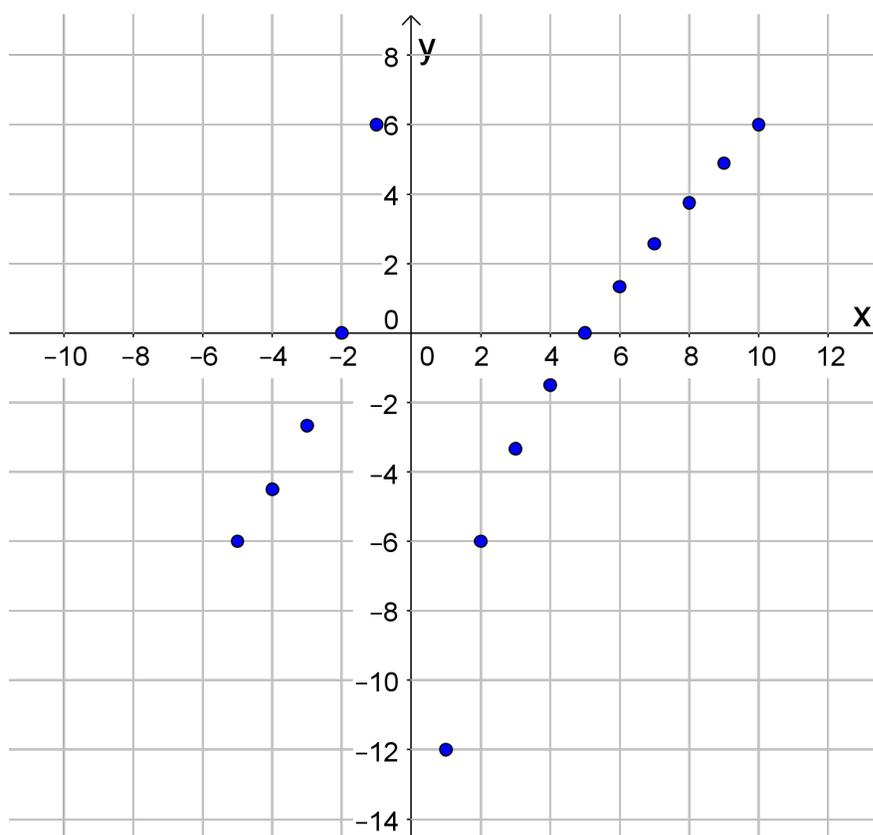


Figure 2.8

We connect the dots to sketch the graph, but we keep in mind that $x = 0$ is not part of the domain. For now we only connect those with values $x \leq -1$ and those with values $x \geq 1$.

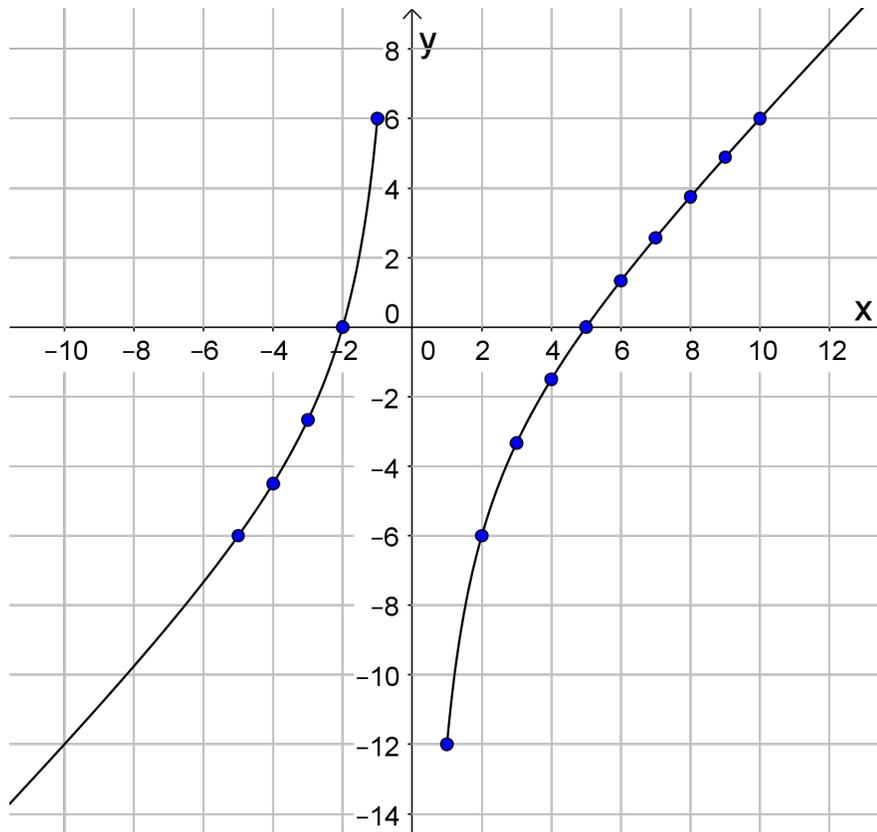


Figure 2.9

Note that $x = -2$ and $x = 5$ are zeroes of the rational function, which means that the function value at these values is zero. These x -values give the x -intercepts of the graph.

The behavior of the function near those values which make the function undefined will be studied in the next few lessons. □

In the previous lesson on rational equations, we discussed a basketball team playing in an interbarangay league. Let us return to that example and expand the scenario to use rational functions.

EXAMPLE 6. In an interbarangay basketball league, the team from Barangay Culiati has won 12 out of 25 games, a winning percentage of 48%. We have seen that they need to win 8 games consecutively to raise their percentage to at least 60%. What will be their winning percentage if they win

- (a) 10 games in a row?
- (b) 15? 20? 30? 50? 100 games?

Can they reach a 100% winning percentage?

Solution. Let x be the number of wins that Barangay Culiati needs to win in a row. Then the percentage p is a function of the number of wins that the team needs to win. The function can be written as:

$$p(x) = \frac{12 + x}{25 + x}$$

Construct a table of values for $p(x)$.

x	10	15	20	30	50	100
$p(x)$	0.63	0.68	0.71	0.76	0.83	0.90

We interpret the table of values as follows:

No. of consecutive wins	Win percentage
8	60%
10	63%
15	68%
20	71%
30	76%
50	83%
100	90%
200	94%
300	96%

Even if the team wins 300 consecutive games, the team still cannot reach a 100% winning percentage. Note that the denominator $25 + x$ is greater than the numerator, so 100% cannot be achieved. This is reasonable, since the team has lost 13 games already, so they cannot be "perfect".

□

EXAMPLE 7. Ten goats were set loose in an island and their population growth can be approximated by the function

$$P(t) = \left\lfloor \frac{60(t + 1)}{t + 6} \right\rfloor$$

where P represents the goat population in year t since they were set loose. Recall that the symbol $\lfloor \cdot \rfloor$ denotes the greatest integer function.

- How many goats will there be after 5 years?
- What is the maximum goat population that the island can support?

Solution. (a) Evaluate the function for $t = 5$:

$$P(5) = \left\lfloor \frac{60(5+1)}{5+6} \right\rfloor = \lfloor 32.726 \rfloor = 32$$

There will be 32 goats after 5 years.

(b) Construct a table of values for $P(x)$:

t	5	10	15	20	50	300	1000
$P(t)$	32	41	45	48	54	59	59

Observe that even if t is increased, the function does not exceed 59. The model suggests that the island cannot support only up to 59 goats. (Note that since the model is just an approximation, there may be errors and the number 59 may not be exact).

□

(C) Seatwork and Homework

Seatwork 1. The electric current I (in amperes) passing through a resistor is related to the voltage V across a resistor (in volts) and the resistance R (in ohms) by the formula $I = \frac{V}{R}$. If the resistor is connected across the terminals of a car battery (12 volts), construct a table of values of the current across the resistor for different resistor values from 2000 ohms to 5,000 ohms. Use intervals of 1000 ohms. Plot these values as points in the Cartesian plane and connect the points. Is this a smooth curve or a straight line?

Answer:

R	2,000	3,000	4,000	5,000
I	0.006	0.004	0.003	0.0024

; smooth curve

Seatwork 2. Construct a table of values for the function $f(x) = \frac{x-3}{x+4}$ for $-6 \leq x \leq 2$, x taking on integer values. Identify values of x where the function will be undefined. Plot the points corresponding to values in the table. Connect these points with a smooth curve. Explain why the points cannot be joined by a single curve. Identify the zeroes of this function.

Answer:

x	-6	-5	-4	-3	-2	-1	0	1	2
$f(x)$	4.5	8	und.	-6	-2.5	-1.33	-0.75	-0.4	-0.167

The points cannot be joined by a single curve because f is not defined at $x = -4$. It has a vertical asymptote there.

Seatwork 3. For the function $f(x) = \frac{x^2 + x - 6}{x^2 + x - 20}$:

(a) Identify values where the function will be undefined.

Answer: $x = 4, -5$

(b) Construct a table of values for $-6 \leq x \leq 2$ where x takes on integer values;

Answer:

x	-6	-5	-4	-3	-2	-1	0	1	2
$f(x)$	2.4	und	-0.75	0	0.22	0.3	0.3	0.22	0

(c) Identify the zeroes of the function.

Answer: $x = -3, 2$

(d) Plot and sketch the graph of the function.

LESSON 8: Graphing Rational Functions

TIME FRAME: 120 minutes

CODE: M11GM-Ib-5, c-1, c-2, and c-3

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to find the domain and range, intercepts, zeroes, asymptotes of rational functions, graph rational functions, and solve problems involving rational functions.

LESSON OUTLINE:

1. Review: domain, range, zeroes, intercepts of functions
 2. Domain and intercepts of functions
 3. Using a table of signs to introduce the idea of asymptotes
 4. Definition of a vertical asymptote
 5. Definition of a horizontal asymptote
 6. Identifying asymptotes of a rational function
 7. Graphing a rational function using its asymptotes, intercepts and table of signs.
 8. Identifying the range of a rational function through its graph
-

DEVELOPMENT OF THE LESSON

(A) Recall of important concepts

Start the lesson by recalling the following important properties of functions:

- (a) The **domain** of a function is the set of all values that the variable x can take.
- (b) The **range** of a function is the set of all values that $f(x)$ can take.
- (c) The **zeroes** of a function are the values of x which make the function zero. The real numbered zeroes are also **x -intercepts** of the graph of the function.
- (d) The **y -intercept** is the function value when $x = 0$.

(B) Lesson Development

In sketching the graph of a rational function, we will make use of the x and y -intercepts of the function. In addition, we will observe how the function behaves near the values where the function is undefined, as well as in the extreme values of x .

Let us take a detailed look at the steps in graphing a simple rational function and develop a process for graphing other rational functions.

EXAMPLE 1. Consider the function $f(x) = \frac{x-2}{x+2}$. (a) Find its domain, (b) intercepts, (c) sketch its graph and (d) determine its range.

Solution. (a) The **domain of** $f(x)$ is $\{x \in \mathbb{R} \mid x \neq -2\}$.

Explanation: Observe that the function is undefined at $x = -2$. This means that $x = -2$ is not part of the domain of $f(x)$. In addition, no other values of x will make the function undefined.

(b) The **x -intercept** of $f(x)$ is 2 and its **y -intercept** is -1 .

Explanation: Recall that the x -intercepts of a rational function are the values of x that will make the function zero. A rational function will be zero if its numerator is zero. Therefore the zeroes of a rational function are the zeroes of its numerator.

The numerator $x - 2$ will be zero at $x = 2$. Therefore $x = 2$ is a **zero** of $f(x)$. Since it is a real zero, it is also an **x -intercept**.

The **y -intercept** of a function is equal to $f(0)$. In this case, $f(0) = \frac{-2}{2} = -1$.

(c) To sketch the graph of $f(x)$, let us look at what happens to the graph near the values of x which make the denominator undefined. Recall that in the previous lesson, we simply skipped connecting the points at integer values. Let us see what happens when x takes on values that takes the denominator closer to zero.

The denominator is zero when $x = -2$. Let us look at values of x **close to -2 on its left side** (i.e. $x < -2$, denoted -2^-) and values of x **close to -2 on its right side** (i.e. $x > -2$, denoted -2^+).

i. Table of values for x approaching -2^- .

x	-3	-2.5	-2.1	-2.01	-2.001	-2.0001	As x approaches -2^-
$f(x)$	5	9	41	401	4001	40001	$f(x)$ increases without bound

Notation: We use the notation ' $f(x) \rightarrow +\infty$ as $x \rightarrow -2^-$ ' to indicate that $f(x)$ increases without bound as x approaches -2 from the left.

ii. Table of values for x approaching -2^+ .

x	-1	-1.5	-1.9	-1.99	-1.999	-1.9999	As x approaches -2^+
$f(x)$	-3	-7	-39	-399	-3999	-39999	$f(x)$ decreases without bound

Notation: We use the notation ' $f(x) \rightarrow -\infty$ as $x \rightarrow -2^+$ ' to indicate that $f(x)$ decreases without bound as x approaches -2 from the right.

After plotting the points corresponding to these values on the Cartesian plane, we obtain

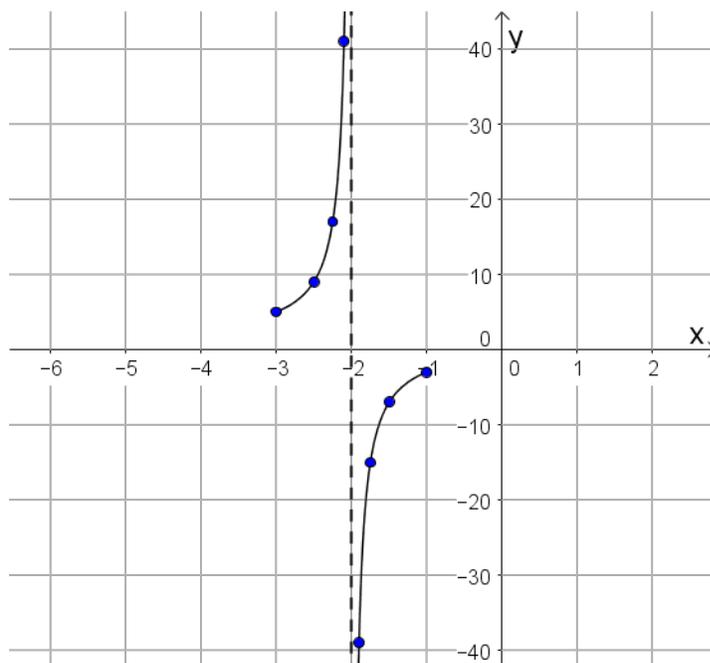


Figure 2.10: Note that the axes do not have the same scale.

Observe that as x approaches -2 from the left and from the right, the graph gets closer and closer to the line $x = -2$, indicated in the figure with a dashed line. We call this line a *vertical asymptote*, which we formally define as follows.

Definition.

The vertical line $x = a$ is a **vertical asymptote** of a function f if the graph of f either increases or decreases without bound as the x -values approach a from the right or left.

Finding the Vertical Asymptotes of a Rational Function

- (a) Reduce the rational function to lowest terms by cancelling out the common factor/s in the numerator and denominator.
- (b) Find the values a that will make the denominator of the reduced rational function equal to zero.
- (c) The line $x = a$ is a vertical asymptote.

We will also look how the function behaves as x increases or decreases without bound. We first construct a table of values for $f(x)$ as x increases without bound, or in symbols, as $x \rightarrow +\infty$.

iii. Table of values for $f(x)$ as $x \rightarrow +\infty$.

x	5	10	100	1000	10000	As $x \rightarrow +\infty$
$f(x)$	0.43	0.67	0.96	0.9960	0.99960	$f(x)$ approaches 1^-

Next, construct a table of values for $f(x)$ as x decreases without bound, or in symbols, as $x \rightarrow -\infty$.

iv. Table of values for $f(x)$ as $x \rightarrow -\infty$.

x	-5	-10	-100	-1000	-10000	As $x \rightarrow -\infty$
$f(x)$	2.33	1.5	1.041	1.00401	1.0004001	$f(x)$ approaches 1^+

After plotting the points corresponding to these values on the Cartesian plane, we obtain

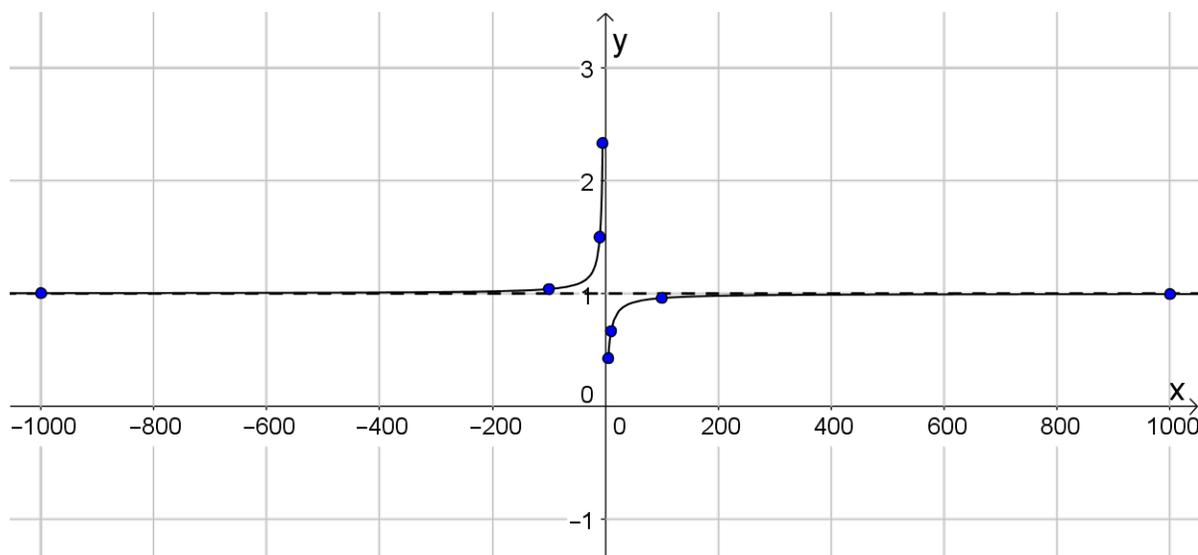


Figure 2.11: Note that the axes do not have the same scale.

Observe that as x increases or decreases without bound, $f(x)$ gets closer and closer to 1. The line $y = 1$ is indicated in the figure with a dashed line. We call this line a *horizontal asymptote*, which we formally define as follows.

Definition.

The horizontal line $y = b$ is a **horizontal asymptote** of the function f if $f(x)$ gets closer to b as x increases or decreases without bound ($x \rightarrow +\infty$ or $x \rightarrow -\infty$).

A rational function may or may not cross its horizontal asymptote. If the function does not cross the horizontal asymptote $y = b$, then b is not part of the range of the rational function.

Now that we know the behavior of the function as x approaches -2 (where the function is undefined), and also as $x \rightarrow +\infty$ or $x \rightarrow -\infty$, we can complete the sketch of the graph by looking at the behavior of the graph at the zeroes.

Construct a table of signs to determine the sign of the function on the intervals determined by the zeroes and the vertical asymptotes. That is, construct a table of signs using the values of x that make either the numerator or denominator 0 as boundaries. In this example, the boundaries are $x = -2$ and $x = 2$.

Interval	$x < -2$	$-2 < x < 2$	$x > 2$
Test point	$x = -3$	$x = 0$	$x = 3$
$x - 2$	-	-	+
$x + 2$	-	+	+
$\frac{x - 2}{x + 2}$	\oplus	\ominus	\oplus
Graph	above the x -axis	below the x -axis	above the x -axis

Observe that at the x -intercept $x = 2$, the graph changes from negative to positive.

Plot the zeroes, the y -intercept, and the asymptotes. From the table of signs and the previous graphs, we know that $f(x) < 1$ as $x \rightarrow +\infty$ and $f(x) > 1$ as $x \rightarrow -\infty$. Draw a short segment across $(2, 0)$ to indicate that the function transitions from negative to positive at this point.

We also know that $f(x)$ increases without bound as $x \rightarrow -2^-$ and $f(x)$ decreases without bound as $x \rightarrow -2^+$. Sketch some arrows near the asymptotes to indicate this information.

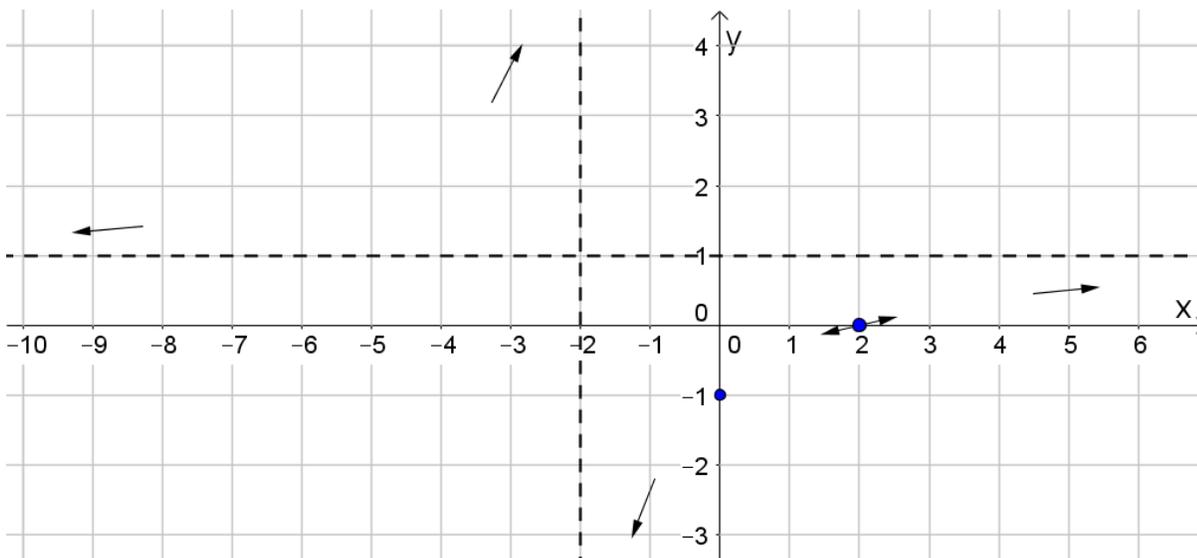


Figure 2.12: Zeroes and asymptotes of $f(x)$.

Trace the arrowheads along with the intercepts using smooth curves. Do not cross the vertical asymptote.

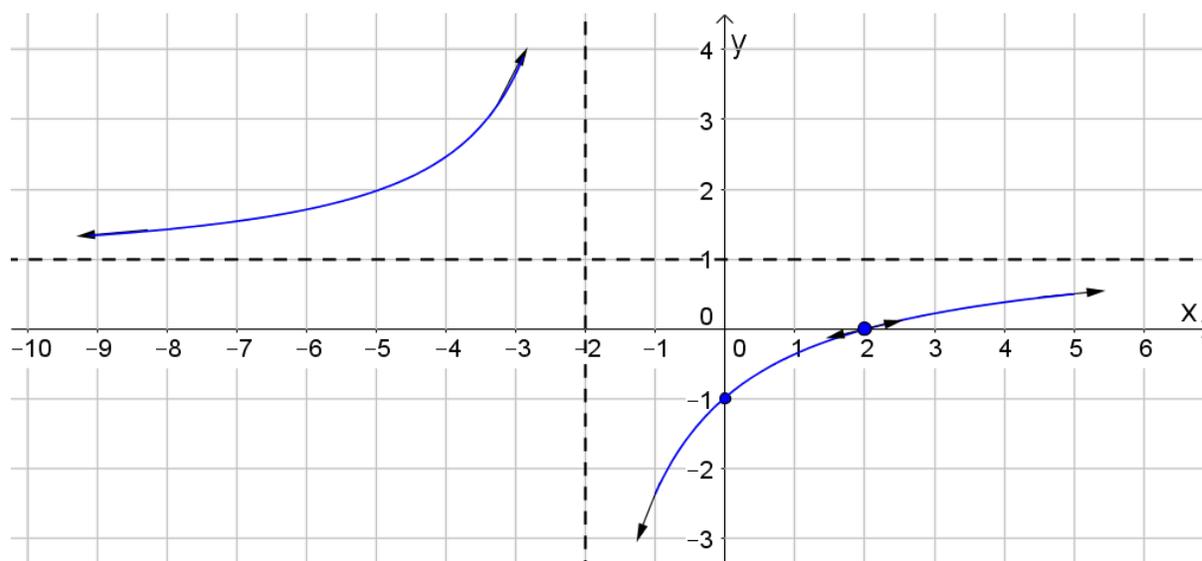


Figure 2.13: Tracing with smooth curves.

This graph shows the correct behavior of the function, but it may not show the correct heights. To do so, one would need to plot more points. The actual graph is shown below. This graph was obtained through the use of a graphing software.

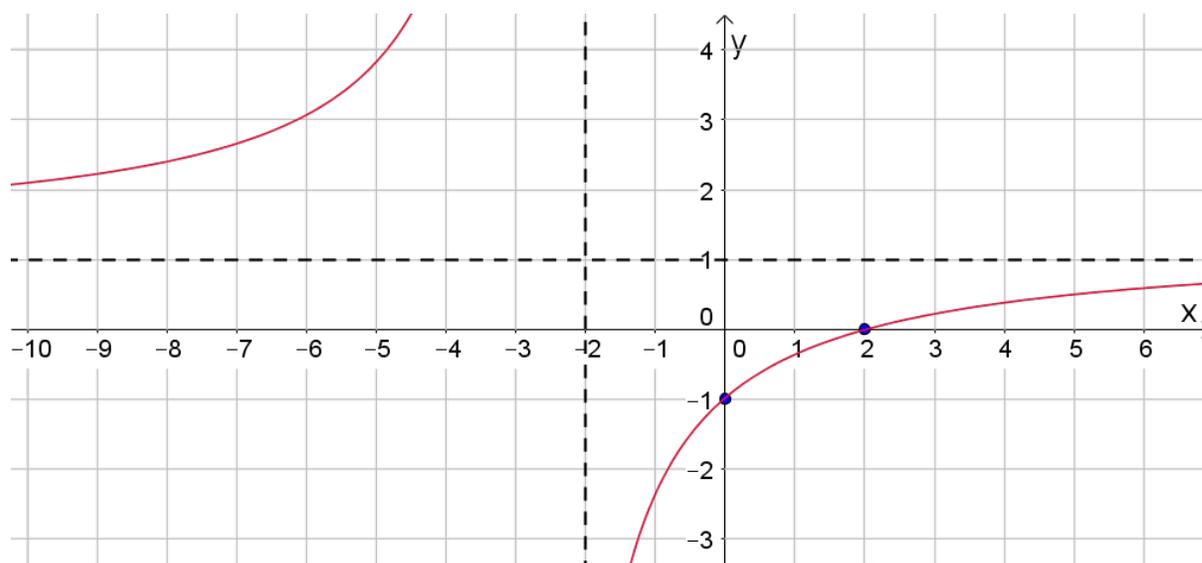


Figure 2.14: The actual sketch of the graph of $y = \frac{x-2}{x+2}$ for reference.

(d) From the graph of the rational function, we observe that the function does not cross the horizontal asymptote. We also observe that the function increases and decreases without bound, and is asymptotic to the line $y = 1$. Therefore only the value 1 is not included in the range of $f(x)$. The range of

$f(x)$ is $(-\infty, 1) \cup (1, +\infty)$.

□

EXAMPLE 2. Find the horizontal asymptote of $f(x) = \frac{4x^2 + 4x + 1}{x^2 + 3x + 2}$.

Solution. We have seen from the previous example that the horizontal asymptotes can be determined by looking at the behavior of rational functions at extreme values of x ($x \rightarrow -\infty$ or $x \rightarrow +\infty$).

However, at extreme values of x , the value of a polynomial can be approximated using the value of the leading term. For example, if $x = 1000$, the value of $4x^2 + 4x + 1$ is 4,004,001. A good approximation is the value of $4x^2$, which is 4,000,000.

Similarly, for extreme values of x , the value of $x^2 + 3x + 2$ can be approximated by x^2 . Thus, for extreme values of x , then $f(x)$ approaches can be approximated by $\frac{4x^2}{x^2} = 4$, and therefore $f(x)$ approaches 4 for extreme values of x . This means that we have a horizontal asymptote at $y = 4$. □

EXAMPLE 3. Find the horizontal asymptote of $f(x) = \frac{2x^2 - 5}{3x^2 + x - 7}$.

Solution. Following the idea from the previous example, the value of $\frac{2x^2 - 5}{3x^2 + x - 7}$ can be approximated by $\frac{2x^2}{3x^2} = \frac{2}{3}$ for extreme values of x . Thus, the horizontal asymptote is $y = \frac{2}{3}$. □

EXAMPLE 4. Find the horizontal asymptote of $f(x) = \frac{3x + 4}{2x^2 + 3x + 1}$.

Solution. Again based on the idea from the previous example, the value of $\frac{3x + 4}{2x^2 + 3x + 1}$ can be approximated by $\frac{3x}{2x^2} = \frac{3}{2x}$ for extreme values of x . If we substitute extreme values of x in $\frac{3}{2x}$, we obtain values very close to 0. Thus, the horizontal asymptote is $y = 0$. □

EXAMPLE 5. Show that $f(x) = \frac{4x^3 - 1}{3x^2 + 2x - 5}$ has no horizontal asymptote.

Solution. For large values of x , the value of $\frac{4x^3 - 1}{3x^2 + 2x - 5}$ can be approximated by $\frac{4x^3}{3x^2} = \frac{4x}{3}$. If we substitute extreme values of x in $\frac{4x}{3}$, we obtain extreme values as well. Thus, if x takes on extreme values, then y also takes on extreme values and do not approach a particular finite number. The function has no horizontal asymptote. □

We summarize the results from the previous examples as follows.

Horizontal asymptotes of a rational function.

Let n be the degree of the numerator and m be the degree of the denominator:

- If $n < m$, the horizontal asymptote is $y = 0$.
- If $n = m$, the horizontal asymptote is $y = \frac{a}{b}$, where a is the leading coefficient of the numerator and b is the leading coefficient of the denominator.
- If $n > m$, there is no horizontal asymptote.

Finally, the table below provides a summary to the entire lesson.

How to find the	Do the following:
domain	Find all x values which do not make the denominator 0.
y -intercept	If 0 is not in the domain, then there is no y -intercept. If 0 is in the domain, the y -intercept is the value of the function at $x = 0$.
x -intercept	Find the values of x where the numerator is zero but the denominator is non-zero.
vertical asymptotes	Find the values of x where the denominator of the reduced rational function is zero.
horizontal asymptote	Use the degree of the polynomial in the numerator and denominator, as indicated above.
range	Graph the function. (However, there may be rational functions where more advanced techniques like calculus are needed)

EXAMPLE 6. Sketch the graph of $f(x) = \frac{3x^2 - 8x - 3}{2x^2 + 7x - 4}$. Find its domain and range.

Solution. The domain of the function is all values of x not including those where the function is undefined. Therefore, the domain of $f(x)$ is the set $(-\infty, -4) \cup (-4, \frac{1}{2}) \cup (\frac{1}{2}, +\infty)$.

The numerator and denominator of $f(x)$ can be factored as follows:

$$f(x) = \frac{3x^2 - 8x - 3}{2x^2 + 7x - 4} = \frac{(3x + 1)(x - 3)}{(2x - 1)(x + 4)}$$

We also obtain the following:

- y -intercept: $f(0) = \frac{0 - 0 - 3}{0 + 0 - 4} = \frac{3}{4}$
- x -intercepts or zeroes: $3x + 1 = 0 \Rightarrow x = -\frac{1}{3}$; $x - 3 = 0 \Rightarrow x = 3$

- vertical asymptotes: $2x - 1 = 0 \Rightarrow x = \frac{1}{2}$; $x + 4 = 0 \Rightarrow x = -4$
- horizontal asymptote: The polynomials in the numerator and denominator have equal degree. The horizontal asymptote is the ratio of the leading coefficients: $y = \frac{3}{2}$

Plot the intercepts and asymptotes on the Cartesian plane.

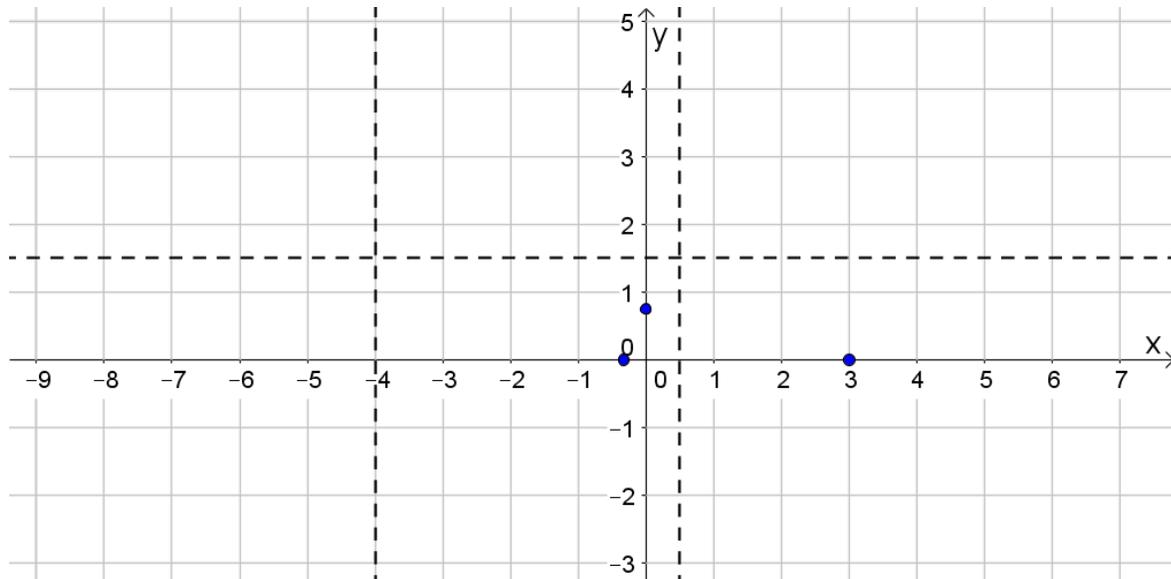


Figure 2.15: Intercepts and asymptotes of $f(x)$.

Construct a table of signs using the values of x that make either the numerator or denominator 0 as boundaries. These are (in increasing order) -4 , $-\frac{1}{3}$, $\frac{1}{2}$, and 3 .

Interval	$x < -4$	$-4 < x < -\frac{1}{3}$	$-\frac{1}{3} < x < \frac{1}{2}$	$\frac{1}{2} < x < 3$	$x > 3$
Test point	$x = -10$	$x = -2$	$x = 0$	$x = 1$	$x = 10$
$3x + 1$	-	-	+	+	+
$x - 3$	-	-	-	-	+
$2x - 1$	-	-	-	+	+
$x + 4$	-	+	+	+	+
$\frac{(3x + 1)(x - 3)}{(2x - 1)(x + 4)}$	⊕	⊖	⊕	⊖	⊕
Graph	above x -axis	below x -axis	above x -axis	below x -axis	above x -axis

Draw sections of the graph through the zeroes indicating the correct transition based on the table of signs.

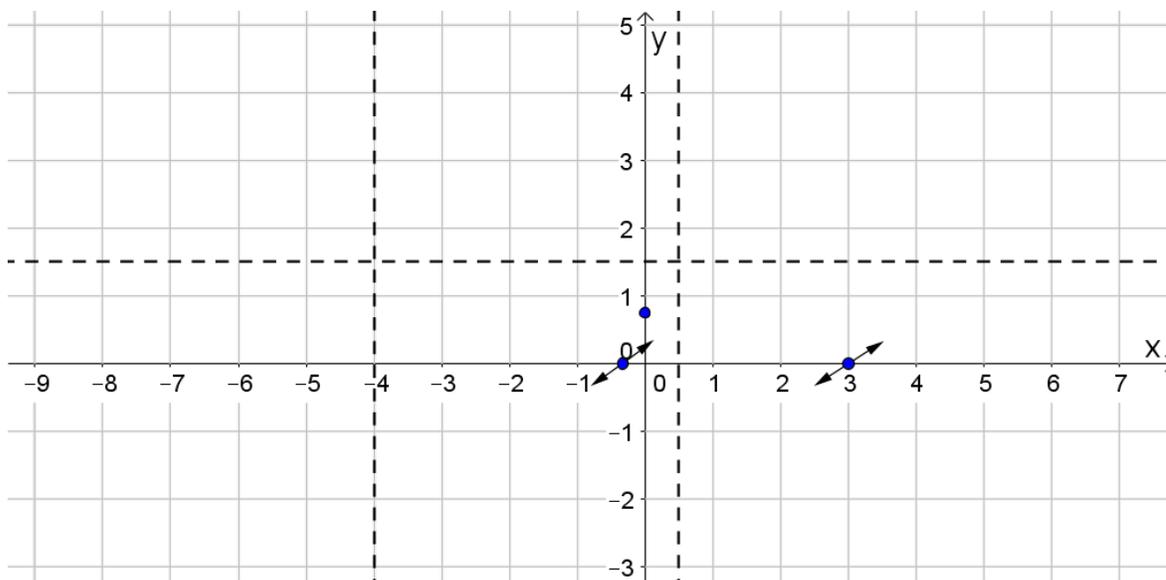


Figure 2.16: Sketch the transitions across the zeroes based on the table of signs.

Draw sections of the graph near the asymptotes based on the transition indicated on the table of signs.

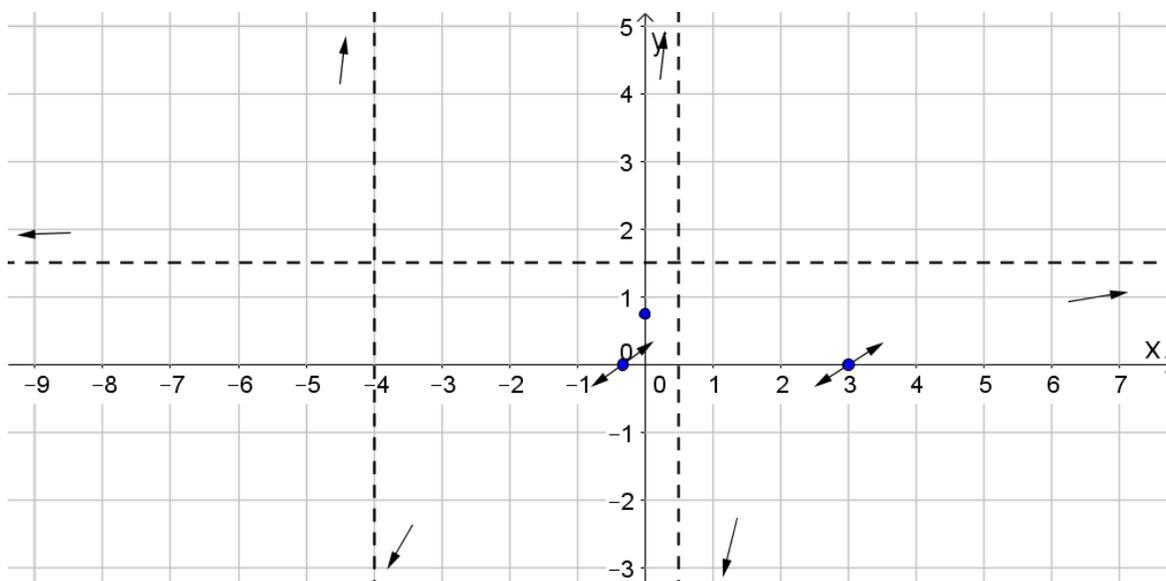


Figure 2.17: Sketch the graph near the asymptotes based on the table of signs.

Complete the sketch by connecting the arrowheads, making sure that the sketch passes through the y -intercept as well. The sketch should follow the horizontal asymptote as the x -values goes to the extreme left and right of the Cartesian plane.

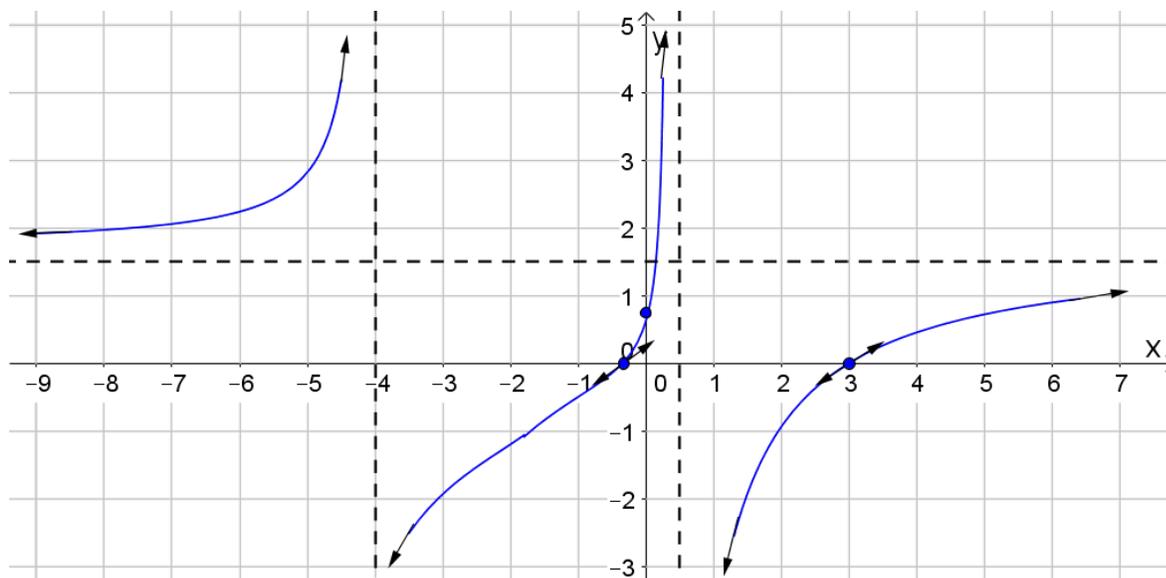


Figure 2.18: Rough sketch of the graph following the information above.

The actual graph, obtained through a graphing software, is shown in the next figure.

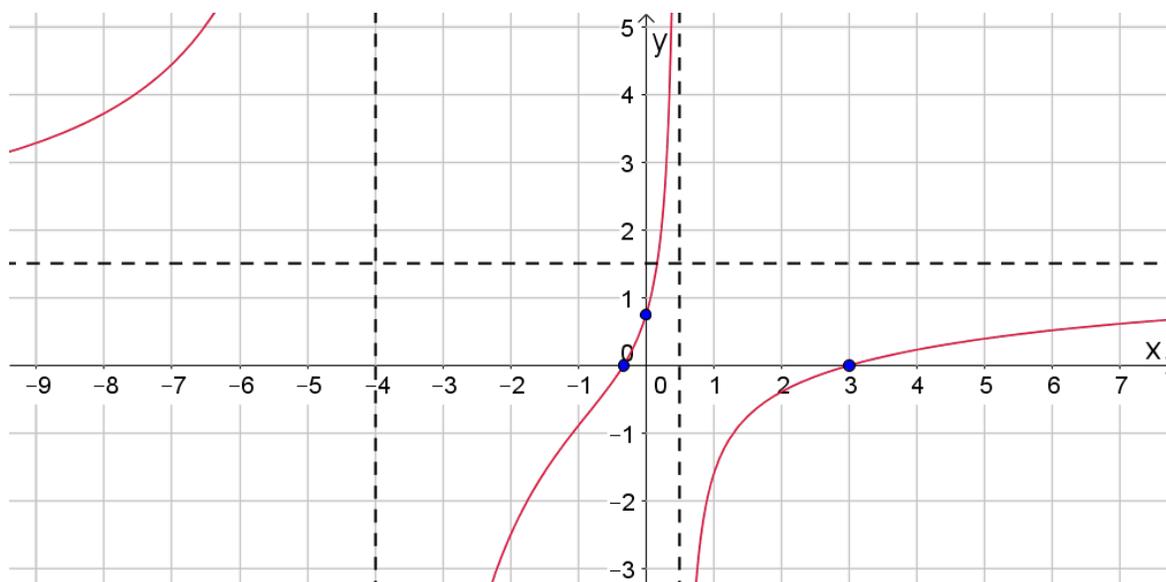


Figure 2.19: Actual sketch of the graph using a software grapher.

From the graph, we observe that the function increases and decreases without bound. The graph also crosses the horizontal asymptote. Therefore the range of the function is the set \mathbb{R} of all real numbers. \square

(C) Seatwork

Seatwork 1. For each, identify the x - and y -intercepts, the asymptotes, and the intervals where the function is above or below the x -axis. Sketch the graph. Identify the domain, and (if possible) the range. You can confirm your work using a graphing software.

(a) $f(x) = \frac{2}{x+1}$

(f) $f(x) = \frac{(5x-2)(x-2)}{(3x-4)(x+2)}$

(b) $f(x) = \frac{2}{x^2+2x+1}$

(g) $f(x) = \frac{x^2-x+6}{x^2-6x+8}$

(c) $f(x) = \frac{3x}{x+3}$

(h) $f(x) = \frac{x^2-4x-5}{x-4}$

(d) $f(x) = \frac{2x+3}{4x-7}$

(i) $f(x) = \frac{x-1}{x^3-4x}$

(e) $f(x) = \frac{(4x-3)(x-1)}{(2x+1)(x+1)}$

(j) $f(x) = \frac{x^2-9}{x^2+4}$

Seatwork 2. Past records from a factory producing electronic components show that on average, new employees can assemble $N(t)$ components per day after t days of training, where

$$N(t) = \frac{75t}{t+5} \quad t \geq 0.$$

Sketch the graph of N on the first quadrant, and include the intercepts and asymptotes. What happens to $N(t)$ as $t \rightarrow \infty$? What does this mean in practical terms? **Answer:** As $t \rightarrow \infty$, $N(t) \rightarrow 75$. Even if a person trains for several days, the maximum number of items that he or she can assemble cannot exceed 75.

Seatwork 3. After a drug is injected into a patient's bloodstream, the concentration c of the drug in the bloodstream t minutes after the injection is given by

$$c(t) = \frac{20t}{t^2+2} \quad t \geq 0.$$

Sketch the graph of $c(t)$ and include the intercepts and asymptotes. What happens to $c(t)$ as $t \rightarrow \infty$? What does this mean in practical terms? **Answer:** As $t \rightarrow \infty$, $c(t) \rightarrow 0$. As time passes by, the drug concentration eventually reaches 0.

Seatwork 4. *A challenging riddle.* I am a rational function having a vertical asymptote at the lines $x = 3$ and $x = -3$, and a horizontal asymptote at $y = 1$. If my only x -intercept is 5, and my y -intercept is $-\frac{5}{9}$, what function am I? **Sample Answer:**

$$f(x) = \frac{(x-5)^2(x^2+1)}{(x+3)(x-3)(x^2+5)}$$

Performance Task

Consider the rational function $p = \frac{5125000V^2 - 449000V + 19307}{125V^2(1000V - 43)}$. This function is based on the van der Waals equation for predicting the pressure p of a gas as a function of V at a fixed temperature. The function above models the pressure p of carbon dioxide in terms of volume V if the temperature is 500 kelvins.¹

1. What are the vertical asymptote(s) of the function? *Ans:* $V = 0$ and $V = 43/1000$
2. What is the horizontal asymptote of the function? *Ans:* $p = 0$
3. What are the p -intercept(s) of the function? *Ans:* None
4. What is the V -intercept of the function? *Ans:* None
5. Use a table of signs to determine when the function is positive or negative.

Students will process their responses from the performance task to answer various questions, such as:

1. Sketch the graph of the function. What do you think is the domain of the function given this context?
Hint: Remember that pressure and volume cannot be negative.
2. Explain the significance of the vertical asymptote(s) in practical terms. *Ans:* There is a vertical asymptote at $V = 0.043$ liters. If we try to "squeeze" in the carbon dioxide a space that is almost small as this, the pressure will be extremely high.
3. Explain the significance of the horizontal asymptote. *Ans:* As the volume increases, the pressure becomes closer to 0. This makes sense because if there is a very large space to contain the gas, then the pressure becomes very small.

¹Crauder, B., Evans, B., & Noell, A. (2008). *Functions and change: A modeling approach to college algebra and trigonometry*

Chapter 3

One-to-One and Inverse Functions

LESSON 9: One-to-one Functions

TIME FRAME: 60 minutes

CODE: M11GM-Id-1 and g-2

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to represent real-life situations using one-to-one functions.

LESSON OUTLINE:

1. Definition of a one-to-one function.
 2. Real-life situations that can be represented by one-to-one functions.
 3. Properties of one-to-one functions.
 4. Horizontal line test.
-

DEVELOPMENT OF THE LESSON

(A) Motivation Activity

In 1999, NASA lost a \$125 million Mars orbiter satellite because a Lockheed Martin engineering team used English units of measurement while NASA used SI units for an important spacecraft operation.

The satellite was planned to be the first weather observer on another planet, but instead of slipping into a stable orbit, it disappeared from the scientists tracking it.

It turned out that the software controlling the orbiter's thrusters calculated the force needed to exert in *pounds* of force, while another piece of software interpreted the resulting calculation in *Newtons*.

The resulting miscalculation brought the spacecraft too close to the planet, making it disintegrate as it entered the Martian atmosphere.

Group the class into pairs and have them perform the following activity:

Each pair is tasked to coordinate an engineering team based in the United States who uses English units of measurement with an engineering team in the Philippines using SI units. The two engineering teams are tasked with designing and constructing an engine for a motorcycle that can be used underwater. Listed below are the conversion factors:

English unit	SI unit	English unit	SI unit
1 pound-force	4.448 Newtons	1 Newton	0.2248 pounds-force
1 foot	0.3048 meters	1 meter	3.281 feet
1 mile	1.609 kilometers	1 kilometer	0.6214 miles
1 gallon	3.785 liters	1 liter	0.2642 gallons

The following formulas for converting Fahrenheit to Celsius and vice versa are also given:

$$T_{\circ C} = \frac{5}{9}(T_{\circ F} - 32)$$

$$T_{\circ F} = \frac{9}{5}T_{\circ C} + 32$$

The first task is to convert and translate the English units in the following message from the US team to metric:

‘As you can see, the engine should be able to provide **850 pounds** of force at half-throttle even when it is **47 feet** underwater. Having sections of the engine casing exposed to the water is great for keeping it cooled down, but we should have a warning sensor whenever the engine temperature reaches **200° F**. The design also guarantees that the motorcycle can travel from **12 to 16 miles** underwater at a full tank of **3.4 gallons**.’

Having converted this message, the Philippine team responded with the following message. Your team is tasked to perform the translations from metric to English units this time:

‘Our test conditions restrict us to working up to **12 meters** underwater, and at this depth, we observe the engine providing **185 Newtons** of force. Also, the temperature you specified is still well within the safe operating temperature of the engine, and we recommend that the warning sensor triggers at **120° C** instead. We were also able to extend the distance covered by the motorcycle to **4.5 kilometers per liter** of fuel.’

Verify the students’ answers using the answer key:

850 lb-f	3780.08 N	12 m	39.37 ft
47 feet	14.3256 m	185 N	41.59 lb-f
200° <i>F</i>	93.33° <i>C</i>	120° <i>C</i>	248° <i>F</i>
12 to 16 miles	19.31 to 25.74 km	4.5 km	2.796 miles
3.4 gallons	12.87 L	1 L	0.2642 gallons

Have the class observe that in each case, the conversion is invertible, that is, conversion can be done in both directions.

(B) Lesson Development

Definition:

The function f is **one-to-one** if for any x_1, x_2 in the domain of f , then $f(x_1) \neq f(x_2)$.
That is, the same y -value is never paired with two different x -values.

The unit conversions performed in the motivation activity represent one-to-one functions as we cannot get two different SI measurements for a given English measurement. The reverse is also true.

In Examples 1-5, ask the class to determine whether the given relation is a function. If it is a function, determine whether it is one-to-one.

EXAMPLE 1. The relation pairing an SSS member to his or her SSS number.

Solution. Each SSS member assigned a unique SSS number. Thus, the relation is a function. Further, two different members cannot be assigned the same SSS number. Thus, the function is one-to-one \square

EXAMPLE 2. The relation pairing a real number to its square.

Solution. Each real number has a unique perfect square. Thus, the relation is a function. However, two different real numbers such as 2 and -2 may have the same square. Thus, the function is not one-to-one. \square

EXAMPLE 3. The relation pairing an airport to its airport code. Airport codes are three letter codes used to uniquely identify airports around the world and prominently displayed on checked-in bags to denote the destination of these bags. Here are some examples of airport codes:

- MNL - Ninoy Aquino International Airport (All terminals)
- CEB - Mactan-Cebu International Airport
- DVO - Francisco Bangoy International Airport (Davao)
- JFK - John F. Kennedy International Airport (New York City)

- CDG - Charles de Gaulle International Airport (Paris, France)

Airport codes can be looked up at <https://www.world-airport-codes.com>

Solution. Since each airport has a unique airport code, then the relation is a function. Also, since no two airports share the same airport code, then the function is one-to-one. \square

EXAMPLE 4. The relation pairing a person to his or her citizenship.

Solution. The relation is not a function because a person can have dual citizenship (i.e., citizenship is not unique). \square

EXAMPLE 5. The relation pairing a distance d (in kilometers) traveled along a given jeepney route to the jeepney fare for traveling that distance.

Solution. The relation is a function since each distance traveled along a given jeepney route has an official fare. In fact, as shown in Lesson 1, the jeepney fare may be represented by a piecewise function, as shown below:

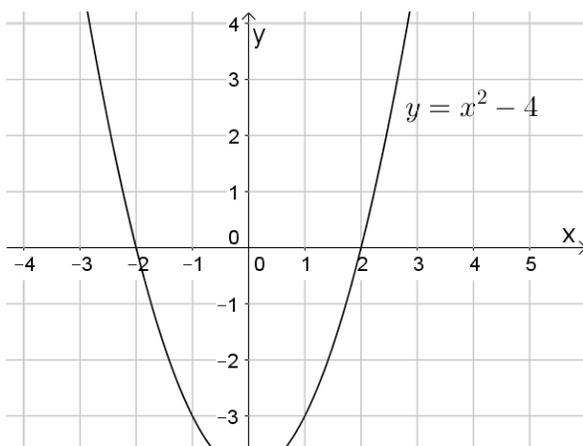
$$F(d) = \begin{cases} 8.00 & \text{if } 0 < d \leq 4 \\ (8.00 + 1.50 \lfloor d \rfloor) & \text{if } d > 4 \end{cases}$$

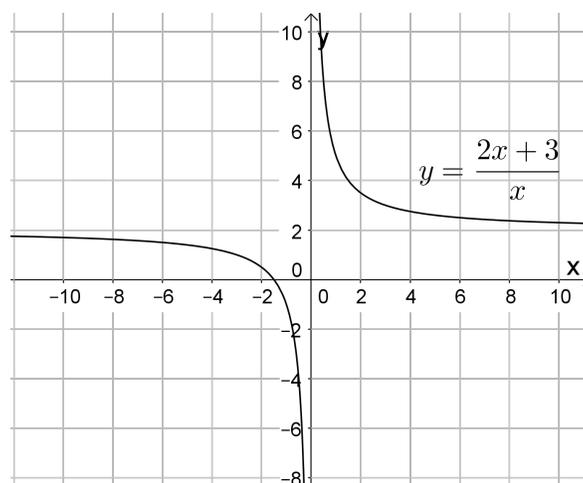
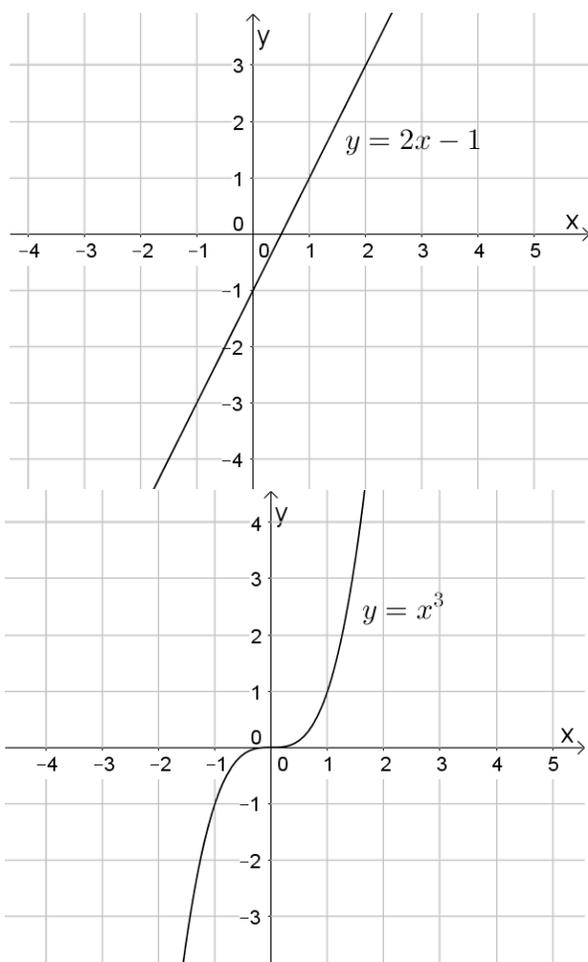
Note that $\lfloor d \rfloor$ is the floor function applied to d .

If the distance to be traveled is 3 kilometers, then $F(3) = 8$. However, the function is not one-to-one because different distances (e.g., 2, 3 or 4 kilometers) are charged the same rate (₱8.00). That is, because $F(3) = F(2) = F(3.5) = 8$, then F is not one-to-one. \square

Graphs of One-to-One Functions

Show the following graphs to the class. Ask them if the given function is one-to-one or not by having them check if there is a y -value with more than one corresponding x -value.





The students with the teacher's guidance should identify that the second, third and fourth graphs represent one-to-one functions. The first graph is a quadratic function. It is not one-to-one because the y -value 0 is paired with two x -values, namely 2 and -2 .

A simple way to determine if a given graph is that of a one-to-one function is by using the Horizontal Line Test.

Horizontal Line Test

A function is one-to-one if each horizontal line does **not** intersect the graph at more than one point.

The first graph showing the plot of $y = x^2 - 4$ fails the horizontal line test because some lines intersect the graph at more than one point. The remaining three graphs pass the horizontal line test, because all horizontal lines intersect the graph at most once.

The Vertical and Horizontal Line Tests

All functions satisfy the vertical line test. All one-to-one functions satisfy *both* the vertical and horizontal line tests.

LESSON 10: Inverse of One-to-one Functions**TIME FRAME:** 120 minutes**CODE:** M11GM-Id-2, d-3, and e-2

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to determine the inverse of a one-to-one function, represent an inverse function through its table of values, and solve problems involving inverse functions.

LESSON OUTLINE:

1. Inverse of a one-to-one function.
2. Properties of the inverse of a one-to-one function.

DEVELOPMENT OF THE LESSON

(A) Motivation

Consider the table of values for the function given by the equation $y = 2x - 1$ given below:

x	-4	-3	-2	-1	0	1	2	3	4
y	-9	-7	-5	-3	-1	1	3	5	7

Verify that it is a one-to-one function by showing that no two y -values share the same x -value.

Let us invert the values for x and y :

x	-9	-7	-5	-3	-1	1	3	5	7
y	-4	-3	-2	-1	0	1	2	3	4

Does this table still represent a function?

We should see that it can still represent a function because each x value is associated with only one y value.

Next consider the table of values for another function below:

x	-4	-3	-2	-1	0	1	2	3	4
y	-1	-1	-1	-1	0	1	1	1	1

Show that the table does not represent a function because there are some y -values that are paired with more than one x -value. For example, $y = 1$ is paired with $x = 1, 2, 3, 4$.

Invert the values for x and y . Will the resulting table still represent a function?

x	-1	-1	-1	-1	0	1	1	1	1
y	-4	-3	-2	-1	0	1	2	3	4

The resulting table does **not** represent a function since $x = 1$ is paired with more than one y -value; namely, 1, 2, 3 and 4.

'Inverting' functions

The previous discussion shows that

- if the x - and y -values of a one-to-one function are interchanged, the result is a function, but
- if the x - and y -values of a function that is not one-to-one are inverted, the result is no longer a function.

(B) Lesson Development

Define the inverse of a one-to-one function.

Definition

Let f be a one-to-one function with domain A and range B . Then the **inverse of f** , denoted f^{-1} , is a function with domain B and range A defined by $f^{-1}(y) = x$ if and only if $f(x) = y$ for any y in B .

A function has an inverse if and only if it is one-to-one.

As shown earlier, 'inverting' the x - and y -values of a function results in a function if and only if the original function is one-to-one.

To determine the inverse of a function from its equation

In light of the definition, the inverse of a one-to-one function can be interpreted as the same function **but in the opposite direction**, that is, it is a function from a y -value back to its corresponding x -value.

To find the inverse of a one-to-one function,

- (a) write the function in the form $y = f(x)$;
- (b) interchange the x and y variables;
- (c) solve for y in terms of x .

This is because we are interchanging the input and output values of a function.

For the next examples, we use the definition of the inverse to verify our answers.

EXAMPLE 1. Find the inverse of $f(x) = 3x + 1$.

Solution. The equation of the function is $y = 3x + 1$.

Interchange the x and y variables: $x = 3y + 1$

Solve for y in terms of x :

$$\begin{aligned} x &= 3y + 1 \\ x - 1 &= 3y \\ \frac{x - 1}{3} &= y \implies y = \frac{x - 1}{3} \end{aligned}$$

Therefore the inverse of $f(x) = 3x + 1$ is $f^{-1}(x) = \frac{x - 1}{3}$.

□

Ask the following questions to the class:

- (a) What is the inverse of the inverse?
- (b) What is $f(f^{-1}(x))$? How about $f^{-1}(f(x))$?

Have the class do these on the example above. Then discuss the following properties that the class should have observed from above:

Property of an inverse of a one-to-one function

Given a one-to-one function $f(x)$ and its inverse $f^{-1}(x)$. Then the following are true:

- (a) The inverse of $f^{-1}(x)$ is $f(x)$.
- (b) $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} .
- (c) $f^{-1}(f(x)) = x$ for all x in the domain of f .

For the second and third properties above, it can be imagined that evaluating a function and its inverse in succession is like reversing the effect of the function. For example, the inverse of a function that multiplies 3 to a number and adds 1 is a function that subtracts 1 and then divides the result by 3.

EXAMPLE 2. Find the inverse of $g(x) = x^3 - 2$.

Solution. The equation of the function is $y = x^3 - 2$.

Interchange the x and y variables: $x = y^3 - 2$.

Solve for y in terms of x :

$$\begin{aligned}x &= y^3 - 2 \\x + 2 &= y^3 \\ \sqrt[3]{x + 2} = y &\implies y = \sqrt[3]{x + 2}\end{aligned}$$

The inverse of $g(x) = x^3 - 2$ is $g^{-1}(x) = \sqrt[3]{x + 2}$.

□

EXAMPLE 3. Find the inverse of the rational function $f(x) = \frac{2x + 1}{3x - 4}$.

Solution. The equation of the function is $y = \frac{2x + 1}{3x - 4}$.

Interchange the x and y variables: $x = \frac{2y + 1}{3y - 4}$

Solve for y in terms of x :

$$\begin{aligned}x &= \frac{2y + 1}{3y - 4} \\x(3y - 4) &= 2y + 1 \\3xy - 4x &= 2y + 1 \\3xy - 2y &= 4x + 1\end{aligned}$$

(Place all terms with y on one side

and those without y on the other side.)

$$\begin{aligned}y(3x - 2) &= 4x + 1 \\y &= \frac{4x + 1}{3x - 2}\end{aligned}$$

Therefore the inverse of $f(x)$ is $f^{-1}(x) = \frac{4x + 1}{3x - 2}$.

□

EXAMPLE 4. Find the inverse of $f(x) = x^2 + 4x - 2$, if it exists.

Solution. The students should recognize that this is a quadratic function with a graph in the shape of a parabola that opens upwards. It is not a one-to-one function as it fails the horizontal line test.

(Optional) We can show that applying the procedure for finding the inverse to this function leads to a result which is **not** a function.

The equation of the function is $y = x^2 + 4x - 2$.

Interchange the x and y variables: $x = y^2 + 4y - 2$

Solve for y in terms of x :

$$\begin{aligned}x &= y^2 + 4y - 2 \\x + 2 &= y^2 + 4y \\x + 2 + 4 &= y^2 + 4y + 4 \text{ (Complete the square.)} \\x + 6 &= (y + 2)^2 \\\pm\sqrt{x + 6} &= y + 2 \\\pm\sqrt{x + 6} - 2 &= y \implies y = \pm\sqrt{x + 6} - 2\end{aligned}$$

The equation $y = \pm\sqrt{x + 6} - 2$ does not represent a function because there are some x -values that correspond to two different y -values (e.g., if $x = 3$, y can be 1 or -5). Therefore the function $f(x) = x^2 + 4x - 2$ has no inverse function.

□

EXAMPLE 5. Find the inverse of $f(x) = |3x|$, if it exists.

Solution. Recall that the graph of $y = |3x|$ is shaped like a “V” whose vertex is located at the origin. This function fails the horizontal line test and therefore has no inverse.

Alternate Solution. We can also show that f^{-1} does not exist by showing that f is not one-to-one. Note that $f(1) = f(-1) = 3$. Since the x -values 1 and -1 are paired to the same y -value, then f is not one-to-one and it cannot have an inverse.

(Optional) Again, if we apply the procedure for finding the inverse of a one-to-one function, a problem occurs:

The equation of the function is $y = |4x|$.

Interchange x and y : $x = |4y|$.

Solve for y in terms of x :

$$\begin{aligned} x &= |4y| \\ x &= \sqrt{(4y)^2} \text{ (Recall the definition } |x| = \sqrt{x^2}\text{)} \\ x^2 &= 4y^2 \\ \frac{x^2}{4} &= y^2 \\ \pm\sqrt{\frac{x^2}{4}} = y &\implies y = \pm\sqrt{\frac{x^2}{4}} \end{aligned}$$

Here, $x = 2$ will correspond to $y = 1$ and $y = -1$, so $y = \pm\sqrt{\frac{x^2}{4}}$ is **not** a function. Therefore $f(x) = |3x|$ has no inverse function. \square

EXAMPLE 6. To convert from degrees Fahrenheit to Kelvin, the function is $k(t) = \frac{5}{9}(t - 32) + 273.15$, where t is the temperature in Fahrenheit (Kelvin is the SI unit of temperature). Find the inverse function converting the temperature in Kelvin to degrees Fahrenheit.

Solution. The equation of the function is $k = \frac{5}{9}(t - 32) + 273.15$.

To maintain k and t as the respective temperatures in Kelvin and Fahrenheit (and lessen confusion), let us not interchange the variables. We just solve for t in terms of k :

$$\begin{aligned} k &= \frac{5}{9}(t - 32) + 273.15 \\ k - 273.15 &= \frac{5}{9}(t - 32) \\ \frac{9}{5}(k - 273.15) &= t - 32 \\ \frac{9}{5}(k - 273.15) + 32 &= t \implies t = \frac{9}{5}(k - 273.15) \end{aligned}$$

Therefore the inverse function is $t(k) = \frac{9}{5}(k - 273.15)$ where k is the temperature in Kelvin. \square

(C) Seatwork

The following activities can be given as either homework or seatwork.

Seatwork 1. Give 3 examples of situations that can be represented as a one-to-one function and two examples of situations that are not one-to-one.

Sample Answer: vehicles to plate numbers, movie tickets to seat numbers, presidents or prime ministers to countries, mayors to cities or towns

Seatwork 2. Choose a situation or scenario that can be represented as a one-to-one function and explain why it is important that the function in that scenario is one-to-one.

Sample Answer: A person must have only one tax identification number (TIN) so that all the taxes he pays can be accurately recorded. If he has two TINs, the BIR might think that he did not pay all his taxes if his payments are split between multiple TINs. If a single TIN has two persons associated to it, then it would not be possible to ascertain which person is paying proper taxes and which is not.

Seatwork 3. Find the inverse functions of the following one-to-one functions:

(a) $f(x) = \frac{1}{2}x + 4$

Answer: $f^{-1}(x) = 2x - 8$

(b) $f(x) = (x + 3)^3$

Answer: $f^{-1}(x) = \sqrt[3]{x} - 3$

(c) $f(x) = \frac{3}{x - 4}$

Answer: $f^{-1}(x) = \frac{4x + 3}{x}$

(d) $f(x) = \frac{x + 3}{x - 3}$

Answer: $f^{-1}(x) = \frac{3x + 3}{x - 1}$

(e) $f(x) = \frac{2x + 1}{4x - 1}$

Answer: $f^{-1}(x) = \frac{x + 1}{4x - 2}$

Seatwork 4. Show that $f(x) = |x - 1|$ is **not** a one-to-one function.

LESSON 11: Graphs of Inverse Functions

TIME FRAME: 120 minutes

CODE: M11GM-Id-3, d-4, e-1, and e-2

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to represent an inverse function through its graph, find the domain and range of an inverse function, graph inverse functions, and solve problems involving inverse functions.

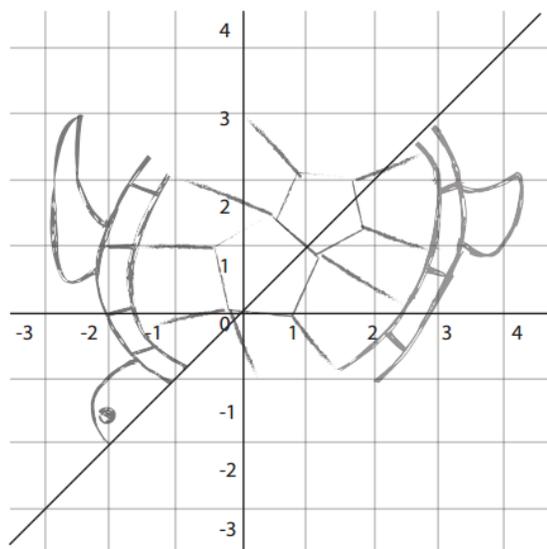
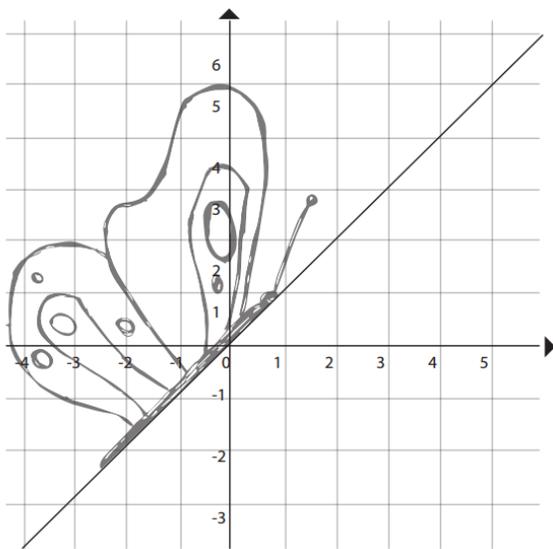
LESSON OUTLINE:

1. Graphical method of determining the inverse of a function
 2. Algebraic methods= of determining the inverse of a function
 3. Domain and range of the inverse of a function
 4. Problems involving inverse functions
-

DEVELOPMENT OF THE LESSON**(A) Motivation**

In this lesson, we will use the property of one-to-one functions and their inverses that the graph of the inverse function is the reflection of the graph of the original function across the line $y = x$. Assign the following activity as homework to practice the idea of drawing reflections across a line. This is suggested as a homework as the activity itself does not make use of the idea of functions but merely makes the reflection transformation more natural to the student.

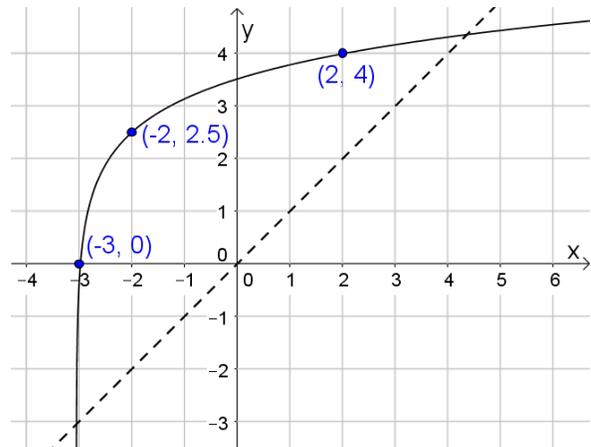
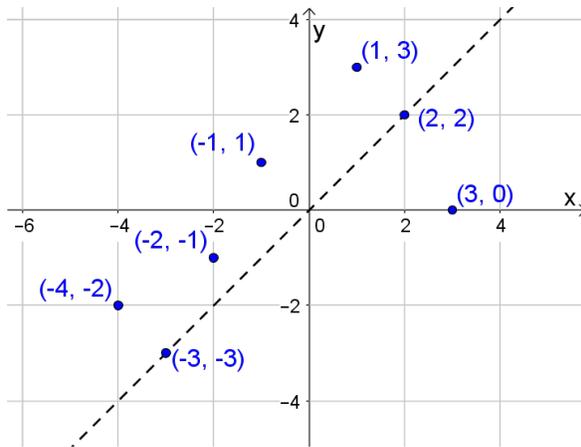
- (a) Complete the drawings by sketching the reflection of the figure across the line $y = x$. For an accurate figure, fold the paper across the line $y = x$ and trace the figure underneath.



Images are based on <http://artforkidshub.com/5-free-symmetry-art-activity/> and were edited for this activity.

- (b) Consider the points on the first image below. Can these points represent a one-to-one function? Draw the reflection of the points below across the line $y = x$. What can you observe about the coordinates of the points on the figure and the coordinates of the points on the image?

- (c) Consider the graph on the second image below. Is this a graph of a one-to-one function? Draw the reflection of the graph across the line $y = x$.



Elicit from the students the following ideas:

- The coordinates of the points in the reflected image are reversed in relation to those in the original image.
- If a function is **not** one-to-one, the reflection of its graph about the line $y = x$ does not produce a function.

(B) Lesson Development

The coordinates of points in a graph of the inverse of a one-to-one function are the coordinates of points in the original function with the x and y -coordinates interchanged.

Recall that f and f^{-1} satisfy the property that $f^{-1}(f(x)) = x$. This means that the inverse $f^{-1}(x)$ sends the y -values of points in $f(x)$ back to x .

$$x \xrightarrow{f(x)} y \xrightarrow{f^{-1}(y)} x$$

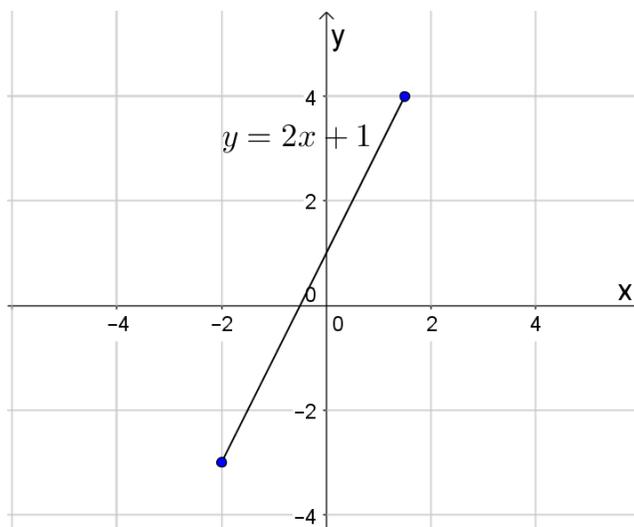
Relationship Between the Graphs of a Function and its Inverse

First we need to ascertain that the given graph corresponds to a one-to-one function by applying the horizontal line test. If it passes the test, the corresponding function is one-to-one.

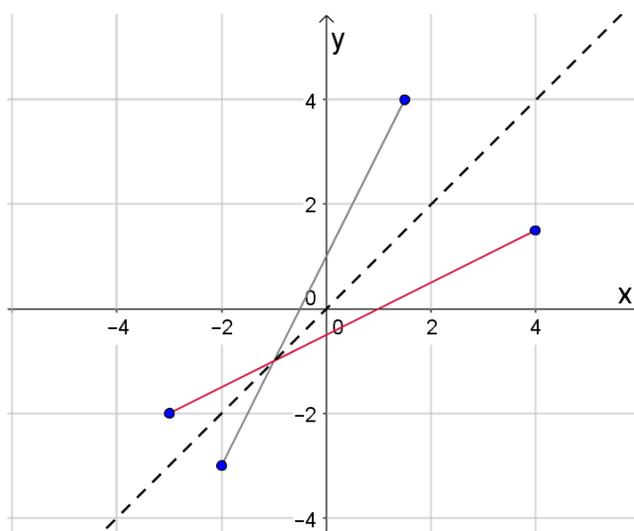
Graphing inverse functions

Given the graph of a one-to-one function, the graph of its inverse can be obtained by reflecting the graph about the line $y = x$.

EXAMPLE 1. Graph $y = f^{-1}(x)$ if the graph of $y = f(x) = 2x + 1$ restricted in the domain $\{x \mid -2 \leq x \leq 1.5\}$ is given below. What is the range of $f(x)$? What is the domain and range of its inverse?



Solution. Take the reflection of the restricted graph of $y = 2x + 1$ across the line $y = x$.



The range of the original function can be determined by inspection of the graph. The range is $\{y \in \mathbb{R} \mid -3 \leq y \leq 4\}$.

Verify using techniques in an earlier lesson that the inverse function is given by $f^{-1}(x) = \frac{x-1}{2}$.

The domain and range of the inverse function can be determined by inspection of the graph:

$$\text{Domain of } f^{-1}(x) = [-3, 4]$$

$$\text{Range of } f^{-1}(x) = [-2, 1.5]$$

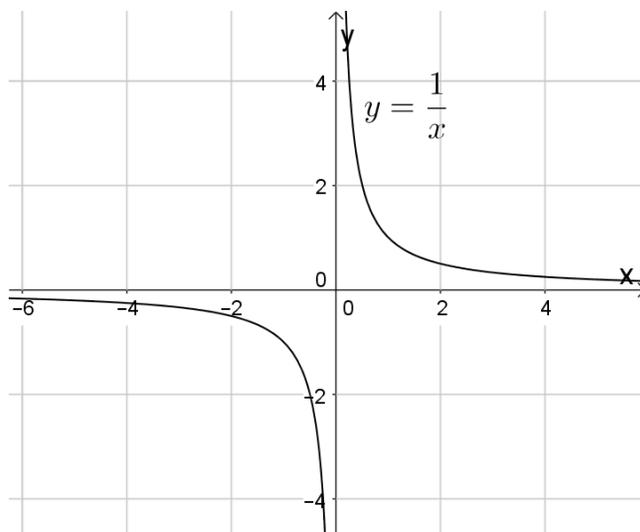
In summary,

	$f(x)$	$f^{-1}(x)$
Domain	$[-2, 1.5]$	$[-3, 4]$
Range	$[-3, 4]$	$[-2, 1.5]$

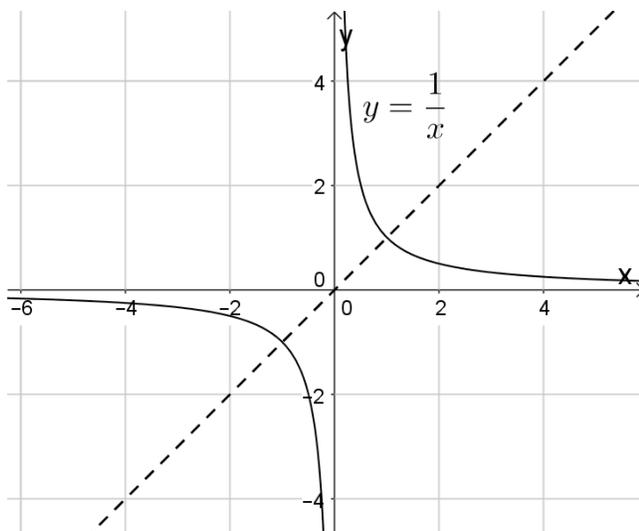
Observe that the domain of the inverse is the range of the original function, and that the range of the inverse is the domain of the original function. Is this true for all one-to-one functions and their inverses?

□

EXAMPLE 2. Find and graph the inverse of $f(x) = \frac{1}{x}$, whose graph is shown below.



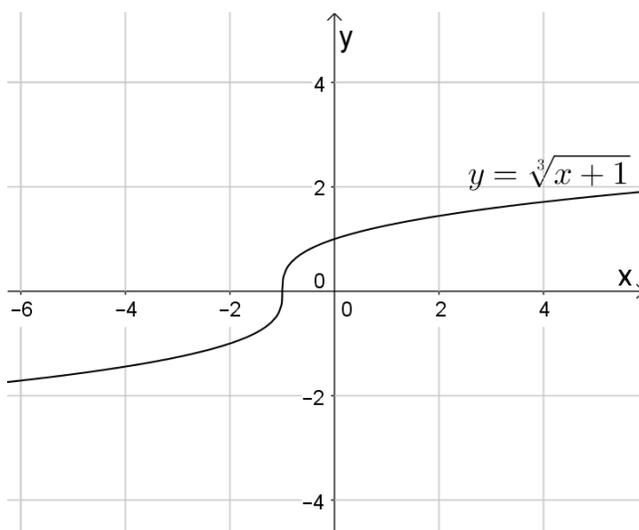
Solution. Applying the horizontal line test we verify that the function is one-to-one. Since the graph of $f(x) = \frac{1}{x}$ is symmetric with respect to the line $y = x$ (indicated by a dashed line), its reflection across the line $y = x$ is itself. Therefore the inverse of $f(x)$ is itself, or $f^{-1}(x) = f(x)$.



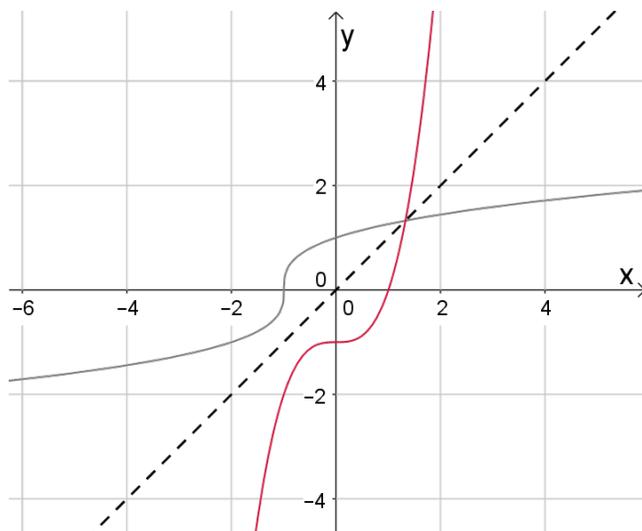
Verify that $f^{-1}(x) = f(x) = \frac{1}{x}$ using the techniques used in the previous lesson.

□

EXAMPLE 3. Find the inverse of $f(x) = \sqrt[3]{x+1}$ using the given graph.



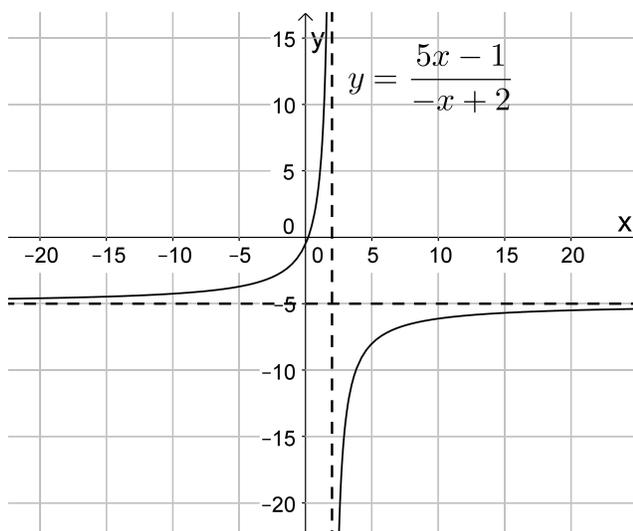
Solution. Applying the horizontal line test we confirm that the function is one-to-one. Reflect the graph of $f(x)$ across the line $y = x$ to get the plot of the inverse function.



The result of the reflection of the graph of $f(x) = \sqrt[3]{x+1}$ is the graph of $y = x^3 - 1$. Therefore $f^{-1}(x) = x^3 - 1$.

□

EXAMPLE 4. Consider the rational function $f(x) = \frac{5x - 1}{-x + 2}$ whose graph is shown below:



- Find its domain and range.
- Find the equation of its asymptotes.
- Find the graph of its inverse.
- Find the domain and range of its inverse.

Solution.

- From our lessons on rational functions, we get the following results:

$$\text{Domain of } f(x) = (-\infty, 2) \cup (2, \infty)$$

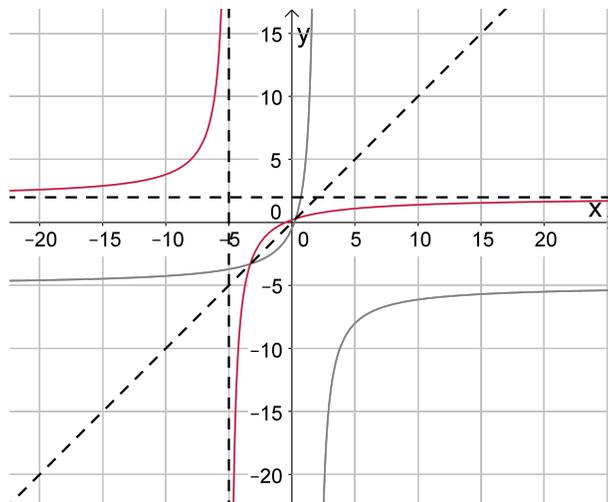
$$\text{Range of } f(x) = (-\infty, -5) \cup (-5, \infty)$$

- Using techniques from the lesson on rational functions, the equations of the asymptotes are

$$\text{Vertical asymptote: } x = 2$$

$$\text{Horizontal asymptote: } y = -5$$

- The inverse can be graphed by taking the reflection of the graph across $y = x$.



Observe that the new asymptotes are the old asymptotes with the x and y values interchanged. In fact, the asymptotes could also be obtained by reflecting the original asymptotes about the line $y = x$.

$$\text{Vertical asymptote: } x = -5$$

$$\text{Horizontal asymptote: } y = 2$$

(d) The domain and range of the function and its inverse are as follows:

	$f(x)$	$f^{-1}(x)$
Domain	$(-\infty, 2) \cup (2, \infty)$	$(-\infty, -5) \cup (-5, \infty)$
Range	$(-\infty, -5) \cup (-5, \infty)$	$(-\infty, 2) \cup (2, \infty)$

We can make the observation that the domain of the inverse is the range of the original function, and the range of the inverse is the domain of the original function.

□

EXAMPLE 5. In the examples above, what will happen if we plot the inverse functions of the inverse functions?

Solution. If we plot the inverse of a function, we reflect the original function about the line $y = x$. If we plot the inverse of the inverse, we just reflect the graph back about the line $y = x$ and end up with the original function.

This result implies that the original function is the inverse of its inverse, or $(f^{-1})^{-1}(x) = f(x)$. □

Solving problems involving inverse functions

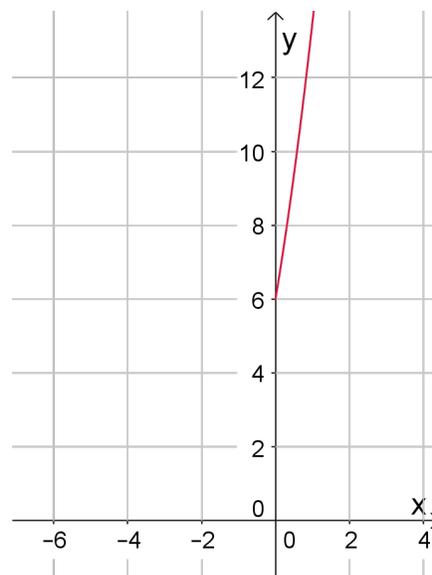
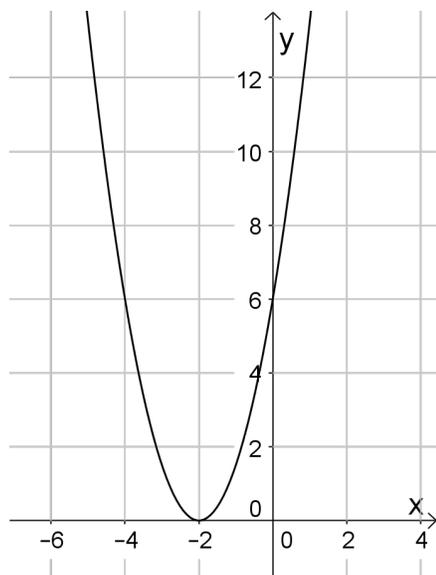
We can apply the concepts of inverse functions in solving word problems involving reversible processes.

EXAMPLE 6. You asked a friend to think of a nonnegative number, add two to the number, square the number, multiply the result by 3 and divide the result by 2. If the result is 54, what is the original number? Construct an inverse function that will provide the original number if the result is given.

Solution. We first construct the function that will compute the final number based on the original number. Following the instructions, we come up with this function:

$$f(x) = (x + 2)^2 \cdot 3 \div 2 = \frac{3(x + 2)^2}{2}$$

The graph is shown below, on the left. This is not a one-to-one function because the graph does not satisfy the horizontal line test. However, the instruction indicated that the original number must be nonnegative. The domain of the function must thus be restricted to $x \geq 0$, and its graph is shown on the right, below.



The function with restricted domain $x \geq 0$ is then a one-to-one function, and we can find its inverse.

Interchange the x and y variables: $x = \frac{3(y + 2)^2}{2}, y \geq 0$

Solve for y in terms of x :

$$\begin{aligned}
 x &= \frac{3(y+2)^2}{2} \\
 \frac{2x}{3} &= (y+2)^2 \\
 \sqrt{\frac{2x}{3}} &= y+2 && \text{Since } y \geq 2 \text{ we do not need to consider } -\sqrt{\frac{2x}{3}} \\
 \sqrt{\frac{2x}{3}} - 2 &= y \implies y = \sqrt{\frac{2x}{3}} - 2
 \end{aligned}$$

Finally we evaluate the inverse function at $x = 54$ to determine the original number:

$$f^{-1}(54) = \sqrt{\frac{2(54)}{3}} - 2 = \sqrt{\frac{108}{3}} - 2 = \sqrt{36} - 2 = 6 - 2 = 4$$

The original number is 4. □

EXAMPLE 7. Engineers have determined that the maximum force t in tons that a particular bridge can carry is related to the distance d in meters between its supports by the following function:

$$t(d) = \left(\frac{12.5}{d}\right)^3$$

How far should the supports be if the bridge is to support 6.5 tons? Construct an inverse function to determine the result.

Solution. The equation of the function is $t = \left(\frac{12.5}{d}\right)^3$.

To lessen confusion in this case, let us not interchange d and t as they denote specific values. Solve instead for d in terms of t :

$$\begin{aligned}
 t &= \left(\frac{12.5}{d}\right)^3 \\
 \sqrt[3]{t} &= \frac{12.5}{d} \\
 d &= \frac{12.5}{\sqrt[3]{t}}
 \end{aligned}$$

The inverse function is $d(t) = \frac{12.5}{\sqrt[3]{t}}$.

Evaluate the function at $t = 6.5$: $d(6.5) = \frac{12.5}{\sqrt[3]{6.5}} = 12.06$.

The supports should be placed at most 12.06 meters apart. □

(C) Seatwork and Exercises

Seatwork 1. Construct a table of values for the function $f(x) = x^2 + 1$ in the restricted domain $\{0, 0.5, 1, 1.5, 2, 2.5, 3\}$. Is this a one-to-one function? If it is a one-to-one function, construct the table of values of the inverse.

Answer:

x	0	0.5	1	1.5	2	2.5	3
$f(x)$	1	1.25	2	3.25	5	7.25	10

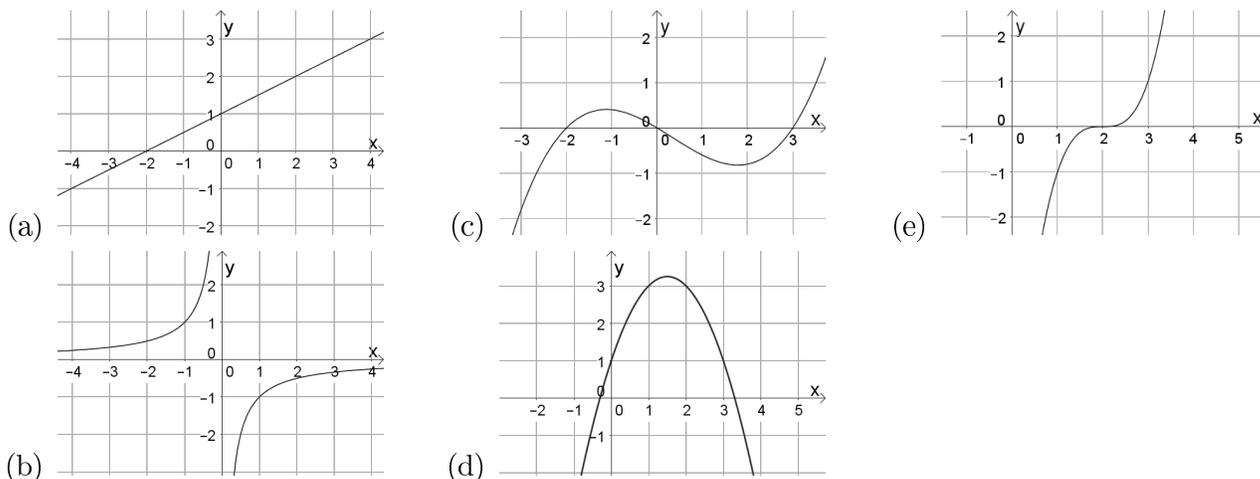
Yes, f is one-to-one in the restricted domain. The table for the inverse is

x	1	1.25	2	3.25	5	7.25	10
$f^{-1}(x)$	0	0.5	1	1.5	2	2.5	3

Seatwork 2. The set $A = \{(-4, 4), (-3, 2), (-2, 1), (0, -1), (1, -3), (2, -5)\}$ of ordered pairs form a function. Find the inverse of this function and plot the inverse on a Cartesian plane.

Answer: $A^{-1} = \{(4, -4), (2, -3), (1, -2), (-1, 0), (-3, 1), (-5, 2)\}$

Seatwork 3. Identify if the given graph can be that of a one-to-one function. Explain briefly why.



Answer: Only (a), (b), and (e) satisfy the horizontal line test.

Seatwork 4. Draw the graph of the inverse function of the one-to-one functions in the previous exercise.

Seatwork 5. The function $f(x) = 2\sqrt{x-2} + 3$ is a one-to-one function.

(a) What is its domain and range?

Answer: $[2, \infty)$

(b) Plot the graph of this function and its inverse.

(c) What is the domain and range of its inverse?

Answer: Domain: $[3, \infty)$, Range: $[2, \infty)$

Seatwork 6. The rational function $f(x) = \frac{3x+2}{x-4}$ is a one-to-one function.

(a) Find the domain and range of its inverse.

Answer: Domain: $(-\infty, 3) \cup (3, \infty)$, Range:

$(-\infty, 4) \cup (4, \infty)$

(b) find the lone zero of its inverse (a value $x = a$ such that $f^{-1}(a) = 0$).

Answer: $-1/2$

Seatwork 7. A particular breed of tilapia has its weight w (in kilograms) related to its length l (in cm) modeled by the function $w = (3.24 \times 10^{-3})l^2$.

Explain why the function is one-to-one, even if it is a quadratic function. [*Possible explanation:* The graph for $l \geq 0$ (length cannot be negative) passes the horizontal line test] Find the inverse of this function and find the approximate length of a single fish if its weight is 400 grams. **Answer:**

$$l = \sqrt{w/(3.4 \times 10^{-3})}; 11.11 \text{ cm}$$

Chapter 4

Exponential Functions

LESSON 12: Representing Real-Life Situations Using Exponential Functions

TIME FRAME: 150 minutes

CODE: M11GM-Ie-3 and g-2

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to represent real-life situations using exponential functions.

LESSON OUTLINE:

1. Exponential functions and their transformations
 2. Exponential growth and decay
 3. Compound interest
 4. The natural exponential function
-

DEVELOPMENT OF THE LESSON

(A) Motivation

Exponential functions occur in various real world situations. Exponential functions are used to model real-life situations such as population growth, radioactive decay, carbon dating, growth of an epidemic, loan interest rates, and investments.

Group Activity. This activity can help introduce the concept of an exponential function.

Materials. One 2-meter string and a pair of scissors for each group

- At Step 0, there is 1 string.
- At Step 1, fold the string into two equal parts and then cut at the middle. How many strings of equal length do you have? Enter your answer in the table below.
- At Step 2, again fold each of the strings equally and then cut. How many strings of equal length do you have? Enter your answer in the table below.
- Continue the process until the table is completely filled-up.

Step	0	1	2	3	4	5	6	7
Number of Strings	1							

Questions.

- What pattern can be observed from the data?
- Define a formula for the number of strings as a function of the step number.

Answers.

Step	0	1	2	3	4	5	6	7
Number of Strings	1	2	4	8	16	32	64	128

It can be observed that as the step number increases by one, the number of strings doubles. If n is the number of strings and s is the step number, then $n = 2^s$.

(B) Lesson Proper

Definition.

An **exponential function** with base b is a function of the form $f(x) = b^x$ or $y = b^x$, where $b > 0$, $b \neq 1$.

EXAMPLE 1. Complete a table of values for $x = 3, 2, 1, 0, 1, 2$ and 3 for the exponential functions $y = \left(\frac{1}{3}\right)^x$, $y = 10^x$, and $y = (0.8)^x$.

Solution. The solution is as follows:

x	-3	-2	-1	0	1	2	3
$y = \left(\frac{1}{3}\right)^x$	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$
$y = 10^x$	$\frac{1}{1000}$	$\frac{1}{100}$	$\frac{1}{10}$	1	10	100	1000
$y = (0.8)^x$	1.953125	1.5625	1.25	1	0.8	0.64	0.512

□

EXAMPLE 2. If $f(x) = 3^x$, evaluate $f(2)$, $f(-2)$, $f\left(\frac{1}{2}\right)$, $f(0.4)$, and $f(\pi)$.

Solution.

$$f(2) = 3^2 = 9$$

$$f(-2) = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

$$f\left(\frac{1}{2}\right) = 3^{1/2} = \sqrt{3}$$

$$f(0.4) = 3^{0.4} = 3^{2/5} = \sqrt[5]{3^2} = \sqrt[5]{9}$$

Since $\pi \approx 3.14159$ is irrational, the rules for rational exponents are not applicable. We define 3^π using rational numbers: $f(\pi) = 3^\pi$ can be approximated by $3^{3.14}$. A better approximation is $3^{3.14159}$. Intuitively, one can obtain any level of accuracy for 3^π by considering sufficiently more decimal places of π . Mathematically, it can be proved that these approximations approach a unique value, which we define to be 3^π . □

Definition.

Let b be a positive number not equal to 1. A **transformation** of an exponential function with base b is a function of the form

$$g(x) = a \cdot b^{x-c} + d$$

where a , c , and d are real numbers.

Many applications involve transformations of exponential functions. Some of the most common applications in real-life of exponential functions and their transformations are **population growth**, **exponential decay**, and **compound interest**.

(a) Population

On several instances, scientists will start with a certain number of bacteria or animals and watch how the population grows. For example, if the population doubles every 3 days, this can be represented as an exponential function.

EXAMPLE 3. Let $t =$ time in days. At $t = 0$, there were initially 20 bacteria. Suppose that the bacteria doubles every 100 hours. Give an exponential model for the bacteria as a function of t .

Initially, at $t = 0$	Number of bacteria = 20
at $t = 100$	Number of bacteria = $20(2)$
at $t = 200$	Number of bacteria = $20(2)^2$
at $t = 300$	Number of bacteria = $20(2)^3$
at $t = 400$	Number of bacteria = $20(2)^4$

Solution. An exponential model for this situation is $y = 20(2)^{t/100}$. □

Exponential Models and Population Growth.

Suppose a quantity y doubles every T units of time. If y_0 is the initial amount, then the quantity y after t units of time is given by $y = y_0(2)^{t/T}$.

(b) Exponential Decay

Exponential functions can be used to model radioactive decay.

Definition.

The **half-life** of a radioactive substance is the time it takes for half of the substance to decay.

EXAMPLE 4. Suppose that the half-life of a certain radioactive substance is 10 days and there are 10g initially, determine the amount of substance remaining after 30 days.

Solution. Let $t =$ time in days. We use the fact that the mass is halved every 10 days (from definition of half-life). Thus, we have:

Initially, at $t = 0$	Amount of Substance = 10 g
at $t = 10$ days	Amount of Substance = 5 g
at $t = 20$ days	Amount of Substance = 2.5 g
at $t = 30$ days	Amount of Substance = 1.25 g

An exponential model for this situation is $y = 10 \left(\frac{1}{2}\right)^{t/10}$.

□

(c) Compound Interest

A starting amount of money (called the **principal**) can be invested at a certain interest rate that is earned at the end of a given period of time (such as one year). If the interest rate is **compounded**, the interest earned at the end of the period is added to the principal, and this new amount will earn interest in the next period. The same process is repeated for each succeeding period: interest previously earned will also earn interest in the next period. Compound interest will be discussed in more depth in Lessons 23 and 25.

EXAMPLE 5. Mrs. De la Cruz invested ₱100,000. in a company that offers 6% interest compounded annually. How much will this investment be worth at the end of each year for the next five years?

Solution. Let t be the time in years. Then we have:

Initially, at $t = 0$	Investment = ₱100,000
at $t = 1$	Investment = ₱100,000(1.06) = ₱106,000
at $t = 2$	Investment = ₱106,000(1.06) = ₱112,360
at $t = 3$	Investment = ₱112,360(1.06) \approx ₱119,101.60
at $t = 4$	Investment = ₱119,101.60(1.06) \approx ₱126,247.70
at $t = 5$	Investment = ₱126,247.70(1.06) \approx ₱133,822.56

An exponential model for this situation is $y = 100,000(1.06)^t$.

□

Compound Interest.

If a principal P is invested at an annual rate of r , compounded annually, then the amount after t years is given by $A = P(1 + r)^t$.

EXAMPLE 6. Referring to Example 5, is it possible for Mrs. De la Cruz to double her money in 8 years? in 10 years?

Solution. Using the model $y = 100000(1.06)^t$, substitute $t = 8$ and $t = 10$:

If $t = 8$, then $y = ₱100,000(1.06)^8 \approx ₱159,384.81$. If $t = 10$, then $y = ₱100,000(1.06)^{10} \approx ₱179,084.77$. Since her money still has not reached ₱200,000 after 10 years, then she has not doubled her money during this time.

□

The Natural Exponential Function

While an exponential function may have various bases, a frequently used based is the irrational number $e \approx 2.71828$. The enrichment in Lesson 27 will show how the number e arises from the concept of compound interest. Because e is a commonly used based, the natural exponential function is defined having e as the base.

Definition.

The **natural exponential function** is the function $f(x) = e^x$.

EXAMPLE 7. A large slab of meat is taken from the refrigerator and placed in a pre-heated oven. The temperature T of the slab t minutes after being placed in the oven is given by $T = 170165e^{0.006t}$ degrees Celsius. Construct a table of values for the following values of t : 0, 10, 20, 30, 40, 50, 60, and interpret your results. Round off values to the nearest integer.

Solution. The solution is as follows:

t	0	10	20	30	40	50	60
T	5	15	24	32	40	47	54

The slab of meat is increasing in temperature at roughly the same rate. □

(C) Seatwork

Seatwork 1. Suppose that a couple invested ₱50,000 in an account when their child was born, to prepare for the child's college education. If the average interest rate is 4.4% compounded annually, (a) give an exponential model for the situation, and (b) will the money be doubled by the time the child turns 18 years old?

Solution. An exponential model for the situation is $A = 50,000(1.044)^t$, where t is the number of years since the amount was invested. If $t = 18$, then $A = 50,000(1.044)^{18} \approx \text{₱}108,537.29$. By the time the child turns 18 years old, the money has more than doubled since the amount exceeded ₱100,000. □

Seatwork 2. You take out a ₱20,000 loan at a 5% interest rate. If the interest is compounded annually, (a) give an exponential model for the situation, and (b) how much will you owe after 10 years?

Solution. An exponential model for this situation is $A = \text{₱}20,000(1.05)^t$ where t is the number of years since the amount was loaned. If $t = 10$, $A = \text{₱}20,000(1.05)^{10} \approx \text{₱}32,577.89$. Thus, the total amount you owe after 10 years is ₱32,577.89. □

Seatwork 3. Suppose that the half-life of a substance is 250 years. If there were initially 100 g of the substance, (a) give an exponential model for the situation, and (b) how much will remain after 500 years?

Solution. The situation can be modeled by $y = 100 \left(\frac{1}{2}\right)^{x/250}$, where $x = 0$ corresponds to the time when there were 100 g of substance. If $x = 500$, then $y = 100 \left(\frac{1}{2}\right)^{500/250} = 100 \left(\frac{1}{2}\right)^2 = 25$. Thus, there will be 25 g remaining after 500 years. \square

Alternate Solution. After 250 years, the substance reduces to half (50 g); after another 250 years, the substance reduces to half again (25 g). \square

(D) Evaluation

Solve the given problems and check.

- (a) A population starts with 1,000 individuals and triples every 80 years. **(a)** Give an exponential model for the situation. **(b)** What is the size of the population after 100 years?

Solution. The situation can be modeled by $y = 1,000(3)^{x/80}$, where $x = 0$ corresponds to the time when there were 1,000 individuals. If $x = 100$, then $y = 1,000(3)^{100/80} = 3,948.22 \approx 3,948$. Thus, there will be 3,948 individuals after 100 years. \square

- (b) ₱10,000 is invested at 2% compounded annually. **(a)** Give an exponential model for the situation. **(b)** What is the amount after 12 years?

Solution. The situation can be modeled by $A = 10,000(1.02)^t$, where t is the number of years since the amount was invested.

$$A = 10,000(1.02)^{12} = 12,682.42$$

Thus, the amount after 12 years is ₱12,682.42. \square

LESSON 13: Exponential Functions, Equations, and Inequalities

TIME FRAME: 30 minutes

CODE: M11GM-Ie-4

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to distinguish among exponential functions, exponential equations, and exponential inequalities.

LESSON OUTLINE:

1. Exponential expressions
2. Definition and examples of rational functions, rational equations, and rational inequalities

DEVELOPMENT OF THE LESSON

(A) Motivation

Ask students to share their ideas about how A, B, and C are similar, and how they are different.

(a) $4^{x-1} = 16x$

(b) $y = 2^x$

(c) $2^x \geq 26$

Possible responses. Items (a), (b), and (c) all involve expressions with a variable in the exponent. Items (a) and (c) involve only one variable, but (b) expresses a relationship between two variables. Items (a) and (b) involve an equality, but (0c) involves an inequality.

(B) Lesson Proper

Definition.

An **exponential expression** is an expression of the form $a \cdot b^{x-c} + d$, where $b > 0$, $b \neq 1$.

The definitions of exponential equations, inequalities, and functions are shown below.

	Exponential Equation	Exponential Inequality	Exponential Function
Definition	An equation involving exponential expressions	An inequality involving exponential expressions	Function of the form $f(x) = b^x$, where $b > 0$, $b \neq 1$.
Example	$7^{2x-x^2} = \frac{1}{343}$	$5^{2x} - 5^{x+1} \leq 0$	$f(x) = (1.8)^x$ or $y = (1.8)^x$

An exponential equation or inequality can be solved for all x values that satisfy the equation or inequality (Lesson 14). An exponential function is not ‘solved’ Rather, it expresses a relationship between two variables (such as x and y), and can be represented by a table of values or a graph (Lessons 15 and 16).

(C) Seatwork

Seatwork 1. Determine whether the given is an exponential function, an exponential equation, an exponential inequality or none of these.

(a) $f(x) = 2x^3$

Answer: None of these

(b) $f(x) = 2^x$

Answer: Exponential function

(c) $y = e^x$

Answer: Exponential function

(d) $2^2(5^{x+1}) = 500$

Answer: Exponential equation

(e) $625 \geq 5^{x+8}$

Answer: Exponential inequality

LESSON 14: Solving Exponential Equations and Inequalities

TIME FRAME: 120 minutes

CODE: M11GM-Ie-f-1

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to solve exponential equations and inequalities.

LESSON OUTLINE:

1. Review: Properties of exponents
 2. Solving exponential equations
 3. Solving exponential inequalities
-

DEVELOPMENT OF THE LESSON

(A) Motivation

Recall the following definitions and theorems.

Definition.

Let $a \neq 0$. We define the following:

- (1) $a^0 = 1$
- (2) $a^{-n} = \frac{1}{a^n}$

Theorem.

Let r and s be rational numbers. Then

- (1) $a^r a^s = a^{r+s}$
- (2) $\frac{a^r}{a^s} = a^{r-s}$
- (3) $(a^r)^s = a^{rs}$
- (4) $(ab)^r = a^r b^r$
- (5) $\left(\frac{a}{b}\right)^r = \frac{a^r}{b^r}$

(B) Lesson Proper

Ask students which of the following are exponential equations.

(a) $49 = 7^{x+1}$

(b) $7 = 2x + 3$

(c) $3^x = 3^{2x-1}$

(d) $5^{x-1} = 125$

(e) $8x = x^2 - 9$

(f) $x^2 = 3x^3 + 2x - 1$

(g) $2x + 3 > x - 1$

(h) $2^{x-2} > 8$

Answer. Only (a), (c), and (d) are exponential equations.

Some exponential equations can be solved by using the fact that exponential functions are one-to-one.

One-to-one Property of Exponential Functions

If $x_1 \neq x_2$, then $b^{x_1} \neq b^{x_2}$. Conversely, if $b^{x_1} = b^{x_2}$ then $x_1 = x_2$.

 **Teaching Tip**

One strategy to solve exponential equations is to write both sides of the equation as powers of the same base.

EXAMPLE 1. Solve the equation $4^{x-1} = 16$.

Solution. We write both sides with 4 as the base.

$$4^{x-1} = 16$$

$$4^{x-1} = 4^2$$

$$x - 1 = 2$$

$$x = 2 + 1$$

$$x = 3$$

□

Alternate Solution. We can also write both sides with 2 as the base.

$$4^{x-1} = 16$$

$$(2^2)^{x-1} = 2^4$$

$$2^{2(x-1)} = 2^4$$

$$2(x - 1) = 4$$

$$2x - 2 = 4$$

$$2x = 6$$

$$x = 3$$

You can verify that $x = 3$ is a solution by substituting it back to the original equation: $4^{3-1} = 4^2 = 16$. \square

This example shows that there may be more than one way to solve an exponential equation. The important thing is to write both sides using the same base.

EXAMPLE 2. Solve the equation $125^{x-1} = 25^{x+3}$.

Solution. Both 125 and 25 can be written using 5 as the base.

$$\begin{aligned} 125^{x-1} &= 25^{x+3} \\ (5^3)^{x-1} &= (5^2)^{x+3} \\ 5^{3(x-1)} &= 5^{2(x+3)} \\ 3(x-1) &= 2(x+3) \\ 3x-3 &= 2x+6 \\ x &= 9 \end{aligned}$$

\square

EXAMPLE 3. Solve the equation $9^{x^2} = 3^{x+3}$.

Solution. Both 9 and 3 can be written using 3 as the base.

$$\begin{aligned} (3^2)^{x^2} &= 3^{x+3} \\ 3^{2x^2} &= 3^{x+3} \\ 2x^2 &= x+3 \\ 2x^2 - x - 3 &= 0 \\ (2x-3)(x+1) &= 0 \\ 2x-3 = 0 &\text{ or } x+1 = 0 \\ x = \frac{3}{2} &\text{ or } x = 1 \end{aligned}$$

\square

Exponential inequalities can be solved using the following property.

Property of Exponential Inequalities

If $b > 1$, then the exponential function $y = b^x$ is increasing for all x . This means that $b^x < b^y$ if and only if $x < y$.

If $0 < b < 1$, then the exponential function $y = b^x$ is decreasing for all x . This means that $b^x > b^y$ if and only if $x < y$.

Teaching Tip

Students should be careful when solving exponential inequalities such as $b^m < b^n$. The resulting direction of the inequality ($m < n$ or $m > n$) is based on whether the base b is greater than 1 or less than 1.

EXAMPLE 4. Solve the inequality $3^x < 9^{x-2}$.

Solution. Both 9 and 3 can be written using 3 as the base.

$$\begin{aligned} 3^x &< (3^2)^{x-2} \\ 3^x &< 3^{2(x-2)} \\ 3^x &< 3^{2x-4} \end{aligned}$$

Since the base $3 > 1$, then this inequality is equivalent to

$$\begin{aligned} x &< 2x - 4 \quad (\text{the direction of the inequality is retained}) \\ 4 &< 2x - x \\ 4 &< x \end{aligned}$$

Thus, the solution set is $(4, +\infty]$. (You can verify that $x = 5$ and 6 are solutions, but $x = 4$ and 3 are not.) \square

EXAMPLE 5. Solve the inequality $\left(\frac{1}{10}\right)^{x+5} \geq \left(\frac{1}{100}\right)^{3x}$.

Solution. Since $\frac{1}{100} = \left(\frac{1}{10}\right)^2$, then we write both sides of the inequality with $\frac{1}{10}$ as the base.

$$\begin{aligned} \left(\frac{1}{10}\right)^{x+5} &\geq \left(\frac{1}{100}\right)^{3x} \\ \left(\frac{1}{10}\right)^{x+5} &\geq \left(\frac{1}{10^2}\right)^{3x} \\ \left(\frac{1}{10}\right)^{x+5} &\geq \left(\frac{1}{10}\right)^{6x} \end{aligned}$$

Since the base $\frac{1}{10} < 1$, then this inequality is equivalent to

$$x + 5 \leq 6x \text{ (the direction of the inequality is reversed)}$$

$$5 \leq 6x - x$$

$$5 \leq 5x$$

$$1 \leq x$$

Thus, the solution set is $[1, +\infty)$. (You can verify that $x = 1$ and 2 are solutions, but $x = 0$ and 1 are not.) \square

EXAMPLE 6. The half-life of Zn-71 is 2.45 minutes.¹ Initially, there were y_0 grams of Zn-71, but only $\frac{1}{256}$ of this amount remains after some time. How much time has passed?

Solution. Using exponential models in Lesson 12, we can determine that after t minutes, the amount of Zn-71 in the substance is $y = y_0 \left(\frac{1}{2}\right)^{t/2.45}$. We solve the equation $y_0 \left(\frac{1}{2}\right)^{t/2.45} = \frac{1}{256}y_0$:

$$\begin{aligned} \left(\frac{1}{2}\right)^{t/2.45} &= \frac{1}{256} \\ \left(\frac{1}{2}\right)^{t/2.45} &= \left(\frac{1}{2}\right)^8 \\ \frac{t}{2.45} &= 8 \\ t &= 19.6 \end{aligned}$$

Thus, 19.6 minutes have passed since $t = 0$. \square

(C) Seatwork

Seatwork 1. John and Peter are solving $(0.6)^{x-3} > (0.36)^{-x-1}$. Did anyone get the correct solution? If not, spot the error or errors.

¹<http://www.periodictable.com/Isotopes/030.71/index.p.full.html>

John

$$(0.6)^{x-3} > (0.36)^{-x-1}$$

$$(0.6)^{x-3} > (0.6^2)^{-x-1}$$

$$(0.6)^{x-3} > (0.6)^{2(-x-1)}$$

$$(0.6)^{x-3} > (0.6)^{-2x-2}$$

$$x - 3 > -2x - 2$$

$$3x > 1$$

$$x > \frac{1}{3}$$

Peter

$$(0.6)^{x-3} > (0.36)^{-x-1}$$

$$(0.6)^{x-3} > (0.6^2)^{-x-1}$$

$$(0.6)^{x-3} > (0.6)^{2(-x-1)}$$

$$(0.6)^{x-3} > (0.6)^{-2x-2}$$

$$x - 3 < -2x - 2$$

$$3x < 1$$

$$x < \frac{1}{3}$$

Solution. Peter is correct and John is *not*. John should have reversed the direction of the inequality when he transformed $(0.6)^{x-3} > (0.6)^{-2x-2}$ to $x - 3 > -2x - 2$ because the base 0.6 is less than 1. The correct answer is $x > \frac{1}{3}$, which was reached by Peter. \square

Seatwork 2. Solve for x .

(a) $16^{2x-3} = 4^{x+2}$

Answer: $x = \frac{8}{3}$

(b) $\left(\frac{1}{2}\right)^{2x} = 2^{3-x}$ *Hint:* Write $\frac{1}{2}$ as 2^{-1}

Answer: $x = -3$

(c) $4^{2x+7} \leq 32^{2x-3}$

Answer: $\left[\frac{29}{6}, +\infty\right)$

(d) $\left(\frac{2}{5}\right)^{5x-1} \geq \frac{25}{4}$ *Hint:* Write $\frac{25}{4}$ as $\left(\frac{2}{5}\right)^{-2}$

Answer: $(-\infty, -\frac{1}{5}]$

(D) Evaluation.

Solve for x .

(a) $7^{x+4} = 49^{2x-1}$

Answer: $x = 2$

(b) $4^{x+2} = 8^{2x}$

Answer: $x = 1$

(c) $\left(\frac{2}{3}\right)^{5x+2} = \left(\frac{3}{2}\right)^{2x}$

Answer: $\left(-\frac{2}{7}, +\infty\right)$

LESSON 15: Graphing Exponential Functions**TIME FRAME:** 120 minutes**CODE:** M11GM-f-2, f-3, f-4, and g-1

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to represent an exponential function through its table of values, graph, and equation, find the domain, range, intercepts, zeroes, and asymptotes of an exponential function, and graph exponential functions.

LESSON OUTLINE:

1. Graphs of $f(x) = b^x$ if $b > 1$
 2. Graphs of $f(x) = b^x$ if $0 < b < 1$
 3. Domain, range, intercepts, zeroes, and asymptotes of an exponential function
-

DEVELOPMENT OF THE LESSON

(A) Introduction

The graph of an exponential function is a necessary tool in describing its behavior and characteristics – its intercepts, asymptotes, and zeroes. A graph can also provide insights as to real-life situations that can be modeled by exponential functions.

(B) Lesson Proper

We first graph by plotting a few points. Results will be generalized later on.

EXAMPLE 1. Sketch the graph of $f(x) = 2^x$.

Solution.

Step 1. Construct a table of values of ordered pairs for the given function. The table of values for $f(x)$ is as follows:

x	-4	-3	-2	-1	0	1	2	3
$f(x)$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Step 2. Plot the points found in the table and connect them using a smooth curve.

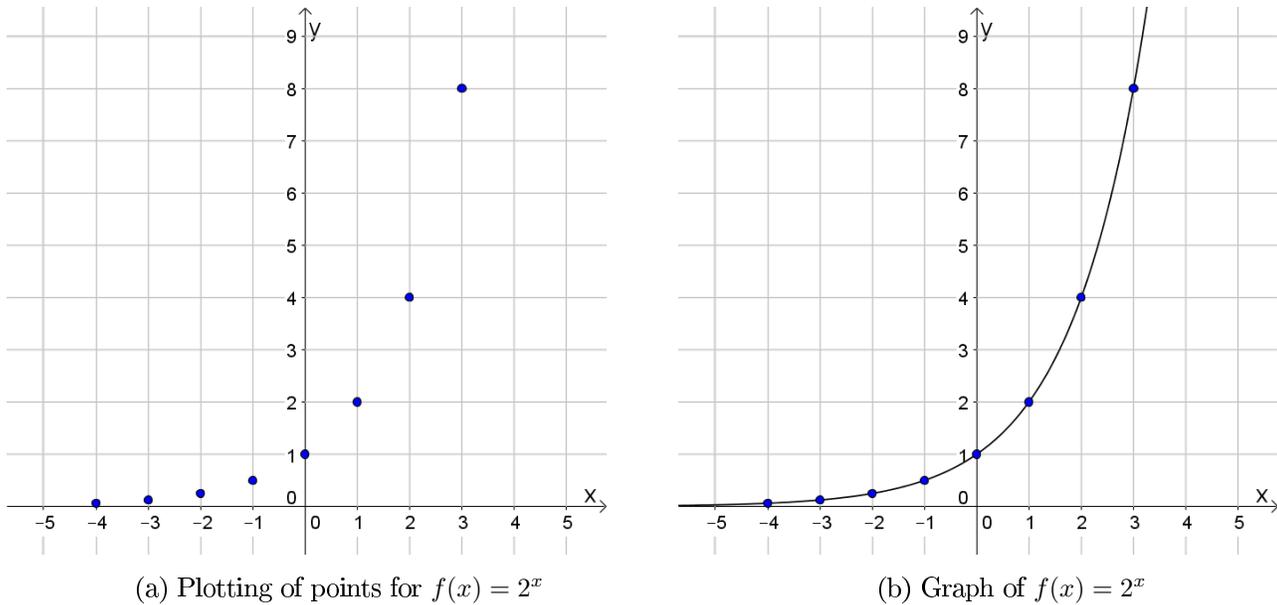


Figure 4.1

It can be observed that the function is defined for all values of x , is strictly increasing, and attains only positive y -values. As x decreases without bound, the function approaches 0. That is, the line $y = 0$ is a horizontal asymptote. \square

EXAMPLE 2. Sketch the graph of $g(x) = \left(\frac{1}{2}\right)^x$.

Solution.

Step 1. The table of values for $g(x)$ is as follows:

x	-3	-2	-1	0	1	2	3	4
$f(x)$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

Step 2. Plot the points found in the table and connect them using a smooth curve.

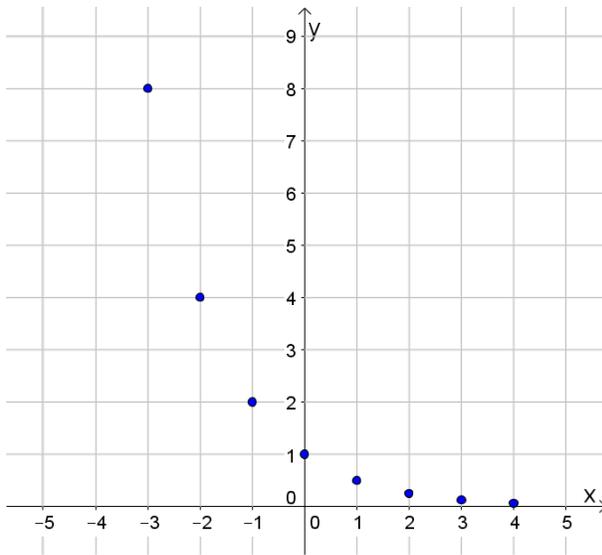
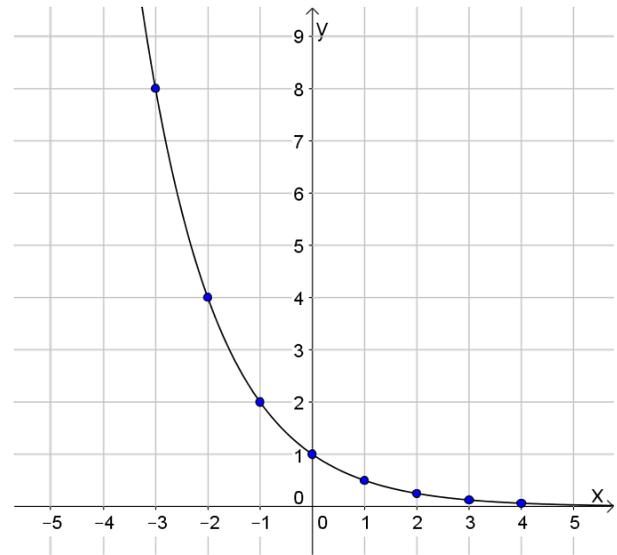
(a) Plotting of points for $g(x) = \left(\frac{1}{2}\right)^x$ (b) Graph of $g(x) = \left(\frac{1}{2}\right)^x$

Figure 4.2

It can be observed that the function is defined for all values of x , is strictly decreasing, and attains only positive values. As x increases without bound, the function approaches 0. That is, the line $y = 0$ is a horizontal asymptote. \square

In general, depending on whether $b > 1$ or $0 < b < 1$, the graph of $f(x) = b^x$ has the following behavior:

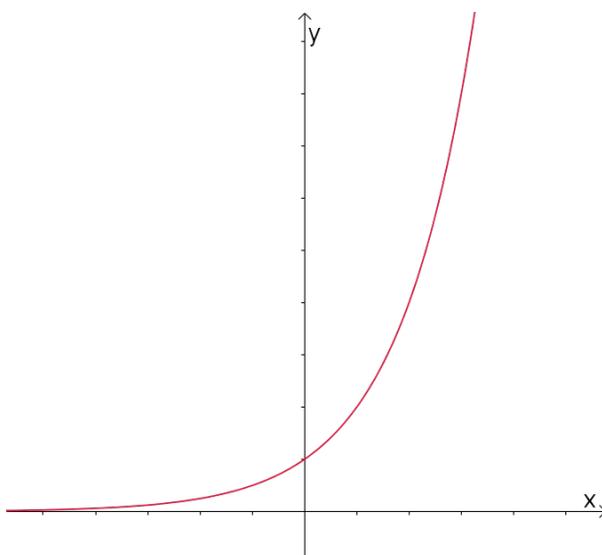
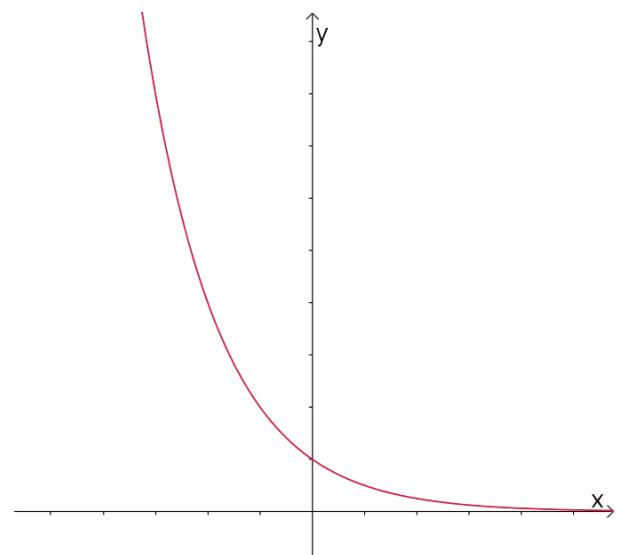
(a) $b > 1$ (b) $0 < b < 1$

Figure 4.3

Properties of Exponential Functions

The following properties of $f(x) = b^x$, where $b > 0$ and $b \neq 1$, can be observed from the graph:

- The domain is the set \mathbb{R} .
- The range is the set $(0, +\infty)$.
- It is a one-to-one function. It satisfies the Horizontal Line Test.
- The y -intercept is 1. There is no x -intercept.
- The horizontal asymptote is the line $y = 0$ (or the x -axis). There is no vertical asymptote.



Teaching Tip

Although graphs of exponential functions generally look like one of the two graphs above, the graph changes as the base changes. Give the following seatworks to help students investigate how the base affects the shape of the graph.

(C) Seatwork

Seatwork 1.

- Complete the following table of values for $f(x) = 2^x$ and $g(x) = 3^x$.

x	-4	-3	-2	-1	0	1	2	3
$f(x)$								
$g(x)$								

- Construct a coordinate plane showing x from -4 to 4 and y from -1 to 30 . Sketch the graphs of $f(x)$ and $g(x)$ on this coordinate plane. Compare both graphs.
- For each function, identify the domain, range, y -intercept, and horizontal asymptote.

Seatwork 2.

- Complete the following table of values for $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \left(\frac{1}{3}\right)^x$.

x	-4	-3	-2	-1	0	1	2	3
$f(x)$								
$g(x)$								

- Construct a coordinate plane showing x from -4 to 4 , and y from -1 to 30 . Sketch the graphs of $f(x)$ and $g(x)$ on this coordinate plane. Compare both graphs.

(c) For each function, identify the domain, range, y -intercept, and horizontal asymptote.

Seatwork 3. Graph the function $f(x) = 5^{-x}$. Identify the domain, range, y -intercept, and horizontal asymptote. *Hint:* Note that $5^{-x} = \frac{1}{5^x} = \left(\frac{1}{5}\right)^x$.

LESSON 16: Graphing Transformations of Exponential Functions

TIME FRAME: 120 minutes

CODE: M11GM-f-2, f-3, f-4, and g-1

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to represent a transformation of an exponential function through its table of values, graph, and equation, and graph transformations of exponential functions.

LESSON OUTLINE:

1. Vertical reflection
 2. Vertical stretching or shrinking
 3. Vertical and horizontal shifts
-

DEVELOPMENT OF THE LESSON

(A) Motivation

Divide the students into groups. Assign each group one of the functions below. For their assigned function, they need to do the following:

- Complete a table of values (use integer x -values from -3 to 3).
- Sketch the graph. Each group should use a coordinate plane showing x from -4 to 4 , and y from -1 to 30 .
- After graphing, each group explains how their graph compares with the graph of $y = 2^x$.

Functions:

(a) $y = 2^x$

(c) $y = 3 \cdot 2^x$

(e) $y = 2^x + 1$

(g) $y = 2^{x+1}$

(b) $y = -2^x$

(d) $y = \frac{2}{5} \cdot 2^x$

(f) $y = 2^x - 1$

(h) $y = 2^{x-1}$

Answers.

x	-3	-2	-1	0	1	2	3	How the graph relates to $y = 2^x$.
(a) $y = 2^x$	0.125	0.25	0.5	1	2	4	8	Not Applicable
(b) $y = -2^x$	-0.125	-0.25	-0.5	-1	-2	-4	-8	Each y is multiplied by -1 .
(c) $y = 3 \cdot 2^x$	0.375	0.75	1.5	3	6	12	24	Each y is multiplied by 3.
(d) $y = \frac{2}{5} \cdot 2^x$	0.05	0.1	0.2	0.4	0.8	1.6	3.2	Each y is multiplied by $2/5$.
(e) $y = 2^x + 1$	1.125	1.25	1.5	2	3	5	9	Graph shifts up by 1 unit
(f) $y = 2^x - 1$	-0.875	-0.75	-0.5	0	1	3	7	Graph shifts down by 1 unit.
(g) $y = 2^{x+1}$	0.25	0.5	1	2	4	8	16	Graph shifts to the left by 1 unit.
(h) $y = 2^{x-1}$	0.0625	0.125	0.25	0.5	1	2	4	Graph shifts to the right by 1 unit

(B) Lesson Proper

Reflecting Graphs

EXAMPLE 1. Use the graph of $y = 2^x$ to sketch the graphs of the functions $y = -2^x$ and $y = 2^{-x}$.

Solution. Some y -values are shown in the following table.

x	-3	-2	-1	0	1	2	3
$y = 2^x$	0.125	0.25	0.5	1	2	4	8
$y = -2^x$	-0.125	-0.25	-0.5	-1	-2	-4	-8
$y = 2^{-x}$	8	4	2	1	0.5	0.25	0.125

The y -coordinate of each point on the graph of $y = -2^x$ is the negative of the corresponding y -coordinate of the graph of $y = 2^x$. Thus, the graph of $y = -2^x$ is the reflection of the graph of $y = 2^x$ about the x -axis.

The value of $y = 2^{-x}$ at x is the same as that of $y = 2^x$ at $-x$. Thus, the graph of $y = 2^{-x}$ is the reflection of the graph of $y = 2^x$ about the y -axis.

The corresponding graphs are shown below:

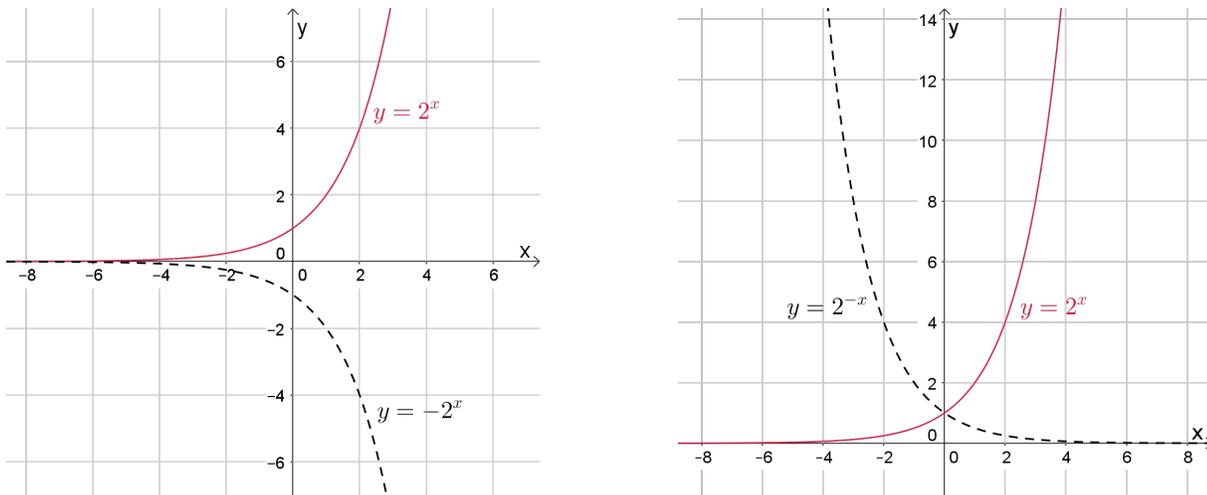


Figure 4.4

□

The results in the Example 1 can be generalized as follows.

Reflection

The graph of $y = -f(x)$ is the *reflection about the x -axis* of the graph of $y = f(x)$.

The graph of $y = f(-x)$ is the *reflection about the y -axis* of the graph of $y = f(x)$.

EXAMPLE 2. Use the graph of $y = 2^x$ to graph the functions $y = 3(2^x)$ and $y = 0.4(2^x)$.

Solution. Some y -values are shown in the following table:

x	-3	-2	-1	0	1	2	3
$y = 2^x$	0.125	0.25	0.5	1	2	4	8
$y = 3(2^x)$	0.375	0.75	1.5	3	6	12	24
$y = 0.4(2^x)$	0.05	0.1	0.2	0.4	0.8	1.6	3.2

The y -coordinate of each point on the graph of $y = 3(2^x)$ is 3 times the y -coordinate of the corresponding point on $y = 2^x$. Similarly, the y -coordinate of each point on the graph of $y = 0.4(2^x)$ is 0.4 times the y -coordinate of the corresponding point on $y = 2^x$. The graphs of these functions are shown below.

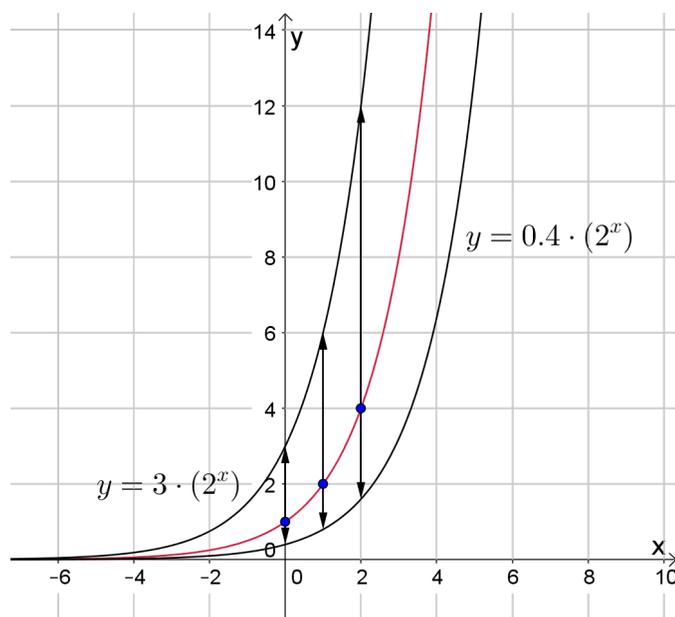


Figure 4.5

Observations

- The domain for all three graphs is the set \mathbb{R} .
- The y -intercepts were also multiplied correspondingly. The y -intercept of $y = 3(2^x)$ is 3, and the y -intercept of $y = 0.4(2^x)$ is 0.4.
- All three graphs have the same horizontal asymptote $y = 0$.
- The range of all three graphs is the set $(0, +\infty)$.

□

The results of Example 2 can be generalized as follows.

Vertical Stretching or Shrinking

Let $a > 0$ be a constant. The graph of $y = af(x)$ can be obtained by multiplying the y -coordinate of each point on the graph of $y = f(x)$ by a . The effect is a *vertical stretching*, if $a > 1$, or a *vertical shrinking*, if $0 < a < 1$, of the graph of $y = f(x)$.

EXAMPLE 3. Use the graph of $y = 2^x$ to graph the functions $y = 2^x - 3$ and $y = 2^x + 1$.

Solution. Some y -values are shown in the following table:

x	-3	-2	-1	0	1	2	3
$y = 2^x$	0.125	0.25	0.5	1	2	4	8
$y = 2^x - 3$	-2.875	-2.75	-2.5	-2	-1	1	5
$y = 2^x + 1$	1.125	1.25	1.5	2	3	5	9

The graphs of these functions are shown below.

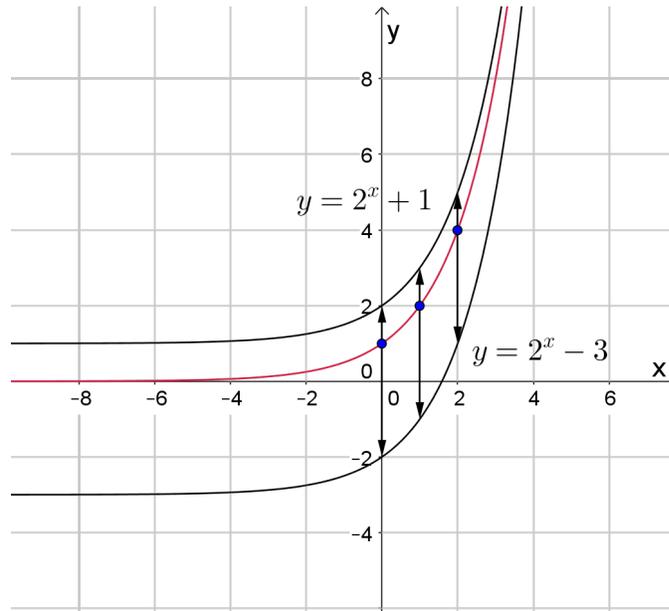


Figure 4.6

Observations

- The domain for all three functions is the set \mathbb{R} .
- The range of $y = 2^x + 1$ is $(1, +\infty)$ and the range of $y = 2^x - 3$ is $(-3, +\infty)$.
- The y -intercepts and horizontal asymptotes were also vertically translated from the y -intercept and horizontal asymptote of $y = 2^x$.
- The horizontal asymptote of $y = 2^x$ is $y = 0$. Shift this 1 unit up to get the horizontal asymptote of $y = 2^x + 1$ which is $y = 1$, and 3 units down to get the horizontal asymptote of $y = 2^x - 3$, which is $y = -3$.

□

The results of Example 3 can be generalized as follows.

Vertical Shifts

Let d be a real number. The graph of $y = f(x) + d$ is a *vertical shift of d units up*, if $d > 0$, or a *vertical shift of d units down* if $d < 0$, of the graph of $y = f(x)$.

EXAMPLE 4. Use the graph of $y = 2^x$ to graph the functions $y = 2^{x-2}$ and $y = 2^{x+4}$.

Solution. Some y -values are shown in the following table:

x	-3	-2	-1	0	1	2	3
$y = 2^x$	0.125	0.25	0.5	1	2	4	8
$y = 2^{x-2}$	0.031	0.063	0.125	0.25	0.5	1	2
$y = 2^{x+4}$	2	4	8	16	32	64	128

The graphs of these functions are shown below.

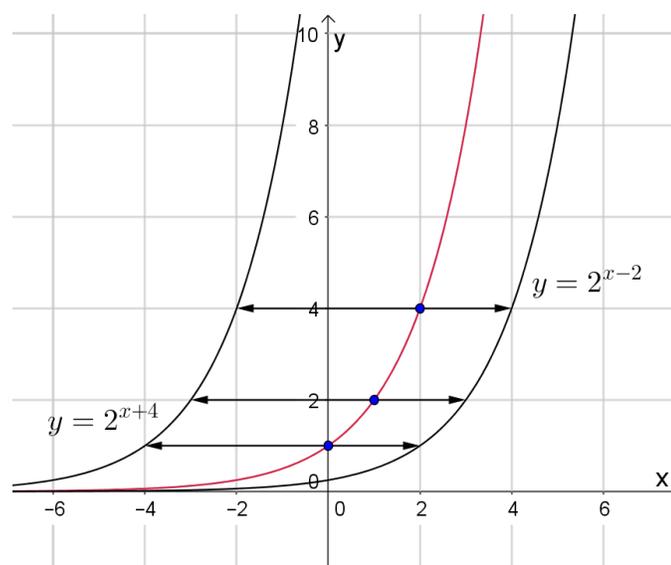


Figure 4.7

Observations

- The domain of all three functions is the set \mathbb{R} .
- The range of all three functions is the set $(0, +\infty)$.
- The y -intercepts changed. To find them, substitute $x = 0$ in the function. Thus, the y -intercept of $y = 2^{x+4}$ is $2^4 = 16$ and the y -intercept of $y = 2^{x-2}$ is $2^{-2} = 0.25$.
- Translating a graph horizontally does not change the horizontal asymptote. Thus, the horizontal asymptote of all three graphs is $y = 0$.

□

The results of Example 4 can be generalized as follows.

Horizontal Shifts

Let c be a real number. The graph of $y = f(x - c)$ is a *horizontal shift of c units to the right*, if $c > 0$, or a *horizontal shift of c units to the left*, if $c < 0$, of the graph of $y = f(x)$.

The examples above can be generalized to form the following guidelines for graphing transformations of exponential functions:

Graph of $f(x) = a \cdot b^{x-c} + d$

- The value of b , where $b > 1$ or $0 < b < 1$, determines whether the graph is increasing or decreasing.
- The value of $|a|$ determines the stretch or shrinking of the graph. Further, if a is negative, there is a reflection of the graph about the x -axis.
- The vertical shift is d units up if $d > 0$, or d units down if $d < 0$.
- The horizontal shift is c units to the right if $c > 0$, or c units to the left if $c < 0$.

(C) Seatwork

Seatwork 1. For each of the following functions, **(a)** use transformations to describe how the graph is related to an exponential function $y = b^x$, **(b)** sketch the graph, **(c)** identify the domain, range, y -intercept, and horizontal asymptote.

(a) $y = 3^x - 4$

(b) $y = \left(\frac{1}{2}\right)^x + 2$

(c) $y = 2^{x-5}$

(d) $y = (0.8)^{x+1}$

(e) $y = 2 \left(\frac{1}{3}\right)^x$

(f) $y = 0.25(3^x)$

(g) $y = 2^{x-3} + 1$

(h) $y = \left(\frac{1}{3}\right)^{x-1} - 2$

Chapter 5

Logarithmic Functions

LESSON 17: Introduction to Logarithms

TIME FRAME: 120 minutes

CODE: M11GM-Ih-1 and j-2

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to represent real-life situations using logarithmic functions, and solve problems involving logarithms.

LESSON OUTLINE:

1. Logarithms and its inverse relationship with exponents
 2. Exponential and logarithmic forms of an equation
 3. Real-life situations that can be represented by logarithmic functions
-

DEVELOPMENT OF THE LESSON

(A) Motivation

Review exponents by asking the class to simplify the given expressions, and find the value of x in each case.

(a) $2^4 = x$

(b) $4^3 = x$

(c) $5^{-1} = x$

(d) $16^{-\frac{1}{2}} = x$

(Answers: 16, 64, $\frac{1}{5}$, $\frac{1}{4}$)

In this lesson, we will look at it from another perspective. What if it is the exponent that is missing?

For example, ask students to figure out the missing number in the following expressions. (You may turn this into a game.)

(a) $5^x = 625$

(c) $7^x = 0$

(b) $3^x = \frac{1}{9}$

(d) $10^x = 100,000$

Finally, present the question: ‘How do we solve for x , aside from observation?’ Mention that there is a mathematical way to present (rewrite) the expressions, and that is through **logarithms**, the inverse process of exponentiation.



Teaching Tip

By mentioning ‘inverse process,’ we are giving the students an impression of what to expect in the next lessons: that the exponential and logarithmic functions are actually inverse functions.

(B) Lesson Proper

- (a) Introduce the logarithm concept.

Think of a **logarithm of x to the base b** (denoted by $\log_b x$ as **the exponent of b that gives x**). For example, $\log_3 81 = 4$ because $3^4 = 81$. Some additional examples are given below:

(a) $\log_2 32 = 5$ because $2^5 = 32$

(b) $\log_5 1 = 0$ because $5^0 = 1$

(c) $\log_6 \left(\frac{1}{6}\right) = (-1)$ because $6^{-1} = \frac{1}{6}$

EXAMPLE 1. Without using a calculator, find the value of the following logarithmic expression.

(a) $\log_2 32$

(c) $\log_5 5$

(e) $\log_7 1$

(b) $\log_9 729$

(d) $\log_{\frac{1}{2}} 16$

(f) $\log_5 \frac{1}{\sqrt{5}}$

Solution.

(a) What exponent of 2 will give 32? **Answer:** 5

(b) What exponent of 9 will give 729? **Answer:** 3

(c) What exponent of 5 will give 5? **Answer:** 1

(d) What exponent of $1/2$ will give 16? **Answer:** -4

(e) What exponent of 7 will give 1? **Answer:** 0

(f) What exponent of 5 will give $\frac{1}{\sqrt{5}}$? **Answer:** $-1/2$

□

(b) Provide the formal definition of logarithms.

Definition

Let a and b be positive real numbers such that $b \neq 1$. The **logarithm of a with base b** , denoted by $\log_b a$, is defined as the number such that $b^{\log_b a} = a$. That is, $\log_b a$ is the exponent that b must be raised to produce a .

Logarithms and exponents allow us to express the same relationship in two different ways. The following table shows how the expressions in Example 1 can be written using two equivalent forms.

Logarithmic Form	Exponential Form
$\log_2 32 = 5$	$2^5 = 32$
$\log_9 729 = 3$	$9^3 = 729$
$\log_5 5 = 1$	$5^1 = 5$
$\log_{1/2} 16 = -4$	$(\frac{1}{2})^{-4} = 16$
$\log_7 1 = 0$	$7^0 = 1$
$\log_5 \frac{1}{\sqrt{5}} = -\frac{1}{2}$	$5^{-1/2} = \frac{1}{\sqrt{5}}$

Exponential and Logarithmic Forms

Logarithmic form: $\log_b a = c$

Exponential form: $b^c = a$

- In both the logarithmic and exponential forms, b is the base.
- In the exponential form, c is an exponent. But $c = \log_b a$. This implies that the logarithm is actually an exponent.
- In the logarithmic form $\log_b a$, a cannot be negative. For example, $\log_2(-8)$ is not defined since 2 raised to any exponent will never result to a negative number.
- The value of $\log_b x$ can be negative. For example, $\log_5 \frac{1}{125} = -3$ because $5^{-3} = \frac{1}{125}$.

Common logarithms are logarithms with base 10; the base 10 is usually omitted when writing common logarithms. This means that $\log x$ is a short notation for $\log_{10} x$.

As mentioned in Lesson 12, the number e (which is approximately 2.71828) has important applications in mathematics. Logarithms with base e are called **natural logarithms**, and are denoted by "ln". In other words, $\ln x$ is another way of writing $\log_e x$.


Teaching Tip

It will be an advantage if students have a good grasp of their table of powers, the same way the mastery of the multiplication table aids in division.

EXAMPLE 2. Rewrite the following exponential equations in logarithmic form, whenever possible.

(a) $5^3 = 125$

(b) $7^{-2} = \frac{1}{49}$

(c) $10^2 = 100$

(d) $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$

(e) $(0.1)^{-4} = 10,000$

(f) $4^0 = 1$

(g) $7^b = 21$

(h) $e^2 = x$

(i) $(-2)^2 = 4$

Solution.

(a) $\log_5 125 = 3$

(b) $\log_7 \frac{1}{49} = -2$ (Remark that it is alright to have a negative exponent, but not a negative base or argument.)

(c) $\log 100 = 2$

(d) $\log_{\frac{2}{3}} \left(\frac{4}{9}\right) = 2$ (Remark that it is alright to have a fractional or decimal base, as long as it is positive.)

(e) $\log_{0.1} 10,000 = -4$

(f) $\log_4 1 = 0$

(g) $\log_7 21 = b$ (Because of the presence of 7 and 21, some students might incorrectly think that $b = 3$. However, $7^3 \neq 21$. The approximate value of b , using a calculator, is 1.5646. You can also ask your students to verify that $7^{1.5645}$ is approximately 21.)

(h) $\ln x = 2$

(i) Cannot be written in logarithmic form since the base is a negative number.

□

EXAMPLE 3. Rewrite the following logarithmic equations in exponential form.

(a) $\log m = n$

(b) $\log_3 81 = 4$

(c) $\log_{\sqrt{5}} 5 = 2$

(d) $\log_{\frac{3}{4}} \frac{64}{27} = -3$

(e) $\log_4 2 = \frac{1}{2}$

(f) $\log_{10} 0.001 = -3$

(g) $\ln 8 = a$

Solution.

(a) $10^n = m$

(b) $3^4 = 81$

(c) $(\sqrt{5})^2 = 5$

(d) $\left(\frac{3}{4}\right)^{-3} = \frac{64}{27}$

(e) $4^{1/2} = 2$

(f) $10^{-3} = 0.001$ or $10^{-3} = \frac{1}{1,000}$

(g) $e^a = 8$

□

- (c) Representation of real-life situations that use logarithms. You can choose just one situation, depending on the interest of your students.


Teaching Tip

The purpose of this lesson is to briefly inform the students on the application of logarithms in situations that are familiar to them, without actually asking students to solve for unknown quantities yet. We will revisit all of these problem types in future lessons.

Logarithms allow us to discuss very large numbers in more manageable ways. For example, 10^{31} , a very large number, may be difficult to work with. But its common logarithm $\log 10^{31} = 31$ is easier to grasp. Because logarithms can facilitate an understanding of very large numbers (or positive numbers very close to zero), it has applications in various situations.

- i. Sample application: The Richter scale. In 1935, Charles Richter proposed a logarithmic scale to measure the intensity of an earthquake. He defined the magnitude of an earthquake as a function of its amplitude on a standard seismograph. The following formula produces the same results, but is based on the energy released by an earthquake.¹

Earthquake Magnitude on a Richter scale

The magnitude R of an earthquake is given by

$$R = \frac{2}{3} \log \frac{E}{10^{4.40}}$$

where E (in joules) is the energy released by the earthquake (the quantity $10^{4.40}$ is the energy released by a very small reference earthquake).

The formula indicates that the magnitude of an earthquake is based on the logarithm of the ratio between the energy it releases and the energy released by the reference earthquake.

EXAMPLE 4. Suppose that an earthquake released approximately 10^{12} joules of energy. (a)

¹Barnett, R. A., Ziegler, M. R., Byleen, K. E., & Sobecki, D. (2008). *Precalculus*. New York: McGraw-Hill.

What is its magnitude on a Richter scale? (b) How much more energy does this earthquake release than that by the reference earthquake?

Solution.

(a) Since $E = 10^{12}$, then $R = \frac{2}{3} \log \frac{10^{12}}{10^{4.40}} = \frac{2}{3} \log 10^{7.6}$.

Since by definition, $\log 10^{7.6} = 7.6$ is the exponent by which 10 must be raised to obtain $10^{7.6}$, then $\log 10^{7.6} = 7.6$.

Thus, $R = \frac{2}{3}(7.6) \approx 5.1$.

(b) This earthquake releases $10^{12}/10^{4.40} = 10^{7.6} \approx 39810717$ times more energy than that by the reference earthquake.

□

We will discuss the earthquakes and the Richter scale in more detail as we learn more about logarithms (see Lesson 21).

Activity (optional; if time permits)

After defining the Richter scale, you may ask the students to conduct a simple research activity on the following: how do earthquakes with various magnitudes compare with each other? If students do not have access to reference materials, you can just give them the

following descriptions of various earthquake magnitudes. ²

²PHIVOLCS, http://www.phivolcs.dost.gov.ph/index.php?option=com_content&task=view&id=45&Itemid=100

Magnitude	Description
1.0	Scarcely Perceptible - Perceptible to people under favorable circumstances.
2.0	Slightly Felt - Hanging objects swing slightly.
3.0	Weak - Vibration is felt like one passing of a light truck. Hanging objects swing moderately.
4.0	Moderately Strong - Felt generally by people indoors and by some people outdoors.
5.0	Strong - Strong shaking and rocking felt throughout building. Hanging objects swing violently.
6.0	Very Strong - Some people lose their balance. Heavy objects or furniture move or may be shifted.
7.0	Destructive - People find it difficult to stand in upper floors. Some cracks may appear. Limited liquefaction, lateral spreading and landslides are observed. Trees are shaken strongly.
8.0	Very Destructive - Many well-built buildings are considerably damaged. Liquefaction and lateral spreading cause man-made structure to sink, tilt or topple. Fissures and faults rupture may be observed.
9.0	Devastating - Most buildings are totally damaged. Bridges and elevated concrete structures are toppled or destroyed. Landslides and liquefaction with lateral spreading and sand-boil are widespread.
10.0	Completely Devastating - Practically all man-made structures are destroyed. Massive landslides and liquefaction, large scale subsidence and uplifting of land forms and many ground fissures are observed. Changes in river courses occur.

This activity can help foster awareness on possible disaster management and risk reduction plans. You may pose the following problem for them to ponder on until the necessary skills have been taught:

The 2013 earthquake in Bohol and Cebu had a magnitude of 7.2, while the 2012 earthquake that occurred in Negros Oriental recorded a 6.7 magnitude. How much more energy was released by the 2013 Bohol/Cebu earthquake compared to that by the Negros Oriental earth-

quake? [Note: This will be solved in Lesson 21 Example 4.]

ii. Sample application: The decibel³

 **Teaching Tip**

While intensity is an objective measure of sound with respect to energy and area (measured in Watts per square meter), loudness is a subjective measure; it depends on how the listener perceives the sound.

Sound Intensity

In acoustics, the decibel (dB) level of a sound is

$$D = 10 \log \frac{I}{10^{-12}}$$

where I is the sound intensity in watts/m² (the quantity 10^{-12} watts/m² is least audible sound a human can hear).

With the use of decibels, a rock concert measuring 60 dB is 10^6 (1000000) times as loud as a normal conversation that is 10 dB.

(**Enrichment:** You may ask students to research on decibel levels that are already harmful to our auditory health, and seek ways to prevent hearing loss.)

Again, if students have limited reference materials, you can provide the following table. ⁴

³Young, C. (2012). College algebra (3rd ed). Hoboken, NJ: John Wiley & Sons.

⁴National Institute on Deafness and Other Communication Disorders (NIDCD). <https://www.nidcd.nih.gov/health/i-love-what-i-hear-common-sounds>

Intensity	Description	Examples
Under 60 dB	Comfortable hearing levels	Normal breathing, whisper, quiet office, normal conversation
60 – 85 dB	Intrusive and interferes with conversation	Vacuum cleaner, washing machine, average city traffic, television
85 dB	Hearing damage begins (unprotected exposure for 8 hours)	Lawnmower, food blender, recreational vehicles
90 – 100 dB	No more than 15 minutes of unprotected exposure recommended	Motorcycle, newspaper press, garbage truck
Over 100 dB	Regular exposure of more than one minute risks permanent damage	Jet flyover (1000 ft.), jackhammer, chainsaw, orchestra, stereos, boom box
125 dB and above	Threshold of pain begins (125 dB)	Rock concerts, jet takeoff (100-200 ft.), shotgun firing, jet engines (near)

EXAMPLE 5. The decibel level of sound in a quiet office is 10^{-6} watts/ m^2 . (a) What is the corresponding sound intensity in decibels? (b) How much more intense is this sound than the least audible sound a human can hear?

Solution.

(a) $D = 10 \log \frac{10^{-6}}{10^{-12}} = 10 \log 10^6$ Hence by definition, $\log 10^6$ is the exponent by which 10 must be raised to obtain 10^6 , then $\log 10^6 = 6$.

Thus, $D = 10(6) = 60$ decibels.

(b) This sound is $\frac{10^{-6}}{10^{-12}} = 10^6 = 100,000$ times more intense than the least audible sound a human can hear.

□

iii. (Optional enrichment for science-track students) pH levels.

Definition

In chemistry, the pH level measures the acidity of a water-based solution that is measured by the concentration of hydrogen ions (written $[H^+]$, in moles per liter) in the solution. It is obtained by

$$pH = -\log[H^+]$$

Note (1): Once the students learn the properties of logarithms, you may go back to this formula and show that it is exactly the same as

$$pH = \log \frac{1}{[H^+]}$$

EXAMPLE 6. A 1-liter solution contains 0.00001 moles of hydrogen ions. Find its pH level.

Solution. Since there are 0.00001 moles of hydrogen ions in 1 liter, then the concentration of hydrogen ions is 10^{-5} moles per liter. The pH level is $-\log 10^{-5}$. Since $\log 10^{-5}$ is the exponent by which 10 must be raised to obtain 10^{-5} , then $\log 10^{-5} = -5$.

Thus, $pH = -\log 10^{-5} = -(-5) = 5$. □

(Enrichment: You may give the students a simple research activity involving acidity. List 10 distinct water-based (i.e. aqueous) solutions, their corresponding pH levels, and rank from least to most acidic. They should be able to deduce that higher pH level means lesser acidity.)

Again, the following table can provide information, in case there are not enough reference materials.

pH Level of Common Solutions⁵

0	Battery acid
1	Concentrated sulfuric acid
2	Lemon juice, vinegar
3	Orange juice, soda
4	Tomato juice, acid rain
5	Black coffee, banana
6	Urine, milk
7	Pure water
8	Sea water, eggs
9	Baking soda
10	Milk of magnesia
11	Ammonia solution
12	Soapy water
13	Bleach, oven cleaner
14	Liquid drain cleaner

(d) Seatwork

Seatwork 1. Find the value of the following logarithmic expressions.

(a) $\log_3 243$

Answer: 5

(b) $\log_6 \left(\frac{1}{216} \right)$

Answer: -3

(c) $\log_{0.25} 16$

Answer: -2

Seatwork 2. Write the following exponential equations in logarithmic form.

(a) $49^x = 7$

Answer: $\log_{49} 7 = x$

(b) $6^{-3} = \frac{1}{216}$

Answer: $\log_6 216 = -3$

(c) $10^2 = 100$

Answer: $\log 100 = 2$

Seatwork 3. Write the following logarithmic equations in exponential form.

(a) $\log_{\frac{11}{2}} \left(\frac{4}{121} \right) = -2$

Answer: $\left(\frac{11}{2} \right)^{-2} = \frac{4}{121}$

(b) $\ln 3 = y$

Answer: $e^y = 3$

(c) $\log 0.001 = -3$

Answer: $10^{-3} = 0.001$

(e) Evaluation/Group Activity

Ask the students to form groups of three or four. Using a Manila paper or cartolina, they should

⁵http://www.sciencebuddies.org/science-fair-projects/project_ideas/Chem_AcidsBasespHScale.shtml

list down 5 exponential equations that cannot be expressed in logarithmic form, and 5 incorrect logarithmic expressions. There should be a variety in their examples (i.e. not all five logarithmic expressions should have negative bases). They may present this in the next meeting, and can serve as the motivation and recall of the introductory lesson.

LESSON 18: Logarithmic Functions, Equations, and Inequalities

TIME FRAME: 30 minutes

CODE: M11GM-Ih-2

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to distinguish logarithmic function, logarithmic equation, and logarithmic inequality.

LESSON OUTLINE:

1. Definition and examples of logarithmic functions, logarithmic equations, and logarithmic inequalities

DEVELOPMENT OF THE LESSON

(A) Introduction

Ask students to share their ideas about how A, B, and C are similar, and how they are different.

$$A) \log_3(x - 2) = 5$$

$$B) \ln x \geq 9$$

$$C) y = -\log_{\frac{1}{2}} x$$

Possible responses: A, B, and C all involve expressions with logarithms. A and B involve only one variable, but C expresses a relationship between two variables. A and C involve an equality, but B involves an inequality.

(B) Lesson Proper The definitions of exponential equations, inequalities and functions are shown below.

	Logarithmic Equation	Logarithmic Inequality	Logarithmic Function
Definition	An equation involving logarithms	An inequality involving logarithms	Function of the form $f(x) = \log_b x$ ($b > 0$, $b \neq 1$).
Example	$\log_x 2 = 4$	$\ln x^2 > (\ln x)^2$	$g(x) = \log_3 x$

A logarithmic equation or inequality can be solved for all x values that satisfy the equation or inequality

(Lesson 21). A logarithmic function expresses a relationship between two variables (such as x and y), and can be represented by a table of values or a graph (Lesson 22).

Brief Introduction to Logarithmic Equations



Teaching Tip

The purpose of this lesson is to help students differentiate among logarithmic equations, inequalities, and functions. Solving is not yet required at this point.

Present the following examples of logarithmic equations:

- $\log_x 2 = 4$
- $\log_2 x = -4$
- $\log_2 4 = x$

In each, a part of the logarithmic equation is missing, and the goal is to find the value of the unknown that will make the left- and the right-hand sides equal.

- For $\log_x 2 = 4$, aid the students in interpreting the equation: "Which value/s of x , when raised to the fourth power, will give us 2?"
- Next, $\log_2 x = -4$, the interpretation is like: "What is the answer when 2 is raised to -4 ?"
- Lastly, for $\log_2 4 = x$, it will be "2 raised to which value/s of x will give us 4?"

For the examples above, students might be able to get the values of just by inspection. However, algebraic techniques will be discussed later.



Teaching Tip

It is important for students to know that before they solve any mathematics problems, they have to check if the given problem "violates" any rule or condition. An equation such as $\log_{-2} x = 4$ cannot be solved since the logarithmic expression on the left-hand side contains a negative base, and is invalid in the first place.

Activity

Divide the class into pairs. Some of the pairs will work on Activity A, while the others on Activity B.

- (a) **Activity A:** Given the exponential function $f(x) = 3^x$, fill in the table of values below:

x	$f(x)$
-4	
-1	
0	
3	
5	

- (b) **Activity B:** Given the logarithmic function $g(x) = \log_3 x$, fill in the table of values below. (Note: You can guide students by asking, "To which exponent should 3 be raised to obtain x ?")

x	$g(x)$
$\frac{1}{81}$	
$\frac{1}{3}$	
1	
27	
243	

- (c) After completing the table, ask two different pairs to compare their results and provide a conclusion. [Note: This activity introduces the exponential function as the inverse of a logarithmic function.]

(C) Assignment

As a precursor to the next lesson, using knowledge in solving exponential equations, solve the following simple logarithmic equations.

(a) $\log_2 x = 4$

(b) $\log_x 16 = 2$

(c) $\log 1000 = -x$

Solution. (a) $\log_2 x = 4$

$$2^4 = x$$

$$x = 16$$

(b) $\log_x 16 = 2$

$$x^2 = 16$$

$$x^2 - 16 = 0$$

$$(x + 4)(x - 4) = 0$$

$$x = -4, +4$$

But the base cannot be negative. Thus, the only solution is $x = 4$.

(c) $\log 1000 = -x$

Using analysis and observation, 10 raised to 3 is 1000.

Hence, $\log 1000$ is 3.

$$3 = -x$$

$$x = -3$$

□

LESSON 19: Basic Properties of Logarithms

TIME FRAME: 60 minutes

CODE: M11GM-Ih-3

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to illustrate the laws of logarithms.

LESSON OUTLINE:

1. Review: Definition of logarithms
2. Three basic properties of logarithms

DEVELOPMENT OF THE LESSON

(A) Introduction

Review logarithms by asking the class to find the value of $\log_3 81$. Guide them by asking ‘3 raised to which power will give us 81?’. The correct answer is 4 since $3^4 = 81$. Provide the following examples, then ask students to simplify and observe carefully:

$$\log_4 1$$

$$\log_3 3$$

$$\log_4 4^2$$

Answers: $\log_4 1 = 0$ because $4^0 = 1$.

$\log_3 3 = 1$ because $3^1 = 3$.

$\log_4 4^2 = 2$ because $4^2 = 4^2$.

Through observing the examples above, we will introduce the basic properties of logarithms.

(B) Lesson Proper

Basic properties of logarithms

Let b and x be real numbers such that $b > 0$ and $b \neq 1$.

(a) $\log_b 1 = 0$

(b) $\log_b b^x = x$

(c) If $x > 0$, then $b^{\log_b x} = x$

Property 1 is straightforward: $\log_b 1$ is the exponent of b that gives 1; in other words $b^? = 1$. This exponent must be 0.

Property 2 is also evident from the definition: $\log_b b^x$ is the exponent of b that gives b^x . Clearly, the exponent must be x .

Property 3:

Remind students that $\log_b x$ is the exponent of b that gives x . So if we make it the exponent of b , then we should get x .

To clarify this, ask students what $\log_2 14$ is. They probably cannot answer since the value is not exact. Ask for estimates: 3 is too small since $2^3 = 8$, but 4 is too large since $2^4 = 16$. The value must be somewhere between 3 and 4 (in fact, closer to 4 because 14 is closer to 16 than to 8).

Now tell them that the approximate value is 3.8074. How can they check? Invite them to use their calculators (or give the value yourself) and see that $2^{3.8074} \approx 14.000$.

This answer should not be surprising since 3.8074 is the **exponent (approximately) of 2 that gives 14**. Thus, if we make it the exponent of 2, then it should give 14. In other words, $2^{\log_2 14} = 14$.

EXAMPLE 1. Use the properties $\log_b 1 = 0$, $\log_b b^x = x$, and $b^{\log_b x} = x$ to find the value of the following logarithmic expressions.

(a) $\log 10$

(b) $\ln e^3$

(c) $\log_4 64$

(d) $\log_5 \left(\frac{1}{125} \right)$

(e) $5^{\log_5 2}$

(f) $\log 1$

Solution.

(a) $\log 10 = \log_{10} 10^1 = 1$ (Property 2)

(b) $\ln e^3 = \log_e e^3 = 3$ (Property 2)

(c) $\log_4 64 = \log_4 4^3 = 3$ (Property 2)

(d) $\log_5 \left(\frac{1}{125} \right) = \log_5 5^{-3} = -3$ (Property 2)

- (e) $5^{\log_5 2} = 2$ (Property 3)
 (f) $\log 1 = 0$ (Property 1)

□

EXAMPLE 2. Suppose you have seats to a concert featuring your favorite musical artist. Calculate the approximate decibel level associated if a typical concert's sound intensity is 10^{-2} W/m². (Refer to Lesson 17 for a discussion of sound intensity).

Solution.

$$D = 10 \log \left(\frac{I}{I_0} \right)$$

$$D = 10 \log \left(\frac{10^{-2}}{10^{-12}} \right)$$

$$D = 10 \log 10^{10}$$

$$D = 10 \cdot 10 \text{ (Property 2)}$$

$$D = 100 \text{ dB}$$

The concert's sound intensity is 100dB.

□

EXAMPLE 3. Calculate the hydrogen ion concentration of vinegar that has a pH level of 3.0. (Refer to Lesson 17 for a discussion of pH levels).

Solution.

$$pH = -\log[H^+]$$

$$3.0 = -\log[H^+]$$

$$-3.0 = \log[H^+]$$

$$10^{-3.0} = 10^{\log H^+}$$

$$10^{-3.0} = [H^+] \text{ (Property 3)}$$

Answer. The hydrogen ion concentration is $10^{-3.0}$ moles per liter.

□

(C) Seatwork

Seatwork 1. Find the value of the following logarithmic expressions.

(a) $\log_7 49$ (Answer: 2)

(b) $\log_{27} 3$ (Answer: $\frac{1}{3}$)

(c) $\ln e$ (Answer: 1)

TIME FRAME: 150 minutes

CODE: M11GM-Ih-3

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to illustrate the laws of logarithms.

LESSON OUTLINE:

1. Laws of logarithms
 2. Change-of-base formula
-

DEVELOPMENT OF THE LESSON

(A) Introduction

On the board, write the 10 logarithmic expressions below (but mix them up). Ask students to find the value of each, and to group the logarithms with the same answer together.

- | | |
|--|-------------------|
| (a) $\log_7(7^3 \cdot 7^8)$ and $\log_7 7^3 + \log_7 7^8$ | Answer: 11 |
| (b) $\log_7\left(\frac{49}{7}\right)$ and $\log_7 49 - \log_7 7$ | Answer: 1 |
| (c) $\log_7 7^5$ and $5 \cdot \log_7 7$ | Answer: 5 |
| (d) $\log_2\left(\frac{2^4}{2^{10}}\right)$ and $\log_2 2^4 - \log_2 2^{10}$ | Answer: -6 |
| (e) $\log_3(27 \cdot 81)$ and $\log_3 27 + \log_3 81$ | Answer: 7 |

After the students group the logarithms into pairs with the same answer, mention that these pairs can be generalized into certain **laws of logarithms**.

(B) Lesson Proper

- (a) Laws of logarithms

Laws of Logarithms

Let $b > 0, b \neq 1$ and let $n \in \mathbb{R}$. For $u > 0, v > 0$, then

1. $\log_b(uv) = \log_b u + \log_b v$
2. $\log_b\left(\frac{u}{v}\right) = \log_b u - \log_b v$
3. $\log_b(u^n) = n \log_b u$

To illustrate, ask students to choose pairs of logarithms from the motivation activity as examples to these properties.

Expected Answer:

Law	Examples from motivation
$\log_b(uv) = \log_b u + \log_b v$	$\log_7(7^3 \cdot 7^8) = \log_7 7^3 + \log_7 7^8$
$\log_b\left(\frac{u}{v}\right) = \log_b u - \log_b v$	$\log_7\left(\frac{49}{7}\right) = \log_7 49 - \log_7 7$ $\log_2\left(\frac{2^4}{2^{10}}\right) = \log_2 2^4 - \log_2 2^{10}$
$\log_b(u^n) = n \log_b u$	$\log_7 7^5 = 5 \cdot \log_7 7$ $\log_3(27 \cdot 81) = \log_3 27 + \log_3 81$

Proof of the Laws of Logarithms.

1. Let $r = \log_b u$ and $s = \log_b v$. Then $u = b^r$ and $v = b^s$.

$$\begin{aligned}\log_b(uv) &= \log_b(b^r b^s) \\ \Rightarrow \log_b(uv) &= \log_b b^{r+s} \\ \Rightarrow \log_b(uv) &= r + s \\ \Rightarrow \log_b(uv) &= \log_b u + \log_b v\end{aligned}$$

2. The proof of the second law is similar to the one above.

3. Let $r = \log_b u$. Then $u = b^r$ and $u^n = b^{rn}$.

$$\begin{aligned}u^n &= b^{rn} \\ \Rightarrow \log_b(u^n) &= \log_b(b^{rn}) \\ \Rightarrow \log_b(u^n) &= rn \\ \Rightarrow \log_b(u^n) &= n \log_b u\end{aligned}$$

Teaching Tip

Mention that $\log_b u^n$ is not the same as $(\log_b u)^n$. In the first expression, n is the exponent of u , but in the second expression, n is the exponent of $\log_b u$.

Emphasize some common mistakes:

1. $\log_2(5 + 2) \neq \log_2 5 + \log_2 2$
2. $\log_2(5 + 2) \neq (\log_2 5)(\log_2 2)$
3. $\log_2(5 - 2) \neq \log_2 5 - \log_2 2$
4. $\log_2(5 - 2) \neq \frac{\log_2 5}{\log_2 2}$
5. $\log_2(5^2 \cdot 2) \neq 2\log_2(5 \cdot 2)$

EXAMPLE 1. Use the properties of logarithms to expand each expression in terms of the logarithms of the factors. Assume each factor is positive.

- (a) $\log(ab^2)$ (c) $\ln[x(x - 5)]$
 (b) $\log_3\left(\frac{3}{x}\right)^3$

Solution.

- (a) $\log(ab^2) = \log a + \log b^2 = \log a + 2 \log b$
 (b) $\log_3\left(\frac{3}{x}\right)^3 = 3 \log_3\left(\frac{3}{x}\right) = 3(\log_3 3 - \log_3 x) = 3(1 - \log_3 x) = 3 - 3 \log_3 x$
 (c) $\ln[x(x - 5)] = \ln x + \ln(x - 5)$

□

EXAMPLE 2. Use the properties of logarithm to condense the expressions as a single logarithm.

- (a) $\log 2 + \log 3$ (c) $\log_5(x^2) - 3 \log_5 x$
 (b) $2 \ln x - \ln y$ (d) $2 - \log 5$

- Solution.** (a) $\log 2 + \log 3 = \log(2 \cdot 3) = \log 6$
 (b) $2 \ln x - \ln y = \ln(x^2) - \ln y = \ln\left(\frac{x^2}{y}\right)$
 (c) $\log_5(x^2) - 3 \log_5 x = \log_5(x^2) - \log_5(x^3) = \log_5\left(\frac{x^2}{x^3}\right) = \log_5\left(\frac{1}{x}\right) = \log_5(x^{-1}) = -\log_5 x$
 (d) $2 - \log 5$

The difficulty here is the missing logarithm for the number 2. We have to first express 2 in terms of a logarithm to base 10.

$$\begin{aligned} 2 &= 2(1) \\ &= 2(\log 10) && \text{using } 1 = \log_b b \\ &= \log 10^2 && \text{using } n \log_b u = \log_b u^n \\ &= \log 100 \end{aligned}$$

$$\text{Thus, } 2 - \log 5 = \log 100 = \log\left(\frac{100}{5}\right) = \log 20$$

□

(b) Change-of-base formula

Ask the students to find the values of the given logarithms, and to try to find a pattern.

- (a) $\log_3 729$ **Answer:** 6
 (b) $\log_9 729$ **Answer:** 3
 (c) $\log_{27} 729$ **Answer:** 2
 (d) $\log_{1/27} 729$ **Answer:** -2
 (e) $\log_{729} 729$ **Answer:** 1
 (f) $\log_{81} 729$ **Answer:** $3/2$; solution below

It might be difficult to find the value of $\log_{81} 729$, or the **exponent of 81 that gives 729**. One strategy is to express 3 in terms of 81; that is, $81^{1/4} = 3$. Since $3^6 = 729$, then $(81^{1/4})^6 = 729$ or $81^{6/4} = 81^{3/2} = 729$. Thus, $\log_{81} 729 = \frac{3}{2}$.

Possible pattern from the activity:

The activity suggests that $\log_{3^n} 729 = \frac{6}{n}$.

This pattern can be generalized to the Change-of-base formula, as given below.

Change-of-Base Formula

Let a , b , and x be positive real numbers, with $a \neq 1, b \neq 1$.

$$\log_b x = \frac{\log_a x}{\log_a b}$$

 **Note:**

The pattern $\log_{3^n} 729 = \frac{6}{n}$ from the previous activity can be derived from the change-of-base formula as follows:

$$\log_{3^n} 729 = \frac{\log_3 729}{\log_3 3^n} = \frac{6}{n}.$$

EXAMPLE 3. Use the change-of-base formula to rewrite the following logarithmic expressions to the indicated base, and then compute the value.

- a. $\log_8 32$ (change to base 2)
 b. $\log_{243} \frac{1}{27}$ (change to base 3)
 c. $\log_{25} \frac{1}{\sqrt{5}}$ (change to base 5)

Solution.

$$\begin{aligned} \text{a. } \log_8 32 &= \frac{\log_2 32}{\log_2 8} = \frac{5}{3} \\ \text{b. } \log_{243} \frac{1}{27} &= \frac{\log_3 \frac{1}{27}}{\log_3 243} = \frac{-3}{5} \\ \text{c. } \log_{25} \frac{1}{\sqrt{5}} &= \frac{\log_5 \frac{1}{\sqrt{5}}}{\log_5 25} = \frac{-1/2}{2} = -\frac{1}{4} \end{aligned}$$

□

EXAMPLE 4. Use the change-of-base formula to rewrite the following logarithmic expressions to the indicated base. Simplify.

(a) $\log_6 4$ (change to base 2)

(b) $\log_{\frac{1}{2}} 2$ (change to base e)

Solution.

(a) $\log_6 4 = \frac{\log_2 4}{\log_2 6} = \frac{2}{\log_2 6}$

(b) $\log_{\frac{1}{2}} 2 = \frac{\ln 2}{\ln \left(\frac{1}{2}\right)} = \frac{\ln 2}{\ln 1 - \ln 2} = \frac{\ln 2}{0 - \ln 2} = \frac{\ln 2}{-\ln 2} = -1$

□

(C) Seatwork

Seatwork 1. Use the properties of logarithm to expand the expressions as a sum, difference, or multiple of logarithms. Simplify.

(a) $\log \left(\frac{x^3}{2}\right)$ (Answer: $3 \log x - \log 2$)

(b) $\ln(2e)^2$ (Hint: $(2e)^2 = 4e^3$; Answer: $2 \ln 2 + 2$)

(c) $\log_4(16a)$ (Answer: $2 + \log_4 a$)

Seatwork 2. Use the properties of logarithm to condense the expressions as a single logarithm.

(a) $\log(x+2) + \log(x-2)$

Answer: $\log(x^2 - 4)$

(b) $2 \log_3 5 + 1$

Answer: $\log_3 75$

(c) $2 \ln \left(\frac{3}{2}\right) - \ln 4$

Answer: $\ln \left(\frac{9}{16}\right)$

Seatwork 3. True or false.

(a) $(\log_3 2)(\log_3 4) = \log_3 8$

Answer: False

(b) $(\log_3 2)(\log_3 4) = \log_3 6$

Answer: False

- (c) $\log 2^2 = (\log 2)^2$ **Answer:** False
- (d) $\log_4(x - 4) = \frac{\log_4 x}{\log_4 4}$ **Answer:** False
- (e) $3 \log_9 x^2 = 6 \log_9 x$ **Answer:** True
- (f) $3(\log_9 x)^2 = 6 \log_9 x$ **Answer:** False
- (g) $\log_3 2x^2 = \log_3 2 + 2 \log_3 x$ **Answer:** True
- (h) $\log_3 2x^2 = 2 \log_3 2x$ **Answer:** False

(D) Exploration / Group Activity

Ask the students to work in pairs and solve the following problems.

- (a) Given $\log_5 2 \approx 0.431$. Use this fact and the laws of logarithms to approximate the values of $\log_5 8$, $\log_5 \left(\frac{1}{16}\right)$, $\log_5 \sqrt{2}$, $\log_{25} 2$, and $\log_{25} 8$.
(Answers: 1.2920, -1.7227 , 0.2153, 0.2153, 0.6460)
- (b) Given $\log 6 \approx 0.778$ and $\log 4 \approx 0.602$. Use these facts and the laws of logarithms to approximate the values of $\log_6 4$, $\log 24$, $\log_4 6$, $\log \left(\frac{2}{3}\right)$, and $\log \left(\frac{3}{2}\right)$.
(Answers: 0.7737, 1.3802, 1.2925, -0.1761 , 0.1761)

LESSON 21: Solving Logarithmic Equations and Inequalities

TIME FRAME: 120 minutes

CODE: M11GM-Ih-i-1 and j-2

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to solve logarithmic equations and inequalities, and solve problems involving logarithmic equations and inequalities.

LESSON OUTLINE:

1. Solving logarithmic equations
2. Solving logarithmic inequalities
3. Real life situations involving logarithmic equations and inequalities

DEVELOPMENT OF THE LESSON

(A) Introduction

In solving logarithmic equations and inequalities, it is important for students to remember the restrictions on the values of b and x in the logarithmic expression $\log_b x$. Recall that:

- b can be any positive real number except 1
- x can be any real number
- the entire expression $\log_b x$ can be any real number (hence, can be negative).

The previous lessons exposed students not only to a brief encounter with logarithmic equations and inequalities, but also to simplification techniques involving logarithmic properties.

Remark that in this lesson, they will be able to apply the previous lessons to actually solve for the values that will make the statements true. This is to give them a sense of objective and direction.

(B) Lesson Proper

(a) Solving logarithmic equations

Some Strategies for solving logarithmic equations

- Rewriting to exponential form
- Using logarithmic properties
- Applying the **One-to-One property** of logarithmic functions, as stated below.

One-to-One property of Logarithmic Functions

For any logarithmic function $f(x) = \log_b x$, if $\log_b u = \log_b v$, then $u = v$.

Another useful property is the **Zero Factor Property**:

$$\text{If } ab = 0, \text{ then } a = 0 \text{ or } b = 0.$$



Teaching Tip

After solving, check if each of the obtained values does not result in undefined expressions in the given equation. If so, then these values would NOT be considered solutions.

EXAMPLE 1. Find the value of x in the following equations.

(a) $\log_4(2x) = \log_4 10$

(d) $\log_2(x + 1) + \log_2(x - 1) = 3$

(b) $\log_3(2x - 1) = 2$

(e) $\log x^2 = 2$

(c) $\log_x 16 = 2$

(f) $(\log x)^2 + 2 \log x - 3 = 0$

Solution.

(a) $\log_4(2x) = \log_4 10$

$$2x = 10 \quad (\text{one-to-one property})$$

$$x = 5$$

Check: 5 is a solution since $\log_4(2 \cdot 5) = \log_4(10)$.

$$\begin{aligned} \text{(b)} \quad & \log_3(2x - 1) = 2 \\ & 2x - 1 = 3^2 \quad (\text{changing into exponential form}) \\ & 2x - 1 = 9 \\ & 2x = 10 \\ & x = 5 \end{aligned}$$

Check: 5 is a solution since $\log_3(2 \cdot (5)1) = \log_3(9) = 2$.

$$\begin{aligned} \text{(c)} \quad & \log_x 16 = 2 \\ & x^2 = 16 \quad (\text{changing into exponential form}) \\ & x^2 - 16 = 0 \\ & (x + 4)(x - 4) = 0 \quad (\text{factorization using } a^2 - b^2 = (a + b)(a - b)) \\ & x = -4, 4 \end{aligned}$$

Check: 4 is a solution since $\log_4(16) = 2$. However, -4 is not a solution since $\log_{-4}(16)$ is not defined (the base cannot be negative).

$$\begin{aligned} \text{(d)} \quad & \log_2(x + 1) + \log_2(x - 1) = 3 \\ & \log_2[(x + 1)(x - 1)] = 3 \quad (\text{using the property } \log_b u + \log_b v = \log_b(uv)) \\ & (x + 1)(x - 1) = 2^3 \quad (\text{Note: Zero Factor Property cannot be used yet}) \\ & x^2 - 1 = 8 \quad (\text{multiplication of two binomials}) \\ & x^2 - 9 = 0 \\ & (x + 3)(x - 3) = 0 \quad (\text{factorization using } a^2 - b^2 = (a + b)(a - b)) \\ & x = -3, 3 \quad (\text{Zero Factor Property}) \end{aligned}$$

Check: 3 is a solution since $\log_2(3 + 1)$ and $\log_2(31)$ are defined. However, -3 is not a solution since $\log_2(3 + 1) = \log_2(2)$ is not defined.

Solution: $x = 3$ only

$$\text{(e)} \quad \log x^2 = 2$$

Method A: changing into exponential form

$$\begin{aligned} \log x^2 &= 2 \\ x^2 &= 10^2 \\ x^2 &= 100 \end{aligned}$$

$$x^2 - 100 = 0$$

$$(x + 10)(x - 10) = 0$$

$$x = -10, 10$$

Check: Both are solutions since $\log(10)^2 = 2$ and $\log(10)^2 = 2$.

Method B: using the one-to-one property

$$\log x^2 = 2$$

$$\log x^2 = \log 10^2 \quad (2 = 2(1) = 2(\log 10) = \log 10^2)$$

$$x^2 = 10^2 \quad (\text{One-to-one property})$$

$$x^2 - 100 = 0$$

$$(x + 10)(x - 10) = 0$$

$$x = -10, 10$$

Check: Both are solutions since $\log(10)^2 = 2$ and $\log(10)^2 = 2$.

Incorrect Method: using the property $\log_b u^n = n \log_b u$ immediately

$$\log x^2 = 2$$

$$2 \log x = 2$$

$$\log x = 1$$

$$x = 10 \quad (\text{we lost one solution})$$

Remark that in this case, that $\log x^2 = 2 \log x$ only if $x > 0$.

(f) $(\log x)^2 + 2 \log x - 3 = 0$

Note that the equation is quadratic in form.

One technique is to let $\log x = A$ for easier analysis.

$$A^2 + 2A - 3 = 0$$

$$(A + 3)(A - 1) = 0$$

$$A = -3 \text{ or } A = 1 \quad (\text{Note that these are not yet the final solutions.})$$

$$\log x = -3 \text{ or } \log x = 1$$

$$x = 10^{-3} = \frac{1}{1000} \text{ or } x = 10$$

Check: Both are solutions since $\log\left(\frac{1}{1000}\right)$ and $\log 10$ are defined.

□

EXAMPLE 2. Use logarithms to solve for the value of x in the exponential equation $2^x = 3$.

Solution. $2^x = 3$

$$\begin{aligned} \log 2^x &= \log 3 && \text{(applying the one-to-one property)} \\ x \log 2 &= \log 3 && \text{(applying } \log_b u^n = n \log_b u \text{ since 2 is positive)} \\ x &= \frac{\log 3}{\log 2} \approx 1.58496 \end{aligned}$$

□

(b) Solving logarithmic inequalities

Introduce the topic by asking students to complete the tables below:

x	$\log_{\frac{1}{2}} x$
$\frac{1}{8}$	
$\frac{1}{4}$	
$\frac{1}{2}$	
1	
2	
4	
8	

x	$\log_2 x$
$\frac{1}{8}$	
$\frac{1}{4}$	
$\frac{1}{2}$	
1	
2	
4	
8	

Answer:

x	$\log_{\frac{1}{2}} x$
$\frac{1}{8}$	3
$\frac{1}{4}$	2
$\frac{1}{2}$	1
1	0
2	-1
4	-2
8	-3

x	$\log_2 x$
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

Ask the following questions:

- The base for the first logarithmic expression is $\frac{1}{2}$, which is between 0 and 1. What do you notice with the value of $\log_{\frac{1}{2}} x$ as x increases? ($\log_{\frac{1}{2}} x$ decreases.)
- The base for the second logarithmic expression is 2, which is greater than 1. What do you notice with the value of $\log_2 x$ as x increases? ($\log_2 x$ increases as well.)

We can generalize the observations we made:

Property of Logarithmic Inequalities

Given the logarithmic expression $\log_b x$,

If $0 < b < 1$, then $x_1 < x_2$ if and only if $\log_b x_1 > \log_b x_2$.

If $b > 1$, then $x_1 < x_2$ if and only if $\log_b x_1 < \log_b x_2$.

Teaching Tip

There are two things to remember when solving logarithmic inequalities.

- The direction of the inequality ($<$ or $>$) is based on whether the base b is greater than 1 or less than 1.
- Check also that the resulting x values do not make any of the logarithms undefined.

EXAMPLE 3. Solve the following logarithmic inequalities.

- (a) $\log_3(2x - 1) > \log_3(x + 2)$
 (b) $\log_{0.2} > -3$
 (c) $-2 < \log x < 2$

Solution.

- (a) $\log_3(2x - 1) > \log_3(x + 2)$

Step 1: Ensure that the logarithms are defined.

Then $2x - 1 > 0$ and $x + 2 > 0$ must be satisfied.

$2x - 1 > 0$ implies $x > \frac{1}{2}$ and $x + 2 > 0$ implies $x > -2$

To make both logarithms defined, then $x > \frac{1}{2}$ (If $x > \frac{1}{2}$, then x is surely greater than -2)

Step 2: Ensure that the inequality is satisfied.

The base 3 is greater than 1.

Thus, since $\log_3(2x - 1) > \log_3(x + 2)$, then:

$$2x - 1 > x + 2$$

$$x > 3 \quad (\text{Subtract } x \text{ from both sides; add 1 to both sides})$$

$$\therefore x > 3$$

Hence, the solution is $(3, +\infty)$.

- (b) $\log_{0.2} > -3$

Step 1: Ensure that the logarithms are defined.

This means that $x > 0$.

Step 2: Ensure that the inequality is satisfied.

We first rewrite -3 as a logarithm to base $\frac{1}{5}$: $-3 = \log_{\frac{1}{5}} \left(\frac{1}{5}\right)^{-3}$.

We obtain the inequality

$$\log_{\frac{1}{5}} x > \log_{\frac{1}{5}} \left(\frac{1}{5}\right)^{-3}$$

The base is $0.2 = \frac{1}{5}$, which is less than 1.

Thus, since $\log_{\frac{1}{5}} x > \log_{\frac{1}{5}} \left(\frac{1}{5}\right)^{-3}$, then $x < \left(\frac{1}{5}\right)^{-3} = 125$.

Also, x should be positive (from Step 1). Thus, $0 < x < 125$.

Hence, the solution is $(0, 125)$.

(c) $-2 < \log x < 2$

Step 1: Ensure that the logarithms are defined.

This means that $x > 0$.

Step 2: Ensure that the inequality is satisfied.

We first rewrite -2 and 2 as logarithms to base 10, which are $\log 10^{-2}$ and $\log 10^2$ respectively, obtaining the inequality:

$$\log 10^{-2} < \log x < \log 10^2$$

We split the compound inequality into two simple inequalities:

$$\log 10^{-2} < \log x \text{ and } \log x < \log 10^2$$

Since the base 10 is greater than 1, we simplify both inequalities as

$$10^{-2} < x \text{ and } x < 10^2$$

Thus obtaining $\frac{1}{100} < x < 100$, which automatically satisfies the condition in Step 1. Hence, the solution is $\left(\frac{1}{100}, 100\right)$.

□

EXAMPLE 4. The 2013 earthquake in Bohol and Cebu had a magnitude of 7.2, while the 2012 earthquake that occurred in Negros Oriental recorded a 6.7 magnitude. How much more energy was released by the 2013 Bohol/Cebu earthquake compared to that by the Negros Oriental earthquake? (Refer to Lesson 17 for a discussion of the Richter scale).

Solution. Let E_B and E_N be the energy released by the Bohol/Cebu and Negros Oriental earthquakes, respectively. We will determine $\frac{E_B}{E_N}$.

Based on the given magnitudes, $7.2 = \frac{2}{3} \log \frac{E_B}{10^{4.4}}$ and $6.7 = \frac{2}{3} \log \frac{E_N}{10^{4.4}}$.

$$\begin{aligned} \text{Solving for } E_B: 7.2 \left(\frac{3}{2} \right) &= \log \frac{E_B}{10^{4.4}} \\ 10.8 &= \log \frac{E_B}{10^{4.4}} \\ 10^{10.8} &= \frac{E_B}{10^{4.4}} \\ E_B &= 10^{10.8} \cdot 10^{4.4} = 10^{15.2} \end{aligned}$$

$$\begin{aligned} \text{Solving for } E_N: 6.7 \left(\frac{3}{2} \right) &= \log \frac{E_N}{10^{4.4}} \\ 10.05 &= \log \frac{E_N}{10^{4.4}} \\ 10^{10.05} &= \frac{E_N}{10^{4.4}} \\ E_N &= 10^{10.05} \cdot 10^{4.4} = 10^{14.45} \\ \text{Thus, } \frac{E_B}{E_N} &= \frac{10^{15.2}}{10^{14.45}} = 10^{0.75} \approx 5.62 \end{aligned}$$

The Bohol/Cebu earthquake released 5.62 times more energy than the Negros Oriental earthquake. \square

EXAMPLE 5. How much more severe is an earthquake with a magnitude of n on a Richter scale, compared to one with a magnitude of $n + 1$?

Solution. Let E_1 and E_2 be the energy released by the earthquakes with magnitude n and $n + 1$, respectively. We will determine $\frac{E_2}{E_1}$.

$$\text{Based on the given magnitudes, } n = \frac{2}{3} \log \frac{E_1}{10^{4.4}} \text{ and } n + 1 = \frac{2}{3} \log \frac{E_2}{10^{4.4}}.$$

$$\begin{aligned} \text{Solving for } E_1: \frac{3}{2}n &= \log \frac{E_1}{10^{4.4}} \\ 10^{3n/2} &= \frac{E_1}{10^{4.4}} \\ E_1 &= 10^{3n/2} \cdot 10^{4.4} = 10^{\frac{3n}{2} + 4.4} \end{aligned}$$

$$\begin{aligned} \text{Solving for } E_2: \frac{3}{2}(n + 1) &= \log \frac{E_2}{10^{4.4}} \\ 10^{3(n+1)/2} &= \frac{E_2}{10^{4.4}} \\ E_2 &= 10^{3(n+1)/2} \cdot 10^{4.4} = 10^{\frac{3(n+1)}{2} + 4.4} \end{aligned}$$

$$\text{Thus, } \frac{E_2}{E_1} = \frac{10^{\frac{3(n+1)}{2} + 4.4}}{10^{\frac{3n}{2} + 4.4}} = 10^{\frac{3}{2}} \approx 31.6$$

These computations indicate that each 1 unit increase in magnitude represents 31.6 times more energy released. (This result may seem to contradict other sources which state that each 1 unit increase in magnitude represents an earthquake that is 10 times stronger. However, those computations use *amplitude* as a measure of strength. The computations above are based on the

energy released by the earthquake). □

EXAMPLE 6. Interest compounded annually

Using the formula $A = P(1 + r)^n$ (Lesson 12, Example 5) where A is the future value of the investment, P is the principal, r is the fixed annual interest rate, and n is the number of years, how many years will it take an investment to double if the interest rate per annum is 2.5%?

Solution. Doubling the principal P , we get $A = 2P$, $r = 2.5\% = 0.025$,

$$A = P(1 + r)^n$$

$$2P = P(1 + 0.025)^n$$

$$2 = (1.025)^n$$

$$\log 2 = \log(1.025)^n$$

$$\log 2 = n \log(1.025)$$

$$n = \frac{\log 2}{\log 1.025} \approx 28.07 \text{ years}$$

Answer: It will take approximately 28 years for the investment to double. □

 **Teaching Tip**

Instead of taking the log of both sides in the example above, you can also take the \ln of both sides (or any other logarithm, as long as you use the same base). You can verify that $\frac{\ln 2}{\ln 1.025}$ also gives the correct answer.

EXAMPLE 7. (Population growth) The population of the Philippines can be modeled by the function $P(x) = 20,000,000 \cdot e^{0.0251x}$, where x is the number of years since 1955 (e.g. $x = 0$ at 1955). Assuming that this model is accurate, in what year will the population reach 200 million?

Solution. Given $P(x) = 200,000,000$

$$200,000,000 = 20,000,000 \cdot e^{0.0251x}$$

$$10 = e^{0.0251x}$$

$$\ln 10 = \ln e^{0.0251x}$$

$$\ln 10 = 0.0251x(\ln e)$$

$$\ln 10 = 0.0251x$$

$$x = \frac{\ln 10}{0.0251} \approx 91 \text{ years}$$

$$1955 + 91 = 2046$$

Answer: Around the year 2046, the Philippine population will reach 200 million.

Trivia: Based on this model, we will reach 100 million in the year 2019. But last July 2014, the Philippines officially welcomed its 100 millionth baby.⁶ Hence mathematical models must always be reviewed and verified against new data. \square

EXAMPLE 8. In a bacteria culture, an initial population of 5,000 bacteria grows to 12,000 after 90 minutes. Assume that the growth of bacteria follows an exponential model $f(t) = Ae^{kt}$ representing the number of bacteria after t minutes. **(a)** Find A and k , and **(b)** use the model to determine the number of bacteria after 3 hours.

Solution. (a) It is given that $f(0) = 5,000$ and $f(90) = 12,000$.

Thus, $f(0) = Ae^{k(0)} = A = 5,000$.

Also, $f(90) = 5,000e^{k(90)} = 12,000 \Rightarrow e^{90k} = \frac{12}{5}$.

Take the \ln of both sides to obtain $\ln e^{90k} = \ln \frac{12}{5} \Rightarrow 90k = \ln \frac{12}{5} \Rightarrow k \approx 0.00973$.

The exponential model is $f(t) = 5,000 \cdot e^{0.00973t}$.

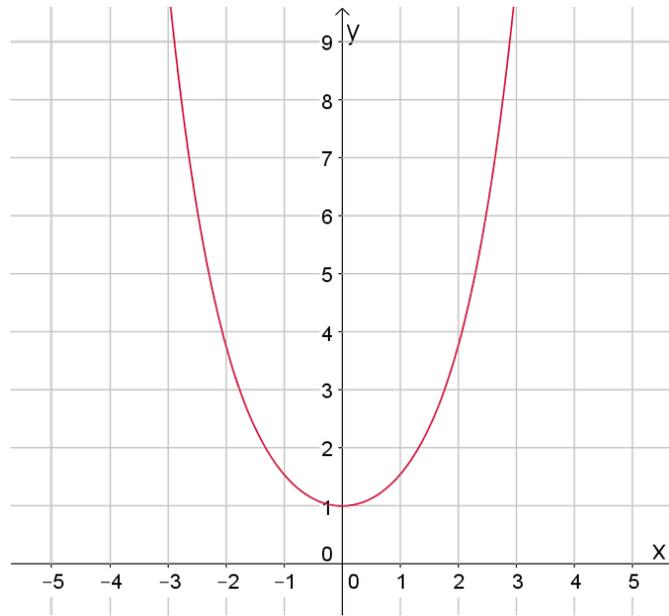
(b) 3 hours = 180 minutes; $f(180) = 5,000 \cdot e^{0.00973(180)} \approx 28,813$ bacteria

\square

EXAMPLE 9. Chains or cables suspended between two points and acted upon by a gravitational force follow the shape of a catenary.⁷ The equation $y = \frac{e^x + e^{-x}}{2}$ is an example of such a curve (see figure). Assuming this is the curve assumed by a suspended cable, how far apart are the cables when $y = 4$? Approximate your answer to two decimal places.

⁶<http://newsinfo.inquirer.net/623749/philippines-welcomes-100-millionth-baby>

⁷Weisstein, Eric W. "Catenary." From MathWorld—A Wolfram Web Resource. <http://mathworld.wolfram.com/Catenary.html>



Solution. We have to solve the equation $4 = \frac{e^x + e^{-x}}{2}$ or $8 = e^x + e^{-x}$.

$$8 = e^x + e^{-x}$$

$$8 = e^x + \frac{1}{e^x}$$

$$8e^x = e^{2x} + 1$$

$$e^{2x} - 8e^x + 1 = 0$$

Let $u = e^x$. Then $u^2 = e^{2x}$, and we obtain $u^2 - 8u + 1 = 0$.

The solutions to this quadratic equation are $u = 4 \pm \sqrt{15}$.

$$\text{Since } u = e^x \text{ then } 4 + \sqrt{15} = e^x \quad \text{or} \quad 4 - \sqrt{15} = e^x$$

$$\ln(4 + \sqrt{15}) = x \quad \text{or} \quad \ln(4 - \sqrt{15}) = x$$

To determine the distance between the cables at $y = 4$, we find the difference between the two obtained x -values. The distance is $\ln(4 + \sqrt{15}) - \ln(4 - \sqrt{15}) \approx 4.13$ \square

(C) Seatwork

Seatwork 1. Solve the following logarithmic equations.

(a) $\log_5(x - 1) + \log_5(x + 3) - 1 = 0$

Answer: 2

(b) $\log_3 x + \log_3(x + 2) = 1$

Answer: 1

Seatwork 2. Use logarithms to solve the exponential equation $3^{x+1} = 10$.

Answer: $\frac{1 - \log 3}{\log 3} \approx 1.0959$

Seatwork 3. Solve the following logarithmic inequalities.

(a) $\ln x > 1$ (Note: $e \approx 2.7183$)

Answer: $(e, +\infty)$

(b) $\log_{0.5}(4x + 1) < \log_{0.5}(1 - 4x)$

Answer: $(0, 1/4)$ **(D) Individual Activity**

Ask the students to carefully "dissect" the expression and solve the equation:

$\log_2 [\log_3 (\log_4 x)] = 0$

Answer: 64**LESSON 22: The Logarithmic Function****TIME FRAME:** 120 minutes**CODE:** M11GM-li2, i-3, i-4, and j-1

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to represent a logarithmic function through its equation, table of values, and graph, find the domain, range, intercepts, zeroes, and asymptotes of logarithmic functions, and graph logarithmic functions.

LESSON OUTLINE:

1. Review that the logarithmic function is the inverse of an exponential function.
2. Represent the logarithmic function through its equation, table of values, and graph.
3. Analyze the basic logarithmic function by identifying its domain, range, intercepts, zeroes, and asymptotes.
4. Sketch the graph of logarithmic functions.

DEVELOPMENT OF THE LESSON

(A) Introduction As was discussed in the preceding lesson, logarithms can be rewritten in exponential

form. Recalling the exponential function $f(x) = b^x$ as taught in the previous lessons, we shall attempt to show that its inverse is a logarithmic function.

First, recall that graphically, using the horizontal line test, it can be determined that $f(x) = b^x$ is a one-to-one function. Hence its inverse is also a function.

To find the inverse, let $y = f(x) = b^x$.

$$y = b^x$$

$$x = b^y \quad (\text{interchange } x \text{ and } y)$$

$$y = \log_b x \quad (\text{rewrite the logarithmic form})$$

$$f^{-1}(x) = \log_b x \quad (\text{the logarithmic function})$$

(B) Lesson Proper

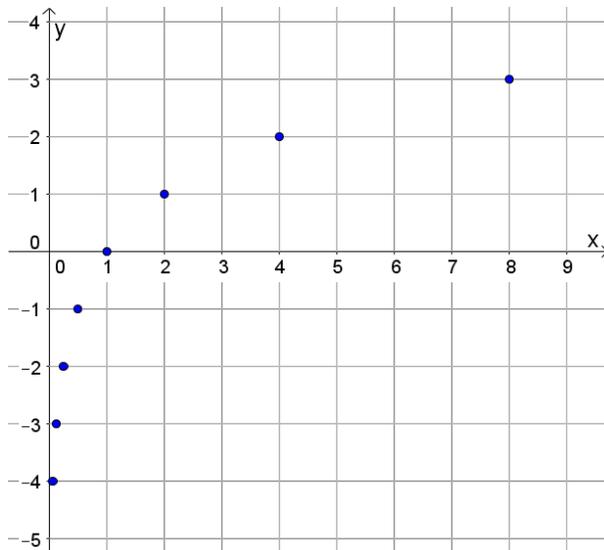
- (a) Representing the logarithmic function (base greater than 1) through its equation, table of values and graph.

EXAMPLE 1. Sketch the graph of $y = \log_2 x$

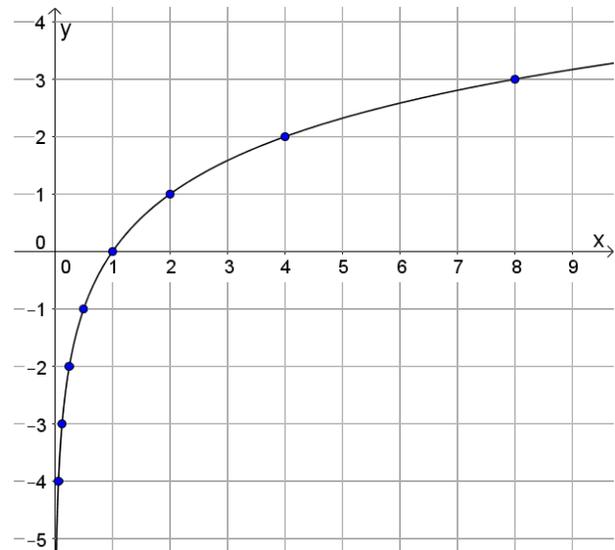
Solution. Step 1: Construct a table of values of ordered pairs for the given function. A table of values for $y = \log_2 x$ is as follows:

x	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
y	-4	-3	-2	-1	0	1	2	3

Step 2: Plot the points found in the table, and connect them using a smooth curve.



(a) Plotting of points for $y = \log_2 x$

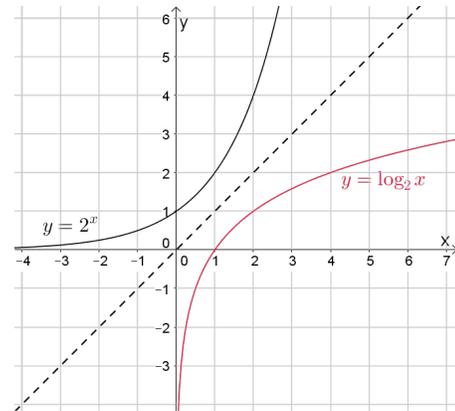


(b) Graph of $y = \log_2 x$

It can be observed that the function is defined only for $x > 0$. The function is strictly increasing, and attains all real values. As x approaches 0 from the right, the function decreases without bound, i.e., the line $x = 0$ is a vertical asymptote. \square

Activity: Comparing the graphs of $y = 2^x$ and $y = \log_2 x$

Ask students to quickly sketch both of these graphs. They should notice that one graph can be obtained from the other by "flipping" the graph about the line $y = x$. This should be expected, since the two functions are **inverses**.



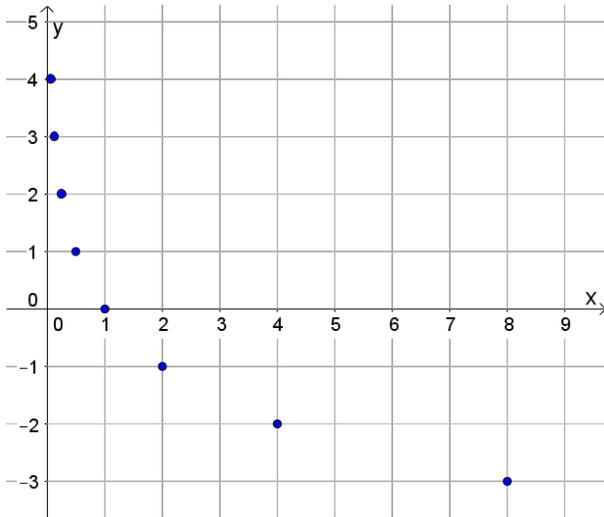
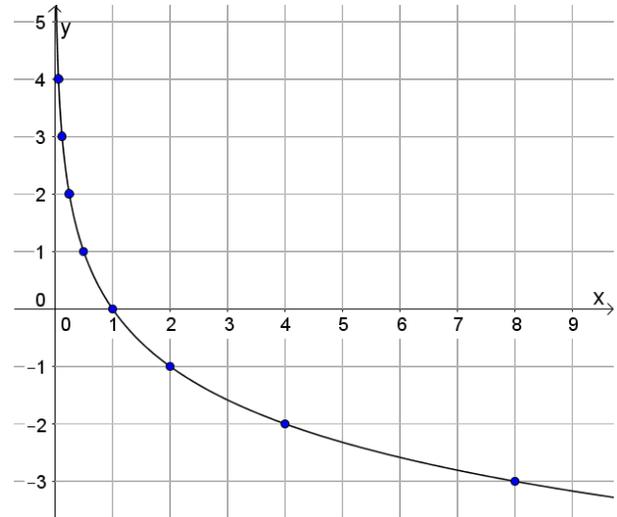
- (b) Representing the logarithmic function (base between 0 and 1) through its equation, table of values and graph.

EXAMPLE 2. Sketch the graph of $y = \log_{\frac{1}{2}} x$

Solution. Step 1: Construct a table of values of ordered pairs for the given function. A table of values for $y = \log_{\frac{1}{2}} x$ is as follows:

x	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
y	4	3	2	1	0	-1	-2	-3

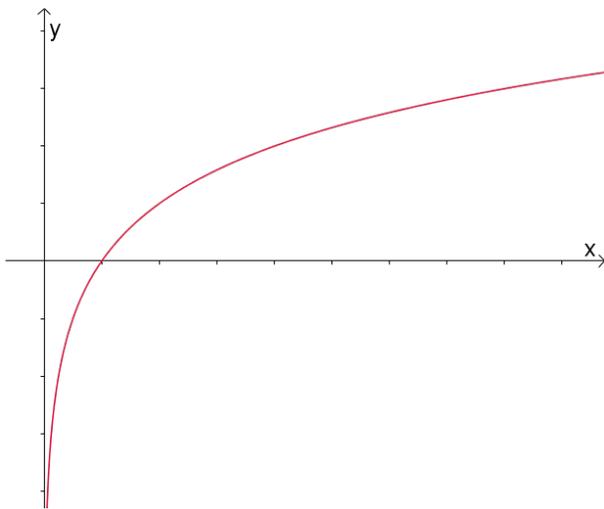
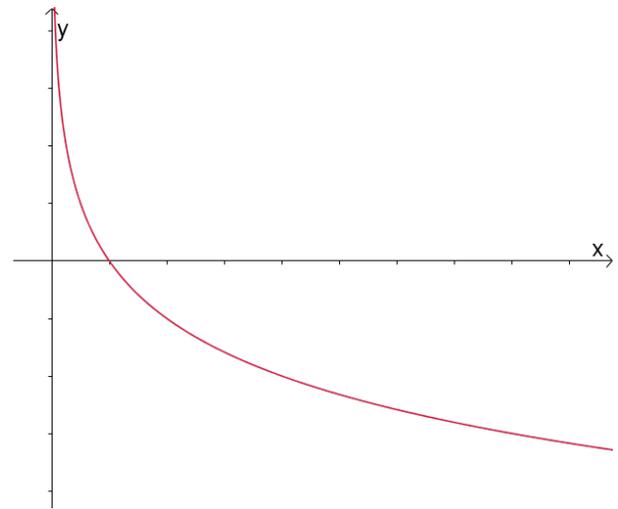
Step 2: Plot the points found in the table, and connect them using a smooth curve.

(a) Plotting of points for $y = \log_{\frac{1}{2}} x$ (b) Graph of $y = \log_{\frac{1}{2}} x$

It can be observed that the function is defined only for $x > 0$. The function is strictly decreasing, and attains all real values. As x approaches 0 from the right, the function increases without bound, i.e., the line $x = 0$ is a vertical asymptote. \square

(c) Analyzing the logarithmic function

The graphs of $y = \log_2 x$ and $y = \log_{\frac{1}{2}} x$ indicate that the graphs of the logarithmic function $y = \log_b x$ depends on the value of b . We generalize the results as follows.

(a) $y = \log_b x (b > 1)$ (b) $y = \log_b x (0 < b < 1)$

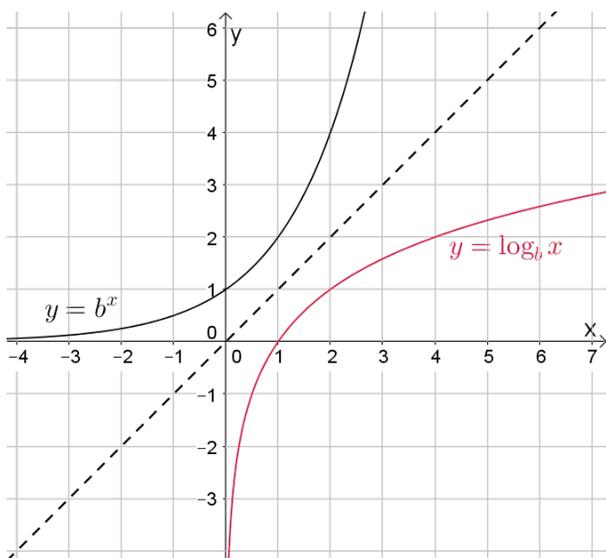
Using the graph of $y = \log_b x$ ($b > 1$ or $0 < b < 1$) as visual cue, you may elicit the following properties from your students:

Properties of Logarithmic Functions

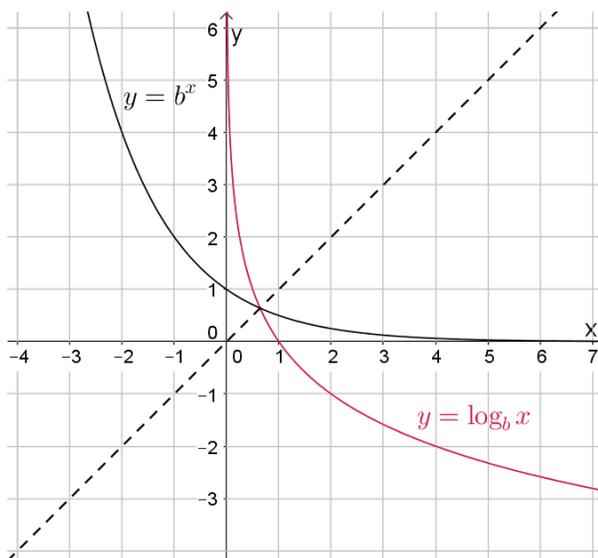
- i. The domain is the set of all positive numbers, or $\{x \in \mathbb{R} | x > 0\}$. Recall that these are precisely the permitted values of x in the expression $\log_b x$.
- ii. The range is the set of all positive real numbers.
- iii. It is a one-to-one function. It satisfies the Horizontal Line Test.
- iv. The x -intercept is 1. There is no y -intercept.
- v. The vertical asymptote is the line $x = 0$ (or the y -axis). There is no horizontal asymptote.

Relationship Between the Graphs of Logarithmic and Exponential Functions

Since logarithmic and exponential functions are inverses of each other, their graphs are reflections of each other about the line $y = x$, as shown below.



(a) $y = b^x$ and $y = \log_b x$ ($b > 1$)



(b) $y = b^x$ and $y = \log_b x$ ($0 < b < 1$)

Activity (Optional) Ask students to sketch the graph of $y = -\log_2 x$, and compare with that of the graph $y = \log_{\frac{1}{2}} x$. (They are the same. Use the laws and properties of logarithms to explain why.)

(d) Graphing logarithmic functions

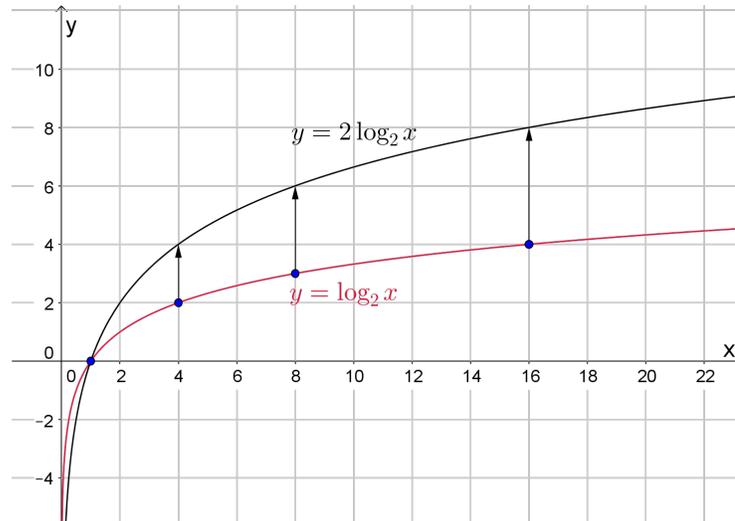
The following examples are given to illustrate graphs of transformations of logarithmic functions.

EXAMPLE 3. Sketch the graphs of $y = 2 \log_2 x$. Determine the domain, range, vertical asymptote, x -intercept, and zero.

Solution. The graph of $y = 2 \log_2 x$ can be obtained from the graph of $y = \log_2 x$ by multiplying each y -coordinate by 2, as the following table of signs shows.

x	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$\log_2 x$	-4	-3	-2	-1	0	1	2	3
$y = 2 \log_2 x$	-8	-6	-4	-2	0	2	4	6

The graph is shown below.



Analysis:

- (a) Domain: $\{x|x \in \mathbb{R}, x > 0\}$
- (b) Range : $\{y|y \in \mathbb{R}\}$
- (c) Vertical Asymptote: $x = 0$
- (d) x -intercept: 1
- (e) Zero: 1

□

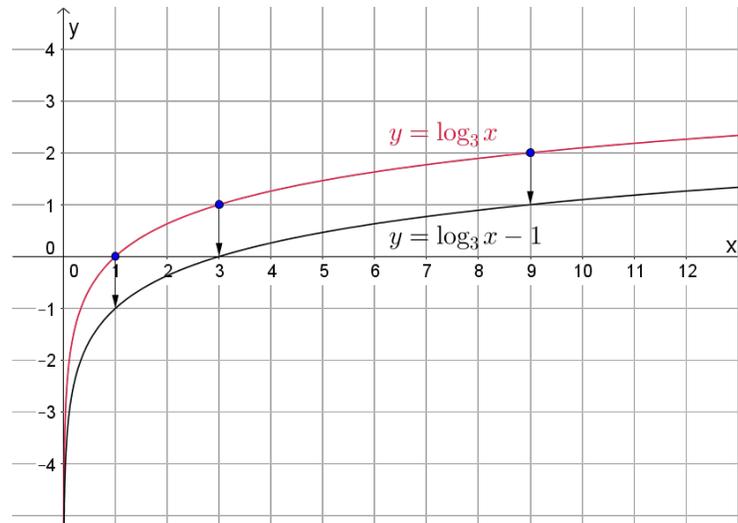
EXAMPLE 4. Sketch the graph of $y = \log_3 x - 1$.

Solution. Sketch the graph of the basic function $y = \log_3 x$. Note that the base $3 > 1$. The "-1" means vertical shift downwards by 1 unit.

Some points on the graph of $y = \log_3 x$ are $(1, 0)$, $(3, 1)$, and $(9, 2)$.

Shift these points 1 unit down to obtain $(1, -1)$, $(3, 0)$, and $(9, 1)$. **Plot these points.**

The graph is shown below.



Analysis:

(a) Domain: $\{x|x \in \mathbb{R}, x > 0\}$

(b) Range : $\{y|y \in \mathbb{R}\}$

(c) Vertical Asymptote: $x = 0$

(d) x -intercept: 3

The x -intercept can be obtained graphically. Likewise, we can solve for the x -intercept algebraically by setting $y = 0$:

$$0 = \log_3 x - 1$$

$$\log_3 x = 1$$

$$x = 3^1 = 3$$

(e) Zero: 3

□

EXAMPLE 5. Sketch the graph of $y = \log_{0.25}(x + 2)$.

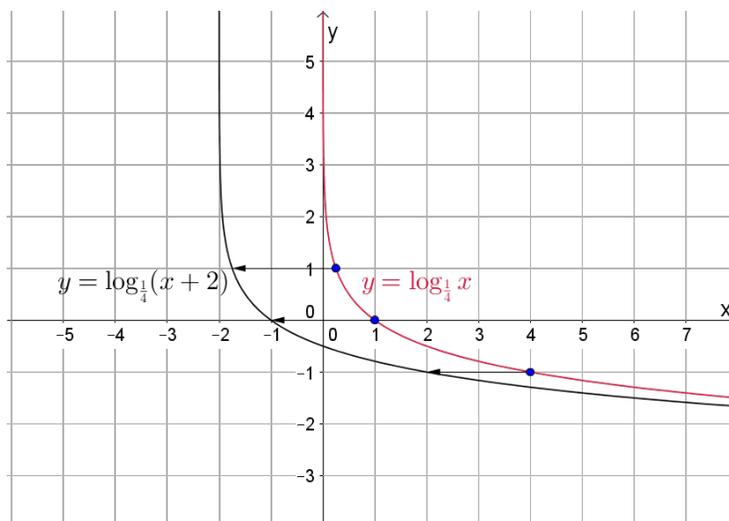
Solution. Sketch the graph of the basic function $y = \log_{0.25} x$. Note that the base $0 < 0.25 < 1$. Rewrite the equation, obtaining $y = \log_{0.25}[x - (-2)]$.

The " -2 " means a horizontal shift of 2 units to the left.

Some points on the graph of $y = \log_{0.25} x$ are $(1, 0)$, $(4, -1)$, and $(0.25, 1)$.

Shift these points 2 units to the left to obtain $(-1, 0)$, $(2, -1)$, and $(-1.75, 1)$. **Plot these points.**

Graph:



Analysis:

(a) Domain: $\{x|x \in \mathbb{R}, x > -2\}$

(The expression $x + 2$ should be greater than 0 for $\log_{0.25}(x + 2)$ to be defined. Hence, x must be greater than -2 .)

(b) Range : $\{y|y \in \mathbb{R}\}$

(c) Vertical Asymptote: $x = -2$

(d) x -intercept: -1

(e) Zero: 1

□

The examples above can be generalized to form the following guidelines for graphing transformations of logarithmic functions:

Graph of $f(x) = a \cdot \log_b(x - c) + d$

- The value of b (either $b > 1$ or $0 < b < 1$) determines whether the graph is increasing or decreasing.
- The value of a determines the stretch or shrinking of the graph. Further, if a is negative, there is a reflection of the graph about the x -axis.
- Based on $f(x) = a \cdot \log_b x$, the vertical shift is d units up (if $d > 0$) or d units down (if $d < 0$), and the horizontal shift is c units to the right (if $c > 0$) or c units to the left (if $c < 0$).

(C) Seatwork

Seatwork 1. For each of the following functions, (a) use transformations to describe how the graph is

related to an logarithmic function $y = \log_b x$, **(b)** sketch the graph, and **(c)** identify the y -intercept, vertical asymptote, domain, and range.

(a) $y = \log_x(x + 3)$

(d) $y = (\log_{0.1} x) - 2$

(b) $y = \log_{\frac{1}{3}}(x - 1)$

(e) $y = \log_{\frac{2}{5}}(x - 4) + 2$

(c) $y = (\log_5 x) + 6$

(f) $y = \log_6(x + 1) + 5$

Performance Task

Provide the common logarithms of some numbers, such as those below. (Or, if you have reference materials, provide students table of logarithms)

$\log 2 \approx 0.3010$	$\log 3 \approx 0.4771$	$\log 5 \approx 0.6990$	$\log 7 \approx 0.8451$
-------------------------	-------------------------	-------------------------	-------------------------

Before calculators were invented, people used a table of logarithms (part of which may be similar to the one shown above) to compute certain numbers, such as $\frac{2^{1/3}}{5^{1/4}}$.

Let the students brainstorm and decide how exponents and logarithms can be used to approximate the value of $\frac{2^{1/3}}{5^{1/4}}$. (Give other numbers, if necessary).

Solution. Let $n = \frac{2^{1/3}}{5^{1/4}}$. Then, using laws of logarithms, $\log n = \frac{1}{3} \log 2 - \frac{1}{4} \log 5$. Use the table above to approximate $\log n$:

$$\log n \approx \frac{1}{3}(0.3010) - \frac{1}{4}(0.6990) \approx -0.0744$$

However, -0.0744 is just the log of n —it is not yet the value of n . That is,

$$\log n \approx -0.0744.$$

Therefore, $n \approx 10^{-0.0744}$. The value of this number can be found using a logarithm table. \square

Chapter 6

Simple and Compound Interest

LESSON 23: Illustrating Simple and Compound Interest

TIME FRAME: 60 minutes

CODE: M11GM-IIa-1 and a-2

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to illustrate simple and compound interest.

LESSON OUTLINE:

1. Simple Interest
 2. Compound Interest
-

DEVELOPMENT OF THE LESSON

(A) Introduction

(a) **Discussion**

Ask students to give their own answers to the question "What are some ways to take care of hard-earned money?" Possible answers: Buy only what you need, try to save some money, invest some money, think of the future.

(b) **Classroom Game (Optional)**

Group the students. For each group, write down a starting amount of cash (e.g., ₱100,000). Prepare some cards that give certain options (e.g, invest in a bank that offers 3% interest, buy clothes for ₱200, and so forth) and even some possible real-life concerns (e.g., pay ₱100 for water,

pay ₱5,000 for medical bills). After each group selects a card, compute the amount of money of that group. End after 4 rounds. The point is to set the tone for thinking about how to prepare for the future.

Try to save money!

Explain that depositing money in a bank is like lending money to the bank in return for which the bank pays interest. By contrast, borrowing money from banks or lending institutions requires payment of interest. Hence, money has present and future values. You may cite successful personalities who have applied good investment mathematics like Bo Sanchez, Warren Buffet, etc.

(B) Lesson Proper

(a) Definition of Terms

- **Lender or creditor** - person (or institution) who invests the money or makes the funds available
- **Borrower or debtor** - person (or institution) who owes the money or avails of the funds from the lender
- **Origin or loan date** - date on which money is received by the borrower
- **Repayment date or maturity date** - date on which the money borrowed or loan is to be completely repaid
- **Time or term (t)** - amount of time in years the money is borrowed or invested; length of time between the origin and maturity dates
- **Principal (P)** - amount of money borrowed or invested on the origin date
- **Rate (r)** - annual rate, usually in percent, charged by the lender, or rate of increase of the investment
- **Interest (I)** - amount paid or earned for the use of money
- **Simple Interest (I_s)** - interest that is computed on the principal and then added to it
- **Compound Interest (I_c)** - interest is computed on the principal and also on the accumulated past interests
- **Maturity value or future value (F)** - amount after t years; that the lender receives from the borrower on the maturity date

(b) Illustration of Simple and Compound Interest

EXAMPLE 1. Suppose you won ₱10,000 and you plan to invest it for 5 years. A cooperative group offers 2% simple interest rate per year. A bank offers 2% compounded annually. Which will you choose and why?

Solution. Investment 1: Simple interest, with annual rate r

Time (t)	Principal (P)	Simple Interest		Amount after t years (Maturity Value)
		Solution	Answer	
1	10,000	$(10,000)(0.02)(1)$	200	$10,000 + 200 = 10,200.00$
2		$(10,000)(0.02)(2)$	400	$10,000 + 400 = 10,200.00$
3		$(10,000)(0.02)(3)$	600	$10,000 + 600 = 10,200.00$
4		$(10,000)(0.02)(4)$	800	$10,000 + 800 = 10,200.00$
5		$(10,000)(0.02)(5)$	1 000	$10,000 + 1,000 = 10,200.00$

Investment 2: Compound Interest, with annual rate r

Time (t)	Amount at the start of year t	Compound Interest		Amount at the end of year t (Maturity Value)
		Solution	Answer	
1	10,000	$(10,000)(0.02)(1)$	200	$10,000 + 200 = 10,200.00$
2	10,200	$(10,200)(0.02)(1)$	204	$10,200 + 204 = 10,404.00$
3	10,404	$(10,404)(0.02)(1)$	208.08	$10,404 + 208.08 = 10,612.08$
4	10,612.08	$(10,612.08)(0.02)(1)$	212.24	$10,612.08 + 212.24 = 10,824.32$
5	10,824.32	$(10,824.32)(0.02)(1)$	216.49	$10,824.32 + 216.49 = 11,040.81$

□

- (c) Let the students compare the interests gained in the two investments.

Simple Interest (in pesos): $11,000 - 10,000 = 1,000$

Compound Interest (in pesos): $11,040.81 - 10,000 = 1,040.81$

Ask the students to distinguish between simple and compound interests based on their illustrations.

Possible answers: Simple interest remains constant throughout the investment term. In compound interest, the interest from the previous year also earns interest. Thus, the interest grows every year.

(C) Homework

Let the students generalize the procedures in finding simple and compound interests. Allow them to express these generalizations using formulas.

LESSON 24: Simple Interest

TIME FRAME: 60 minutes

CODE: M11GM-IIa-b-1 and b-2

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to compute interest, maturity value, and present value in simple interest environment, and solve problems involving simple interest.

LESSON OUTLINE:

1. Finding simple interest
2. Finding the unknown principal, rate, or time
3. Finding maturity value
4. Solving real-life problems involving simple interest

PREREQUISITE SKILLS: Knowledge in Exponential Functions

DEVELOPMENT OF THE LESSON

(A) Introduction

Let the students recall from the previous lesson how to find the interest and maturity value of an amount earning simple interest. Allow students to present their generalizations or formulas.

(B) Lesson Proper

- (a) Discuss the procedure/formula in finding simple interest.

An annual simple interest is based on the 3 factors:

- a. Principal which is the amount invested or borrowed
- b. Simple interest rate, usually expressed in percent
- c. Time or term of loan, in years

Annual Simple Interest

$$I_s = Prt$$

where I_s = simple interest
 P = principal
 r = rate
 t = term or time, in years

EXAMPLE 1. A bank offers 0.25% annual simple interest rate for a particular deposit. How much interest will be earned if 1 million pesos is deposited in this savings account for 1 year?

Solution.Given: $P = 1,000,000$

$$r = 0.25\% = 0.0025$$

$$t = 1 \text{ year}$$

Find: I_s

$$I_s = Prt$$

$$I_s = (1,000,000)(0.0025)(1)$$

$$I_s = 2,500$$

Answer: The interest earned is ₱2,500. □**Teaching Tip**

You may also discuss to students that interest in savings account in the Philippines is subject to 20% withholding tax. If 20% withholding tax will be applied, then the actual interest earned is

$$(2,500)(0.8) = 2,000.$$

EXAMPLE 2. How much interest is charged when P50,000 is borrowed for 9 months at an annual simple interest rate of 10%?**Solution.**Given: $P = 50,000$

$$r = 10\% = 0.10$$

$$t = \frac{9}{12} \text{ year} = 0.75 \text{ year}$$

Find: I_s **Note:**

When the term is expressed in months (M), it should be converted to years by $t = \frac{M}{12}$.

$$I_s = Prt$$

$$I_s = (50,000)(0.10) \left(\frac{9}{12} \right)$$

$$I_s = (50,000)(0.10)(0.75)$$

$$I_s = 3,750$$

Answer: The simple interest charged is ₱3,750. □

EXAMPLE 3. Complete the table below by finding the unknown

Principal (P)	Rate (r)	Time (t)	Interest
(a)	2.5%	4	1,500
36,000	(b)	1.5	4,860
250,000	0.5%	(c)	275
500,000	12.5%	10	(d)

Solution.

(a) The unknown principal can be obtained by

$$P = \frac{I_s}{rt} = \frac{1,500}{(0.025)(4)}$$

$$P = 15,000$$

(b) The unknown rate can be computed by

$$r = \frac{I_s}{Pt} = \frac{4,860}{(36,000)(1.5)}$$

$$r = 0.09 = 9\%$$

(c) The unknown time can be calculated by

$$t = \frac{I_s}{Pr} = \frac{275}{(250,000)(0.005)}$$

$$t = 0.22 \text{ years}$$

(d) The unknown simple interest is given by

$$I_s = Prt = (500,000)(0.125)(10)$$

$$I_s = 625,000$$

□

EXAMPLE 4. When invested at an annual interest rate of 7%, an amount earned P11,200 of simple interest in two years. How much money was originally invested?

Solution.

Given: $r = 7\% = 0.07$

$t = 2$ years

$I_s = 11,200$

Find: P

$$P = \frac{I_s}{rt} = \frac{11,200}{(0.07)(2)}$$

$$P = 80,000$$

Answer: The amount invested is ₱80,000. □

EXAMPLE 5. If an entrepreneur applies for a loan amounting to ₱500,000 in a bank, the simple interest of which is ₱157,500 for 3 years, what interest rate is being charged?

Solution.

Given: $P = 500,000$

$I_s = 157,500$

$t = 3$ years

Find: r

$$r = \frac{I_s}{Pt} = \frac{157,500}{(500,000)(3)}$$

$$r = 0.105 = 10.5\%$$

Answer: The bank charged an annual simple interest rate of 10.5%. □

EXAMPLE 6. How long will a principal earn an interest equal to half of it at 5% simple interest?

Solution.

Given: P

$r = 5\% = 0.05$

$I_s = \frac{1}{2}P = 0.5P$

Find: t

$$t = \frac{I_s}{Pr} = \frac{0.5P}{(P)(0.05)}$$

$$t = 10 \text{ years}$$

Answer: It will take 10 years for a principal to earn half of its value at 5% simple annual interest rate. □

Many persons or institutions are interested to know the amount that a lender will give to the borrower on the maturity date. For instance, you may be interested to know the total amount of money in a savings account after t years at an interest rate r . This amount is called the maturity value or future value F .

Maturity (Future) Value

$$F = P + I_s$$

where F = maturity (future) value
 P = principal
 I_s = simple interest

Substituting I_s by Prt gives

$$F = P + Prt$$

$$F = P(1 + rt)$$

Maturity (Future) Value

$$F = P(1 + rt)$$

where F = maturity (future) value
 P = principal
 r = interest
 t = term / time in years

EXAMPLE 7. Find the maturity value if 1 million pesos is deposited in a bank at an annual simple interest rate of 0.25% after **(a)** 1 year and **(b)** 5 years?

Solution.

Given: $P = 1,000,000$, $r = 0.25\% = 0.0025$

Find: (a) maturity or future value F after 1 year

(b) maturity or future value F after 5 years


Note:

There are two ways to solve the problem.

Method 1: Solve the simple interest I_s first and then add it to P , that is, $F = P + I_s$.

Method 2: Use the derived formula $F = P(1 + rt)$.

a. When $t = 1$, the simple interest is given by

Method 1:

$$I_s = Prt$$

$$I_s = (1,000,000)(0.0025)(1)$$

$$I_s = 2,500$$

The maturity or future value is given by $F = P + I_s$

$$F = 1,000,000 + 2,500$$

$$F = 1,002,500$$

Method 2: To directly solve the future value F ,

$$F = P(1 + rt)$$

$$F = (1,000,000)(1 + 0.0025(1))$$

$$F = 1,002,500$$

Answer: The future or maturity value after 1 year is ₱1,002,500.

b. When $t = 5$,

Method 1:

$$I_s = Prt$$

$$I_s = (1,000,000)(0.0025)(5)$$

$$I_s = 12,500$$

$$F = P + I_s$$

$$F = 1,000,000 + 12,500$$

$$F = 1,012,500$$

Method 2:

$$F = P(1 + rt)$$

$$F = (1,000,000)(1 + 0.0025(5))$$

$$F = 1,012,500$$

Answer: The future or maturity value after 5 years is ₱1,012,500.

□

 **For your information.**

You may tell the students that savings accounts are covered by Philippine Deposit Insurance Corp (PDIC) up to ₱500,000. This means that the money in the savings account is insured up to ₱500,000.

(C) Seatwork

Seatwork 1. Find the unknown principal P , rate r , time t , and interest I by completing the table.

Principal (P)	Rate (r)	Time (t)	Interest (I)
10,000	8%	15	(1)
(2)	2%	5	10,000
360,000	(3)	2	3,600
500,000	10.5%	(4)	175,500
880,000	9.25%	2.5	(5)

Answer: (1) 12,000 (2) 100,000 (3) 0.5% (4) 3.34 (5) 203,500

Seatwork 2. Solve the following problems on simple interest.

(a) What are the amounts of interest and maturity value of a loan for ₱150,000 at $6\frac{1}{2}\%$ simple interest for 3 years?
Answer: $I = ₱29,250$ $F = ₱179,250$

(b) At what simple interest rate per annum will ₱25,000 accumulate to ₱33,000 in 5 years?
Answer: 6.4%

(c) How long will ₱40,000 amount to ₱51,200 if the simple interest rate is at 12% per annum?
Answer: 2.33 years, or 2 years and 4 months

(d) In order to have ₱200,000 in 3 years, how much should you invest if the simple interest is 5.5%?
Answer: 171,673.82

(e) Angel deposited ₱20,000 in a bank that pays 0.5% simple interest. How much will be her money after 6 years?
Answer: ₱20,600

(D) Evaluation

(a) Match the terms in column A with the correct definitions in Column B. You may choose more than one answer from Column B.

Column A	Column B
(1) Principal	A. time money is borrowed
(2) Term	B. amount paid or earned for the use of money
(3) Interest	C. percentage of increase of investment
(4) Maturity value	D. amount of money borrowed or invested
(5) Interest rate	E. amount added by the lender, to be received on repayment date
	F. amount received on repayment date

Answer: (1) D (2) A (3) B, E (4) F (5) C

(b) Complete the table by finding the unknown.

Principal (P)	Rate (r)	Time (t)	Interest (I)	Maturity Value (F)
60,000	4%	15	(1)	(2)
(3)	12%	5	15,000	(4)
50,000	(5)	2	(6)	59,500
(7)	10.5%	(8)	157,500	457,500
1,000,000	0.25%	6.5	(9)	(10)

Answer:

(1) $I = 36,000$ (2) $F = 96,000$ (3) $P = 25,000$ (4) $F = 40,000$ (5) $r = 9.5\%$
 (6) $I = 9,500$ (7) $P = 300,000$ (8) $t = 5$ (9) $I = 16,250$ (10) $F = 1,016,250$

(c) Solve the following problems.

- i. Angel invested a certain amount at 8% simple interest per year. After 6 years, the interest she received amounted to ₱48,000. How much did she invest? **Answer:** ₱100,000
- ii. Justin borrowed ₱5,000 at 5% annual simple interest rate. If he decided to pay after 1 year and 3 months, how much should he pay by then? **Answer:** ₱5,312.50
- iii. How long will an amount of money double at a simple interest rate of 2% per annum? **Answer:** 50 years
- iv. At what simple interest rate will an amount of money double itself in 10 years? **Answer:** 10%
- v. If a person borrowed ₱88,800 at an annual simple interest rate of $10\frac{1}{4}\%$ for 18 months, how much interest should he pay? **Answer:** ₱13,653

LESSON 25: Compound Interest

TIME FRAME: 90 minutes

CODE: M11GM-II-a-b-1 and b-2

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to compute interest, maturity value, and present value in compound interest environment, and solve problems involving compound interest.

LESSON OUTLINE:

1. Finding maturity value
2. Finding present value

PREREQUISITE SKILLS: Knowledge in Simple Interests, Exponential Functions

DEVELOPMENT OF THE LESSON

(A) Introduction

Pose the following situation to the students:

Ella and Thelma each invest ₱10,000 for two years, but under different schemes. Ella's earns 2% of ₱10,000 the first year, which is ₱200, then another ₱200 the second year. Thelma earns 2% of ₱10,000 the first year, which is ₱200, same as Ella's. But during the second year, she earns 2% of the ₱10,000 and 2% of the ₱200 also.

Ask the students the amount in Ella's and Thelma's respective accounts after two years. [₱10,400 and ₱10,404]. Ask them why there is a difference [**Ella just earns 2% of ₱10,000 but Thelma earns 2% of both the ₱10,000 and the previous interest**].

Although the amounts may be quite close, note that the situation only includes two years, with only a 2% interest rate. Mention that the next lesson will formalize the concepts in the given situation.

(B) Lesson Proper

Many bank savings accounts pay compound interest. In this case, the interest is added to the account at regular intervals, and the sum becomes the new basis for computing interest. Thus, the interest earned at a certain time interval is automatically reinvested to yield more interest.

The following table shows the amount at the end of each year if principal P is invested at an annual interest rate r compounded annually. Computations for the particular example $P = ₱100,000$ and $r = 5\%$ are also included.

Year (t)	Principal = P	Principal = ₱100,000
	Int. rate = r, compounded annually	Int. rate = 5%, compounded annually
	Amount at the end of the year	Amount at the end of the year
1	$P(1+r) = P(1+r)$	$100,000 \cdot 1.05 = 105,000$
2	$P(1+r)(1+r) = P(1+r)^2$	$105,000 \cdot 1.05 = 110,250$
3	$P(1+r)^2(1+r) = P(1+r)^3$	$110,250 \cdot 1.05 = 121,550.63$
4	$P(1+r)^3(1+r) = P(1+r)^4$	$121,550.63 \cdot 1.05 = 127,628.16$

Observe that **the amount at the end of each year is just the amount from the previous year multiplied by $(1+r)$** . In other words, $1+r$ is multiplied each time the year ends. This results in the following formula for the amount after t years, given an annual interest rate of r :

Maturity (Future) Value and Compound Interest

$$F = P(1+r)^t$$

where

P = principal or present value
 F = maturity (future) value at the end of the term
 r = interest rate
 t = term / time in years

The compound interest I_c is given by

$$I_c = F - P$$

EXAMPLE 1. Find the maturity value and the compound interest if ₱10,000 is compounded annually at an interest rate of 2% in 5 years.

Solution.

Given: $P = 10,000$

$r = 2\% = 0.02$

$t = 5$ years

Find: (a) maturity value F

(b) compound interest

I_c

Solution:

(a) $F = P(1+r)^t$

$$F = (10,000)(1+0.02)^5$$

$$F = 11,040.081$$

$$(b) I_c = F - P$$

$$I_c = 11,040.81 - 10,000$$

$$I_c = 1,040.81$$

Answer: The future value F is ₱11,040.81 and the compound interest is ₱1,040.81. □



Teaching Tip

Relate the procedure above to the illustration in finding compound interest under Investment 2 in Lesson 1.

EXAMPLE 2. Find the maturity value and interest if ₱50,000 is invested at 5% compounded annually for 8 years.

Solution.

Given: $P = 50,000$

$$r = 5\% = 0.05$$

$$t = 8 \text{ years}$$

Find: (a) maturity value F

(b) compound interest I_c

Solution:

$$(a) F = P(1 + r)^t$$

$$F = (50,000)(1 + 0.05)^8$$

$$F = 73,872.77$$

$$(b) I_c = F - P$$

$$I_c = 73,872.77 - 50,000$$

$$I_c = 23,872.77$$

Answer: The maturity value F is ₱73,872.77 and the compound interest is ₱23,872.77. □

EXAMPLE 3. Suppose your father deposited in your bank account ₱10,000 at an annual interest rate of 0.5% compounded yearly when you graduate from kindergarten and did not get the amount until you finish Grade 12. How much will you have in your bank account after 12 years?

Solution.Given: $P = 10,000$

$$r = 0.5\% = 0.005$$

$$t = 12 \text{ years}$$

Find: F The future value F is calculated by

$$F = P(1 + r)^t$$

$$F = (10,000)(1 + 0.005)^{12}$$

$$F = 10,616.78$$

Answer: The amount will become ₱10,616.77 after 12 years. □ **Teaching Tip**

This is a good time to discuss whether it is advisable to save all your money in an account that earns only 0.5% interest.

The present value or principal of the maturity value F due in t years any rate r can be obtained from the maturity value formula $F = P(1 + r)^t$.Solving for the present value P ,

$$P(1 + r)^t = F$$

$$\frac{P(1 + r)^t}{(1 + r)^t} = \frac{F}{(1 + r)^t}$$

$$P = \frac{F}{(1 + r)^t}$$

or equivalently, $P = F(1 + r)^{-t}$

(C) Seatwork

Present Value P at Compound Interest

$$P = \frac{F}{(1 + r)^t} = F(1 + r)^{-t}$$

where

 P = principal or present value F = maturity (future) value at the end of the term r = interest rate t = term / time in years

EXAMPLE 4. What is the present value of ₱50,000 due in 7 years if money is worth 10% compounded annually?

Solution.

Given: $F = 50,000$

$r = 10\% = 0.1$

$t = 7$ years

Find: P

The present value P can be obtained by

$$P = \frac{F}{(1+r)^t}$$

$$P = \frac{50,000}{(1+0.1)^7}$$

$$P = 25,657.91$$

Answer: The present value is ₱25,657.91. □

EXAMPLE 5. How much money should a student place in a time deposit in a bank that pays 1.1% compounded annually so that he will have ₱200,000 after 6 years?

Solution.

Given: $F = 200,000$

$r = 1.1\% = 0.011$

$t = 6$ years

Find: P

The present value P can be obtained by

$$P = \frac{F}{(1+r)^t}$$

$$P = \frac{200,000}{(1+0.011)^6}$$

$$P = 187,293.65$$

Answer: The student should deposit ₱187,293.65 in the bank. □

(D) Seatwork

Seatwork 1. Find the unknown principal P , rate r , time t , and compound interest I_c by completing

the table.

Principal (P)	Rate (r)	Time (t)	Compound Interest (I_c)	Maturity Value (F)
10,000	8%	15	(1) Ans: 21,721.69	(2) Ans: 31,721.69
3,000	5%	6	(3) Ans: 1,020.29	(4) Ans: 4,020.29
50,000	10.5%	10	(5) Ans: 85,704.04	(6) Ans: 135,704.04
(7) Ans: 45,286.54	2%	5	(8) Ans: 4,713.46	50,000
(9) Ans: 80,157.91	9.25%	2.5	(10) Ans: 19,842.09	100,000

Seatwork 2. Solve the following problems on compound interest.

- (a) What are the amounts of interest and maturity value of a loan for ₱20,000 at 6 % compound interest for 3 years?
Answer: $F_c = ₱23,820.32$ $I_c = ₱3,820.32$
- (b) In order to have ₱50,000 in 5 years, how much should you invest if the compound interest is 5%?
Answer: $P = ₱39,176.31$
- (c) A savings account in a bank yields 0.25% compound interest annually. Accumulate (find the future value of) ₱25,000 for 4 years in this savings account. How much interest will be gained?
Answer: $F_c = ₱25,250.94$ $I_c = ₱250.94$
- (d) In a certain bank, Angel invested ₱88,000 in a time deposit that pays 0.5% compound interest in a year. How much will be her money after 6 years? How much interest will she gain?
Answer: $F_c = ₱90,673.22$ $I_c = ₱2,673.22$
- (e) On the 7th birthday of her daughter, Shirlee deposited an amount in a bank peso bond fund that pays 1.0% interest compounded annually. How much should she deposit if she wants to have ₱100,000 on her daughter's 18th birthday?
Answer: $P = ₱89,632.37$

(E) Evaluation

Answer the following questions.

- (a) Christian deposited ₱5,000 in a bank that pays 2% compounded annually. Complete the table below.

Time (t)	Amount at the start of year t	Rate (r)	Compound Interest I_c	Amount at the end of year t
1	5,000	2%	100	5,100
2		2%	(1) Ans: 202	(2) Ans: 5,202
5		2%	(3) Ans: 520.40	
10	(4) Ans: 5,975.46	2%		(5) Ans: 6,094.97

- (b) In problem 1, Christian made a withdrawal of ₱2,000 after two years. If no further withdrawal is made, how much will be in his account after another 3 years? **Answer: ₱3,397.99**
- (c) How much money must be invested to obtain an amount of ₱30,000 in 4 years if money earns at 8% compounded annually? **Answer: ₱22,050.90**
- (d) A businessman invested ₱100,000 in a fund that pays 10.5% compounded annually for 5 years. How much was in the fund at the end of the term? **Answer: ₱164,744.68**
- (e) What amount must be deposited by a 15-year old student in a bank that pays 1% compounded annually so that after 10 years he will have ₱20,000? **Answer: ₱18,105.74**

LESSON 26: Compounding More than Once a Year

TIME FRAME: 90 minutes

CODE: M11GM-II-a-b-1 and b-2

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to to compute maturity value, interest, and present value, and solve problems involving compound interest when compound interest is computed more than once a year.

LESSON OUTLINE:

1. Compounding more than once a year
2. Finding maturity value, interest, and present value when compound interest is computed more than once a year
3. Enrichment (optional): Continuously compounded interest

PREREQUISITE SKILLS: Knowledge in Compound Interests

DEVELOPMENT OF THE LESSON

(A) Introduction, Motivation

Group the students into two.

Have one representative from each group ‘deposit’ ₱100 into a jar or box.

Group 1: This group will earn compound interest each day at a rate of 5%.

Group 2: This group will earn compound interest *twice a day* at a rate of 2.5%.

Ask the students to discuss whether one group will earn more than the other group. Let each group

compute how much money they will have at the end of 5 days. Let them observe the ‘compounding effect’ by constructing the table below.

Time (in days)	Group 1’s investment	Group 2’s investment
$\frac{1}{2}$	$(100)(1.05) = \mathbf{105.00}$	$(100)(1.025) = \mathbf{102.50}$
1		$(\mathbf{102.50})(1.025) = \mathbf{105.06}$
$1\frac{1}{2}$	$(\mathbf{105.00})(1.05) = \mathbf{110.25}$	$(\mathbf{105.06})(1.025) = \mathbf{107.69}$
2		$(\mathbf{107.69})(1.025) = \mathbf{110.38}$
$2\frac{1}{2}$	$(\mathbf{110.25})(1.05) = \mathbf{115.76}$	$(\mathbf{110.38})(1.025) = \mathbf{113.14}$
3		$(\mathbf{113.14})(1.025) = \mathbf{115.97}$
$3\frac{1}{2}$	$(\mathbf{115.76})(1.05) = \mathbf{121.55}$	$(\mathbf{115.97})(1.025) = \mathbf{118.87}$
4		$(\mathbf{118.87})(1.025) = \mathbf{121.84}$
$4\frac{1}{2}$	$(\mathbf{121.55})(1.05) = \mathbf{127.63}$	$(\mathbf{121.84})(1.025) = \mathbf{124.89}$
5		$(\mathbf{124.89})(1.025) = \mathbf{128.01}$

(B) Lesson Proper

- (a) Allow the students to compare the compound amounts when compounding semi-annually and compounding annually by posing the following example:

EXAMPLE 1. Given a principal of ₱10,000, which of the following options will yield greater interest after 5 years:

OPTION A: Earn an annual interest rate of 2% at the end of the year, or

OPTION B: Earn an annual interest rate of 2% in two portions—1% after 6 months, and 1% after another 6 months?

Solution.

OPTION A: Interest is compounded *annually*

Time (t) in years	Principal = ₱10,000
	Annual Int. rate = 2%, compounded annually
	Amount at the end of the year
1	$(10,000)(1.02) = \mathbf{10,200}$
2	$(\mathbf{10,200})(1.02) = \mathbf{10,404}$
3	$(\mathbf{10,404})(1.02) = \mathbf{10,612.08}$
4	$(\mathbf{10,612.08})(1.02) = \mathbf{10,824.32}$
5	$(\mathbf{10,824.32})(1.02) = \mathbf{11,040.81}$

OPTION B: Interest is compounded semi-annually, or every 6 months.

Under this option, the interest rate per conversion period 1% (2% divided by 2).

Time (t) in years	Principal = ₱10,000
	Annual Int. rate = 2%, compounded semi-annually
	Amount at the end of the year
$\frac{1}{2}$	$(10,000)(1.01) = 10,100$
1	$(10,100)(1.01) = 10,201$
$1\frac{1}{2}$	$(10,201)(1.01) = 10,303.01$
2	$(10,303.01)(1.01) = 10,406.04$
$2\frac{1}{2}$	$(10,406.04)(1.01) = 10,510.10$
3	$(10,510.10)(1.01) = 10,615.20$
$3\frac{1}{2}$	$(10,615.20)(1.01) = 10,721.35$
4	$(10,721.35)(1.01) = 10,828.56$
$4\frac{1}{2}$	$(10,828.56)(1.01) = 10,936.85$
5	$(10,936.85)(1.01) = 11,046.22$

Let the students realize that interest is often compounded more than once a year (semi-annually, quarterly, and daily). If all else is equal, **a more frequent compounding will result in a higher interest**, which is why Option B gives a higher interest than Option A.

The investment scheme in Option B introduces new concepts. Because interest is compounded twice a year, the **conversion period** is 6 months, and the **frequency of conversion** is 2. Because the investment runs for 5 years, the **total number of conversion periods** is 10. The **nominal rate** is 2% and the **rate of interest for each conversion period** is 1%. These terms are defined generally below.

□

(b) Definition of Terms

Let the students define the following additional terms:

- **conversion or interest period** - time between successive conversions of interest
- **frequency of conversion (m)** - number of conversion periods in one year
- **nominal rate ($i^{(m)}$)** - annual rate of interest
- **rate (j) of interest for each conversion period**

$$j = \frac{i^{(m)}}{m} = \frac{\text{annual rate of interest}}{\text{frequency of conversion}}$$

- **total number of conversion periods n**

$$n = tm = (\text{frequency of conversion}) \times (\text{time in years})$$

 **Note on rate notation:** $r, i^{(m)}, j$

In earlier lessons, r was used to denote the interest rate. Now that an interest rate can refer to two rates (either nominal or rate per conversion period), the symbols $i^{(m)}$ and j will be used instead.

- (c) Provide examples of nominal rates and the corresponding frequencies of conversion and interest rate for each period.

$i^{(m)}$ = Nominal Rate (Annual Interest Rate)	m = Frequency of Conversions	j = Interest Rate per conversion period	One conver- sion period
2% compounded annually; $i^{(1)} = 0.02$	1	$\frac{0.02}{1} = 0.02 = 2\%$	1 year
2% compounded semi-annually; $i^{(2)} = 0.02$	2	$\frac{0.02}{2} = 0.01 = 1\%$	6 months
2% compounded quarterly; $i^{(3)} = 0.02$	4	$\frac{0.02}{4} = 0.005 = 0.5\%$	3 months
2% compounded monthly; $i^{(12)} = 0.02$	12	$\frac{0.02}{12} = 0.001\bar{6} = 0.1\bar{6}\%$	1 month
2% compounded daily; $i^{(365)} = 0.02$	365	$\frac{0.02}{365}$	1 day

- (d) Derive the formula in finding compound amount when compounding is computed more than once a year.

Let the students recall (from Lesson 25) how to compute for the compound amount when principal P is invested at an annual interest rate j compounded annually,

$$F = P(1 + j)^t$$

You can modify this formula by noting that:

- the rate for each conversion period is $j = \frac{i^{(m)}}{m}$
- in t years, interest is compounded mt times.

Thus, you obtain the following formula:

Maturity Value, Compounding m times a year

$$F = P \left(1 + \frac{i^{(m)}}{m} \right)^{mt}$$

where F = maturity (future) value

P = principal

$i^{(m)}$ = nominal rate of interest (annual rate)

m = frequency of conversion

t = term / time in years

Teaching Tip

$$F = P(1 + j)^t$$

has the same structure as

$$F = P \left(1 + \frac{i^{(m)}}{m} \right)^{mt}$$

where j and $\frac{i^{(m)}}{m}$ refer to the interest rate per conversion period,

t and mt refer to the number of times

that interest is compounded

- (e) Provide examples on compounding amounts more than once a year.

EXAMPLE 2. Find the maturity value and interest if ₱10,000 is deposited in a bank at 2% compounded quarterly for 5 years.

Solution.Given: $P = 10,000$

$$i^{(4)} = 0.02$$

$$t = 5 \text{ years}$$

$$m = 4$$

Find: (a) F (b) P

Compute for the interest rate in a conversion period by

$$j = \frac{i^{(4)}}{m} = \frac{0.02}{4} = 0.005$$

Compute for the total number of conversion periods given by

$$n = mt = (4)(5) = 20 \text{ conversion periods.}$$

Compute for the maturity value using

$$\begin{aligned} F &= P(1 + j)^n \\ &= (10,000)(1 + 0.005)^{20} \\ F &= 11,048.96 \end{aligned}$$

Answer: The compound interest is given by

$$I_c = F - P = 11,048.96 - 10,000 = \text{P}1,048.96$$

□

EXAMPLE 3. Find the maturity value and interest if ₱10,000 is deposited in a bank at 2% compounded monthly for 5 years.**Solution.**Given: $P = 10,000$

$$i^{(12)} = 0.02$$

$$t = 5 \text{ years}$$

$$m = 12$$

Find: (a) F (b) P

Compute for the interest rate in a conversion period by

$$j = \frac{i^{(12)}}{m} = \frac{0.02}{12} = 0.001\bar{6}$$

Compute for the total number of conversion periods given by

$$n = mt = (12)(5) = 60 \text{ conversion periods.}$$

Compute for the maturity value using

$$\begin{aligned} F &= P(1 + j)^n \\ &= (10,000)(1 + 0.001\bar{6})^{60} \\ F &= \text{P}11,050.79 \end{aligned}$$

Answer: The compound interest is given by

$$I_c = F - P = 11,050.79 - 10,000 = \text{P}1,050.79$$

□

Teaching Tip

Allow the students to compare the compound amounts and the compound interests when compounded annually, semi-annually, quarterly and monthly. Emphasize that as the frequency of conversion periods in a year increases, the larger the compound interest, and so, is the compound amount.

EXAMPLE 4. Cris borrows ₱50,000 and promises to pay the principal and interest at 12% compounded monthly. How much must he repay after 6 years?

Solution.

Given: $P = \text{P}50,000$

$$i^{(12)} = 0.12$$

$$t = 6$$

$$m = 12$$

Find: F

You may also use the other formula to compute for the maturity value

$$F = P \left(1 + \frac{i^{(12)}}{m} \right)^{tm}$$

$$F = (50,000) \left(1 + \frac{0.12}{12} \right)^{(6)(12)}$$

$$F = (50,000)(1.01)^{72}$$

$$F = \text{P}102,354.97$$

Answer: Thus, Cris must pay ₱102,354.97 after 6 years. □

- (f) You may extend the discussion to finding present value when interest is compounded more than once a year.

Present Value P at Compound Interest

$$P = \frac{F}{\left(1 + \frac{i^{(m)}}{m} \right)^{mt}}$$

where F = maturity (future) value

P = principal

$i^{(m)}$ = nominal rate of interest (annual rate)

m = frequency of conversion

t = term / time in years

EXAMPLE 5. Find the present value of ₱50,000 due in 4 years if money is invested at 12% compounded semi-annually.

Solution.

Given: $F = 50,000$

$t = 4$

$i^{(2)} = 0.12$

Find: P

First, compute for the interest rate per conversion period given by

$$j = \frac{i^{(2)}}{m} = \frac{0.12}{2} = 0.06$$

The total number of conversion periods is $n = tm = (4)(2) = 8$.

The present value can be computed by substituting these values in the formula

$$P = \frac{F}{(1 + j)^n}$$

. Thus,

$$P = \frac{50,000}{(1 + 0.06)^8} = \frac{50,000}{(1.06)^8} = \text{P}31,370.62$$

□

EXAMPLE 6. What is the present value of ₱25,000 due in 2 years and 6 months if money is worth 10% compounded quarterly?

Solution.

Given: $F = 25,000$

$$t = 2\frac{1}{2} \text{ years}$$

$$i^{(4)} = 0.10$$

Find: P

The interest rate per conversion period given by

$$j = \frac{i^{(4)}}{m} = \frac{0.10}{4} = 0.025$$

and the total number of conversion periods is

$$n = tm = (2\frac{1}{2})(4) = 10$$

The present value can be computed by substituting these values in the formula

$$P = \frac{F}{(1 + j)^n}$$

. Thus,

$$P = \frac{25,000}{(1 + 0.025)^{10}} = \frac{25,000}{(1.025)^{10}} = \text{P}19,529.96$$

□

(C) Seatwork

Seatwork 1. Complete the table by computing the interest rate per period and total number of conversion periods.

Nominal Rate $i^{(m)}$	Interest Com- pounded	Frequency of conversion (m)	Interest Rate per Con- version Period
12%	Semi-annually	(1) Ans: 2	(2) Ans: 6%
16%	Quarterly	(3) Ans: 4	(4) Ans: 4%
9%	Monthly	(5) Ans: 12	(6) Ans: 0.75%
(7) Ans: 10.95%	Daily	(8) Ans: 365	0.03%

Seatwork 2. Complete the table by computing for compound amounts, compound interests and present values.

Principal	Nominal Rate	Interest com- pounded	Frequency of Conversion	Interest rate per period	Time in Years	Total number of conver- sions	Compound Interest	Compound Amount
10,000	8%	Semi-annually	(1) Ans: 2	(2) Ans: 4%	15	(3) Ans: 30	(4) Ans: 22,433.98	(5) Ans: 32,433.98
3,000	5%	quarterly	(6) Ans: 4	(7) Ans: 1.25%	6 years and 3 months	(8) Ans: 25	(9) Ans: 1,092.58	(10) Ans: 4,092.58
(11) Ans: 15,149.74	12%	monthly	(12) Ans: 12	(13) Ans: 1%	10	(14) Ans: 120	(15) Ans: 34,850.26	50,000

Seatwork 3. Solve the following problems on compound interest.

- (a) Accumulate ₱15,000 for 2 years at 15% compounded monthly. **Answer: ₱20,210.27**
- (b) How much should Kaye set aside and invest in a fund earning 2% compounded quarterly if she needs ₱75,000 in 15 months? **Answer: ₱73,152.80**
- (c) Peter is planning to invest ₱100,000. Bank A is offering 5% compounded semi-annually while Bank B is offering 4.5% compounded monthly. If he plans to invest this amount for 5 years, in which bank should he invest?

Answer: Compound amount after 5 years: Bank A: $F = ₱128,008.45$;
Bank B: $F = ₱125,179.58$; Bank A gives higher compound amount

(D) Evaluation

- (a) Fill in the blanks with the correct answers.
- When money is compounded monthly, the frequency of conversion is _____.
 - When the annual interest rate is 16% compounded quarterly the interest rate in a conversion period is _____.

- c. If the interest rate per conversion period is 1% and money is compounded monthly, the nominal rate is _____.
- d. When the term is 3 years and 6 months and money is compounded semi-annually, the total number of conversion periods is _____
- e. When the total number of conversion periods is 12 and the term is 6 years, then money is compounded _____.

Answer: a. 12 b. 0.04 or 4% c. 0.12 or 12% d. 7 e. semi-annually

(b) Complete the table by computing for the compound amounts, compound interests and present values.

Principal	Nominal Rate	Interest com-pounded	Frequency of Conver-sions	Interest rate per period	Years	Total Number of con-versions per year	Compound Interest (I)	Compound Amount (F)
5,000	6%	Semi-annually	(1) Ans:2	(2) Ans:3%	10	(3) Ans:20	(4) Ans:4,030.56	(5) Ans:9,030.56
30,000	2%	quarterly	(6) Ans:4	(7) Ans:0.5%	3 years and 9 months	(8) Ans:15	(9) Ans:2,330.48	(10) Ans:32,330.48
(11) Ans:60,778.86	10%	monthly	(12) Ans:12	(13) Ans:0.83%	5	(14) Ans:60	(15) Ans:39,221.14	100,000

(c) Solve the following problems on compound interests.

- a. Find the compound amount due in 8 years if ₱200,000 is invested at 12% compounded monthly.
Answer: ₱519,854.59
- b. What present value, compounded quarterly at 6%, will amount to ₱59,780.91 in 3 years?
Answer: ₱50,000.00
- c. Alet borrowed ₱15,000 payable with interest that is compounded semi-annually at 9%. How much must she pay after 3 years?
Answer: ₱19,533.90
- d. How much must Angel deposit in a bank that pays 0.75% compounded quarterly so that she will have ₱200,000 after 15 years?
Answer: ₱178,738.30
- e. Suppose that you have ₱80,000. You decided to deposit it on a bank and will not withdraw from it for 10 years. A bank offers two types of compound interest accounts. The first account offers 6% interest compounded monthly. The second account offers 6.5% interest compounded semi-annually. Which account will you choose if you want your money to earn more?

Answer: First Bank: $F = \text{P}145,551.74$
 Second Bank = $F = \text{P}151,667.03$; Second bank yields more

(E) Enrichment(Optional)

Continuous Compounding

Interest can be compounded continuously like every hour, every minute or even a fraction of a second. If the number of compounding m is to increase without bound, this procedure approaches what is called **continuous compounding**.

The formula for continuous compounding is derived as follows:

Let $i^{(m)}$ be the annual interest rate and let $x = \frac{1}{j} = \frac{m}{i^{(m)}}$. Then $m = x(i^{(m)})$. When interest is compounded continuously, then $m \rightarrow \infty$ so $x = \frac{m}{i^{(m)}} \rightarrow \infty$ (because the numerator m increases without bound).

By substitution $x = \frac{m}{i^{(m)}}$ in the compound interest formula, we obtain

$$F = P \left(1 + \frac{i^{(m)}}{m} \right)^{mt}$$

$$F = P \left(1 + \frac{1}{x} \right)^{xi^{(m)}t}$$

$$F = P \left[\left(1 + \frac{1}{x} \right)^x \right]^{i^{(m)}t}$$

Substitute larger and larger values of x in $\left(1 + \frac{1}{x} \right)^x$.

You will see that as $x \rightarrow \infty$, $\left(1 + \frac{1}{x} \right)^x$ approaches a number. We call this number e , which is approximately 2.71828.

Continuous Compound Interest

If a principal P is invested at annual interest rate $i^{(m)}$ compounded continuously, then the amount F at the end of t years is given by

$$F = Pe^{i^{(m)}t}$$

EXAMPLE 7. Suppose you invested $\text{P}20,000$ at 3% compounded continuously. How much will you have from this investment after 6 years?

Solution.Given: $P = 20,000$

$$i^{(m)} = 0.03$$

$$t = 6 \text{ years}$$

Find: F Use the continuous compounding formula $F = Pe^{i^{(m)}t}$. By substitution,

$$F = Pe^{i^{(m)}t} = 20,000e^{(0.03)(6)} = 20,000e^{0.18} = \text{P}23,944.35$$

Hence, the amount P20,000 will become P23,944.35 if you invest it at 3% compounded continuously for 6 years. \square

LESSON 27: Finding Interest Rate and Time in Compound Interest**TIME FRAME:** 90 minutes**CODE:** M11GM-II-a-b-1 and b-2**LEARNING OUTCOME(S):** At the end of the lesson, the learner is able to solve problems involving rate of interest and time in compound interest.**LESSON OUTLINE:**

1. Interest and time in compound interest
2. Equivalent interest rate

PREREQUISITE SKILLS: Knowledge in Compound Interests, Exponential and Logarithmic Functions

DEVELOPMENT OF THE LESSON

(A) Introduction, Motivation

Ask your students to research on the prevailing interest rates in the various types of deposit accounts in several banks. A sample output of the research may be similar to the data in the table below. (These values are hypothetical, and serve only as an example)

Type of Deposit Account	Required Initial Deposit	Required Daily Balance to earn interest	Interest Rate (per annum)
Easy Savings	₱100	₱1,000	0.25%
Passbook Savings	₱5,000	₱25,000	0.25%
Advanced Savings	₱100,000	₱500,000	None for less than ₱500,000
			1% for ₱5,000,000 to ₱999,999
			1.5% for 1 million and above

Teaching Tip

Emphasize that money deposited in savings account, aside from earning interest, may be safer than keeping them at their houses where money can be stolen or lost. You may also mention that interest earned in a savings account is subject to 20% withholding tax.

(B) Lesson Proper

- (a) Tell the students that investors are interested to find out how much time is needed to accumulate a desired amount in investing a certain principal.

EXAMPLE 1. How long will it take ₱3,000 to accumulate to ₱3,500 in a bank savings account at 0.25% compounded monthly?

Solution.

Given: $P = 3,000$

$$F = 3,500$$

$$i^{(12)} = 0.25\% = 0.0025$$

$$m = 12$$

$$j = \frac{i^{(12)}}{m} = \frac{0.0025}{12}$$

Find: t

Substituting the given values in the maturity value formula

$$F = P(1 + j)^n$$

results to

$$3,500 = 3,000 \left(1 + \frac{0.0025}{12}\right)^n$$

$$\frac{3,500}{3,000} = \left(1 + \frac{0.0025}{12}\right)^n$$

To solve for n , take the logarithm of both sides.

$$\begin{aligned}\log\left(\frac{3,500}{3,000}\right) &= \log\left(1 + \frac{0.0025}{12}\right)^n \\ \log(1.166667) &= n \log\left(1 + \frac{0.0025}{12}\right) \\ n &= 740.00 \text{ periods}\end{aligned}$$

Thus, payments should be made for 740 months, or $t = \frac{n}{m} = \frac{740}{12} = 61.67$ years. □

Teaching Tip

As shown in the previous example, we use logarithms to solve unknowns that appear in the exponent.

EXAMPLE 2. How long will it take ₱1,000 to earn ₱300 if the interest is 12% compounded semi-annually?

Solution.

Given: $F = 1,300$

$$m = 2$$

$$i^{(2)} = 0.12$$

$$j = \frac{i^{(2)}}{2} = \frac{0.12}{2} = 0.06$$

Find: n and t

$$\begin{aligned}F &= P(1 + j)^n \\ 1,300 &= 1,000(1 + 0.06)^n \\ 1.3 &= (1.06)^n \\ \log(1.3) &= \log(1.06)^n \\ \log(1.3) &= n \log(1.06) \\ n &= \frac{\log 1.3}{\log(1.06)} = 4.503 \text{ periods}\end{aligned}$$

Because interest is earned only at the end of the period, then 5 six-month periods are needed so that the interest can reach ₱300.

Thus, $n = 5$ and $t = \frac{n}{m} = \frac{5}{2} = 2.5$ years.

It will take 2.5 years for ₱1,000 to earn ₱300. □


Caution.

The value of n , which refers to the number of periods, should be a whole number. Thus, in the previous example, we used $n = 5$ rather than $n = 4.503$ to compute for t .

- (b) Tell the students that investors are also interested to know the interest rate before investing a certain amount.

EXAMPLE 3. At what nominal rate compounded semi-annually will ₱10,000 accumulate to ₱15,000 in 10 years?

Solution.

Given: $F = 15,000$

$P = 10,000$

$t = 10$

$m = 2$

$n = mt = (2)(10) = 20$

Find: $i^{(2)}$

$$\begin{aligned}
 F &= P(1 + j)^n \\
 15,000 &= 10,000(1 + j)^{20} \\
 \frac{15,000}{10,000} &= (1 + j)^{20} \\
 1.5 &= (1 + j)^{20} \\
 (1.5)^{\frac{1}{20}} &= 1 + j \\
 (1.5)^{\frac{1}{20}} - 1 &= j \\
 j &= 0.0205
 \end{aligned}$$

The interest rate per conversion period is 2.05%.

The nominal rate (annual rate of interest) can be computed by

$$\begin{aligned}
 j &= \frac{i^{(m)}}{m} \\
 0.0205 &= \frac{i^{(2)}}{2} \\
 i^{(2)} &= (0.0205)(2) \\
 i^{(2)} &= 0.0410 \text{ or } 4.10\%
 \end{aligned}$$

Hence, the nominal rate is 4.10%. □

EXAMPLE 4. At what interest rate compounded quarterly will money double itself in 10 years?

Solution.

Given: $F = 2P$

$t = 10$ years

$m = 4$

$n = mt = (4)(10) = 40$

Find: $i^{(4)}$

$$F = P(1 + j)^n$$

$$2P = P(1 + j)^n$$

$$2 = (1 + j)^{40}$$

$$(2)^{1/40} = 1 + j$$

$$(2)^{1/40} - 1 = j$$

$$j = 0.0175 \text{ or } 1.75\%$$

The interest rate in each conversion period is 1.75%.

The nominal rate can be computed by

$$j = \frac{i^{(4)}}{m}$$

$$0.0175 = \frac{i^{(4)}}{4}$$

$$i^{(4)} = (0.0175)(4)$$

$$i^{(4)} = 0.070 \text{ or } 7.00\%$$

Therefore, the nominal rate that will double an amount of money compounded quarterly in 10 years is 7.0%. □

- (c) Differentiate nominal and effective rates. Teach students to convert one interest rate to an equivalent interest rate.

Definition of terms:

- **Equivalent rates** - two annual rates with different conversion periods that will earn the same compound amount at the end of a given number of years

- **Nominal rate** - annual interest rate (may be compounded more than once a year)
- **Effective rate** - the rate compounded annually that will give the same compound amount as a given nominal rate; denoted by $i^{(1)}$

EXAMPLE 5. What effective rate is equivalent to 10% compounded quarterly?

Solution.

Given: $i^{(4)} = 0.10$

$$m = 4$$

Find: effective rate $i^{(1)}$

Since the equivalent rates yield the same maturity value, then

$$F_1 = F_2$$

$$P(1 + i^{(1)})^t = P \left(1 + \frac{i^{(4)}}{m} \right)^{mt}$$

Dividing both sides by P results to

$$(1 + i^{(1)})^t = \left(1 + \frac{i^{(4)}}{m} \right)^{mt}$$

Raise both sides to $\frac{1}{t}$ to obtain:

$$(1 + i^{(1)}) = \left(1 + \frac{0.10}{4} \right)^4$$

$$i^{(1)} = \left(1 + \frac{0.10}{4} \right)^4 - 1 = 0.103813 \text{ or } 10.38\%$$

Hence, the effective rate equivalent to 10% compounded quarterly is 10.38%. □

 **Is our answer reasonable?**

We can note that 10% compounded quarterly cannot be matched by 10% compounded annually—the latter rate would yield less interest because there are less frequent conversion periods. Thus, to match 10% compounded quarterly, the annual rate must be a bit more than 10%, so our answer of 10.38% makes sense.


Number of Decimal Places

As you will see in Lesson 29, when solving for an equivalent rate j , it is important to make it very precise. Thus, when solving for an equivalent rate, say $j = (1.025)^4 - 1$ in Example 5, six or more decimal places are recommended.

EXAMPLE 6. Complete the table by computing for the rates equivalent to the following nominal rates.

Given Interest Rate	Equivalent Interest Rate
12% compounded monthly	_____ compounded annually
8 % compounded semi-annually	_____ compounded quarterly
12% compounded monthly	_____ compounded semi-annually

Solution. The maturity values accumulated by these interest rates at any time t (in particular, at $t = 1$) must be equal. That is, $F_1 = F_2$ for any t , including when $t = 1$.

a. Given: equal P ; equal t

$$\begin{array}{llllll}
 12\% \text{ compounded monthly} & i^{(12)} = 0.12 & m = 12 & P & t \\
 \text{___ compounded monthly} & i^{(1)} = ? & m = 1 & P & t
 \end{array}$$

Let F_1 be the future value when interest is compounded annually, and F_2 be the future value when interest is 12% compounded monthly.

$$\begin{aligned}
 F_1 &= F_2 \\
 P \left(1 + \frac{i^{(1)}}{1} \right)^{(1)t} &= P \left(1 + \frac{i^{(12)}}{12} \right)^{12t} \\
 \left(1 + \frac{i^{(1)}}{1} \right) &= \left(1 + \frac{0.12}{12} \right)^{12} \\
 i^{(1)} &= (1.01)^{12} - 1 \\
 i^{(1)} &= 0.126825\%
 \end{aligned}$$

Answer: 12.6825% compounded annually

b. Given: equal P ; equal t

$$\begin{array}{llllll}
 8\% \text{ compounded semi-annually} & i^{(2)} = 0.08 & m = 2 & P & t \\
 \text{___ compounded quarterly} & i^{(4)} = ? & m = 4 & P & t
 \end{array}$$

Let F_1 be the future value when interest is compounded quarterly, and F_2 be the future value when interest is 8% compounded semi-annually.

$$\begin{aligned}
 F_1 &= F_2 \\
 P \left(1 + \frac{i^{(4)}}{4} \right)^{(4)t} &= P \left(1 + \frac{i^{(2)}}{2} \right)^{(2)t} \\
 \left(1 + \frac{i^{(4)}}{4} \right)^4 &= \left(1 + \frac{0.08}{2} \right)^2 \\
 \left(1 + \frac{i^{(4)}}{4} \right)^4 &= (1.04)^2 \\
 1 + \frac{i^{(4)}}{4} &= [(1.04)^2]^{(1/4)} \\
 1 + \frac{i^{(4)}}{4} &= (1.04)^{1/2} \\
 1 + \frac{i^{(4)}}{4} &= 1.019804 \\
 \frac{i^{(4)}}{4} &= 1.019804 - 1 \\
 \frac{i^{(4)}}{4} &= 0.019804 \\
 i^{(4)} &= (0.019804)(4) \\
 i^{(4)} &= 0.079216\%
 \end{aligned}$$

Answer: 7.9216% compounded quarterly

c. Given: equal P; equal t

$$\begin{array}{llll}
 12\% \text{ compounded monthly} & i^{(12)} = 0.12 & m = 12 & P \quad t \\
 \text{_____ compounded semi-annually} & i^{(2)} = ? & m = 2 & P \quad t
 \end{array}$$

Let F_1 be the future value when interest is compounded semi-annually, and F_2 be the future

value when interest is 12% compounded monthly.

$$\begin{aligned}
 F_1 &= F_2 \\
 P \left(1 + \frac{i^{(2)}}{2}\right)^{(2)t} &= P \left(1 + \frac{i^{(12)}}{12}\right)^{(12)t} \\
 \left(1 + \frac{i^{(2)}}{2}\right)^2 &= \left(1 + \frac{0.12}{12}\right)^{12} \\
 \left(1 + \frac{i^{(2)}}{2}\right)^2 &= (1.01)^{12} \\
 1 + \frac{i^{(2)}}{2} &= [(1.01)^{12}]^{(1/2)} \\
 1 + \frac{i^{(2)}}{2} &= (1.01)^6 \\
 1 + \frac{i^{(2)}}{2} &= 1.061520 \\
 \frac{i^{(2)}}{2} &= 1.061520 - 1 \\
 \frac{i^{(2)}}{2} &= 0.061520 \\
 i^{(2)} &= (0.061520)(2) \\
 i^{(2)} &= 0.12304 \text{ or } 12.304\%
 \end{aligned}$$

Answer: 12.304% compounded semi-annually

□

(C) Seatwork

Seatwork 1. Find the unknown.

(a) $F = 2,000, P = 1,750, m = 2, t = 4$ years, $j = ?$, $i^{(m)} = ?$

Answer: $j = 0.016831504$ or 1.68%, $i^{(m)} = 0.033663008$ or 3.37%

(b) $F = 100,000, P = 10,000, t = 8$ years, money is compounded monthly, $i^{(m)} = ?$, $j = ?$

Answer: $j = 0.024275$ or 2.43%, $i^{(m)} = 0.2913$ or 29.13%

(c) $F = 30,000, P = 10,000, i^{(m)} = 16\%$ compounded quarterly, $j = ?$, $n = ?$, $t = ?$

Answer: $j = 0.04$, $n = 28$ conversion periods, $t = 7$ years

(d) $F = 18,000, P = 12,000, i^{(m)} = 12\%$ compounded semi-annually, $j = ?$, $n = ?$, $t = ?$

Answer: $j = 0.06$, $n = 7$ conversion periods, $t = 3.48$ years

Seatwork 2. Complete the table by computing for the following equivalent interest rates.

Given Interest Rate	Equivalent Interest Rate
2% compounded annually	_____ compounded quarterly Ans: 0.019852
6% compounded quarterly	_____ compounded semi-annually Ans: 0.06045
0.5% compounded semi-annually	_____ compounded annually Ans: 0.005006

Seatwork 3. Solve the following problems.

- (a) What is the nominal rate of interest if ₱4,000 accumulates to ₱10,000 in 8 years with interest compounded quarterly? **Answer:** $j = 0.116191927$ or 11.62%
- (b) At what nominal rate compounded semi-annually will ₱5000 amount to ₱6000 in 2 years? **Answer:** $j = 0.093270279$ or 9.33%
- (c) What rate compounded annually will double an amount of money in 3 years? **Answer:** $j = 0.25992105$ or 26%
- (d) In how many years will it take ₱18,000 accumulate to ₱20,000 when deposited in a savings account that earns 0.250% compounded monthly? **Answer:** 506 periods, so 42 years and 2 months
- (e) How long will a principal double when invested at 10% compounded semi-annually? **Answer:** 15 periods, so 7 years and 6 months

(D) Evaluation

- (a) Complete the table by finding the unknown values (time and rate).

P	Nominal Rate	Interest compounded	Frequency of Conversions	Interest rate per period	Time in Years	Number of conversions	Compound Interest	Compound Amount
2,000	(1) Ans: 17.55%	Semi-annually	(2) Ans: 2	(3) Ans: 8.78%	2	(4) Ans: 4	(5) Ans: 800	2,800
5,000	(6) Ans: 1.82%	quarterly	(7) Ans: 4	(8) Ans: 0.45%	5 years and 3 months	(9) Ans: 21	500	(10) Ans: 5,500
10,000	16%	monthly	(11) Ans: 12	(12) Ans: 1.33%	(13) Ans: 1.083	(14) Ans: 13	(15) Ans: 1,879	11,879

- (b) Solve the following problems on compound interests.

- i. At what nominal rate compounded quarterly will ₱24,000 accumulate to ₱30,000 in 4 years? **Answer:** 1.404% interest rate for each conversion period, 5.62% nominal rate
- ii. How long will it take ₱15,000 to earn ₱5,000 if it is invested at 12% monthly? **Answer:** 29 conversion periods, 2.4 years

- iii. At what annual interest rate compounded semi-annually will a certain amount triple itself in 20 years? **Answer:** 2.78 % interest rate for each conversion period, 5.57% nominal rate
- iv. How much time is needed for money to triple if invested at 9% compounded semi-annually? **Answer:** 25 periods, 12.5 years
- v. Nora borrowed an amount of ₱40 000 which she paid with an interest of ₱2,000 at the end of 3 years. At what nominal rate compounded semi-annually was it invested? **Answer:** 0.82% for each conversion period, 1.63% nominal rate
-
-

Chapter 7

Annuities

LESSON 28: Simple Annuity

TIME FRAME: 120 minutes

CODE: M11GM0IIc-1, c-2, and c-d-1

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to illustrate simple and general annuities, distinguish between simple and general annuities, and find the future and present values of simple annuities.

LESSON OUTLINE:

1. Definition of terms
2. Time diagrams
3. Future value of a simple annuity
4. Present value of a simple annuity
5. Periodic payment of a simple annuity

PREREQUISITE SKILLS: Knowledge of Exponential Functions and Compound Interest

DEVELOPMENT OF THE LESSON

(A) Introduction/Motivation

Ask students examples where people pay by installment.

Possible answers: Insurance payments, major purchases (e.g., appliances, property), or loan payments

Payments by installment are done periodically, and in equal amounts. This payment scheme is called **annuity**.

(B) Lesson Proper

(a) Present definition of terms.

- **Annuity** - a sequence of payments made at equal (fixed) intervals or periods of time
- **Payment interval** - the time between successive payments

Annuities may be classified in different ways, as follows.

	Annuities	
According to payment interval and interest period	<u>Simple Annuity</u> An annuity where the payment interval is the same as the interest period	<u>General Annuity</u> An annuity where the payment interval is not the same as the interest period
According to time of payment	<u>Ordinary Annuity</u> (or Annuity Immediate) A type of annuity in which the payments are made at the end of each payment interval	<u>Annuity Due</u> A type of annuity in which the payments are made at beginning of each payment interval
According to duration	<u>Annuity Certain</u> An annuity in which payments begin and end at definite times	<u>Contingent Annuity</u> An annuity in which the payments extend over an indefinite (or indeterminate) length of time

- **Term of an annuity, t** - time between the first payment interval and last payment interval
- **Regular or Periodic payment, R** - the amount of each payment
- **Amount (Future Value) of an annuity, F** - sum of future values of all the payments to be made during the entire term of the annuity
- **Present value of an annuity, P** - sum of present values of all the payments to be made during the entire term of the annuity

Scope of the Grade 11 curriculum

Annuity Due and Contingent Annuity are NOT emphasized in Grade 11 mathematics. Thus, unless otherwise specified, the term ‘annuity’ refers to an Ordinary Annuity and an Annuity Certain—i.e., an annuity with a definite term, and where payments are made at the end of each period.

- (b) Allow the students to provide real life examples of annuity certain and contingent annuity.
- **Examples of annuity** - installment basis of paying a car, appliance, house and lot, tuition fee, etc.
 - **Examples of contingent annuity** - life insurance, pension payments
- (c) Let the students illustrate simple annuities as shown in the time diagrams below.

Simple Annuity							
P							F
	R	R	R	R	R	\dots	R
0	1	2	3	4	5	\dots	n

- (d) Ask the students to illustrate certain examples of annuities using time diagrams.

An installment payment of an appliance of ₱3000 every month for 6 months						
P						F
	3,000	3,000	3,000	3,000	3,000	3,000
0	1	2	3	4	5	6
	←————— 6 months —————→					
Periodic payment $R = 3,000$ Term $t = 6$ months Payment interval = 1 month						

Ordinary Annuities

Unless mentioned otherwise, all annuities discussed in this lesson are ordinary annuities. That is, the regular payments are assumed to be done at the end of the payment period.

- (e) Let the students illustrate and distinguish between simple and general annuities by providing examples.

Both simple and general annuities have a time diagram for its cash flow as shown below. The main difference is that in a **simple annuity** the payment interval is the same as the interest period while in a **general annuity** the payment interval is not the same as the interest period.

P								F
	R	R	R	R	R	\dots		R
0	1	2	3	4	5	\dots		n

- **Example of a simple annuity** - Installment payment for an appliance at the end of each month with interest compounded monthly
 - **Example of a general annuity** - Installment payment for an appliance at the end of each month with interest compounded annually (to be discussed in the next lesson)
- (f) Discuss how to compute the amount (future value) of a **simple annuity immediate**.

EXAMPLE 1. Suppose Mrs. Remoto would like to save ₱3,000 at the end of each month, for six months, in a fund that gives 9% compounded monthly. How much is the amount or future value of her savings after 6 months?

Solution.

Given: $R = ₱3,000$

term $t = 6$ months

interest rate per annum $i^{(12)} = 0.09$

number of conversions per year $m = 12$

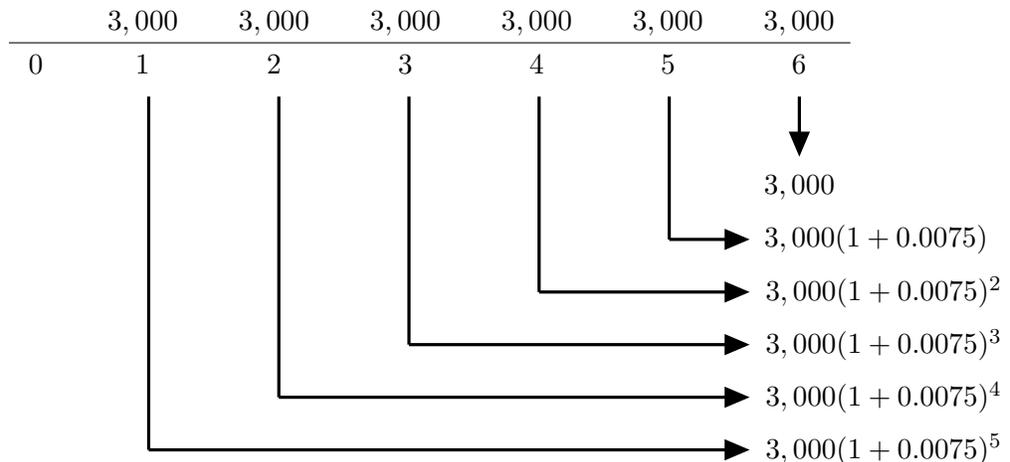
interest rate per period $j = \frac{0.09}{12} = 0.0075$

Find: amount (future value) at the end of the term, F

- (1) Illustrate the cash flow in a time diagram

	3,000	3,000	3,000	3,000	3,000	3,000
0	1	2	3	4	5	6

- (2) Find the future value of all the payments at the end of the term ($t = 6$)



(3) Add all the future values obtained from the previous step.

$$\begin{aligned}
 3,000 &= 3,000 \\
 (3,000)(1 + 0.0075) &= 3,022.5 \\
 (3,000)(1 + 0.0075)^2 &= 3,045.169 \\
 (3,000)(1 + 0.0075)^3 &= 3,068.008 \\
 (3,000)(1 + 0.0075)^4 &= 3,091.018 \\
 (3,000)(1 + 0.0075)^5 &= 3,114.20 \\
 \hline
 F &= 18,340.89
 \end{aligned}$$

Thus, the amount of this annuity is ₱18,340.89. □

Teaching Tip

One basic concept is the **time value of money**. An amount such as ₱3,000 at $t = 1$ has a different value than ₱3,000 at $t = 2$, or ₱3,000 at $t = 3$, and so on.

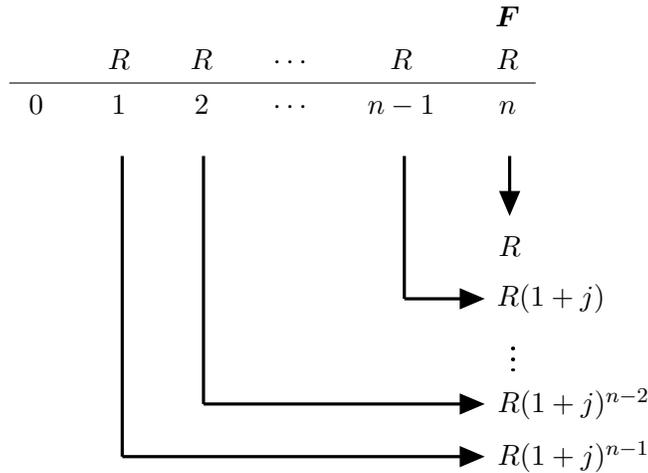
This means that if we consider interest, we cannot just multiply $\text{₱}3,000 \times 6 = \text{₱}18,000$ because each of the individual ₱3,000 payments is valued differently from the others.

The correct way to find the total savings at the end of 6 months is to compute the maturity value of each ₱3,000 at the end of 6 months. By the end of 6 months, the ₱3,000 at $t = 1$ would have earned interest for 5 months, the ₱3,000 at $t = 2$ would have earned interest for 4 months, and so on.

(g) Explain how the formula for finding the amount (future value) F of an ordinary annuity is derived.

The derivation of the formula in finding the amount of an ordinary annuity is similar to the solution of Example 1.

Illustrate the cash flow in a time diagram.



$$F = R + R(1 + j) + R(1 + j)^2 + \dots + R(1 + j)^{n-2} + R(1 + j)^{n-1} \tag{7.1}$$

Multiply both sides by $1 + j$ to get

$$F(1 + j) = R(1 + j) + R(1 + j)^2 + R(1 + j)^3 + \dots + R(1 + j)^{n-1} + R(1 + j)^n \tag{7.2}$$

From equation (7.2), subtract Equation (7.1) to produce

$$\begin{aligned} F(1 + j) - F &= R(1 + j)^n - R \\ F[(1 + j) - 1] &= R[(1 + j)^n - 1] \\ F(j) &= R[(1 + j)^n - 1] \\ F &= R \frac{(1 + j)^n - 1}{j} \end{aligned}$$

The expression $\frac{(1 + j)^n - 1}{j}$ is usually denoted by the symbol $s_{\overline{n}|j}$ (this is read as ‘s angle n’).

Amount (Future Value) of an Ordinary Annuity (Annuity Immediate)

	R	R	R	R	R	\dots	R
0	1	2	3	4	5	\dots	n

The amount of an annuity immediate is given by

$$F = Rs_{\overline{n}|j} = R \frac{(1 + j)^n - 1}{j},$$

where R is the regular payment,
 j is the interest rate per period, and
 n is the number of payments

- (h) Let the student solve Example 1 using the derived formula.

$$\begin{aligned}
 F &= R \frac{(1+j)^n - 1}{j} \\
 &= 3,000 \frac{(1+0.0075)^6 - 1}{0.0075} \\
 &= 18,340.89
 \end{aligned}$$

 **Teaching Tip**

Emphasize that the formula for F was derived based on the procedure on how to accumulate the payment of each period to the end of the term.

EXAMPLE 2. In order to save for her high school graduation, Marie decided to save ₱200 at the end of each month. If the bank pays 0.250% compounded monthly, how much will her money be at the end of 6 years?

Solution.

Given: $R = 200$

$$m = 12$$

$$i^{(12)} = 0.250\% = 0.0025$$

$$j = \frac{0.0025}{12} = 0.000208\bar{3}$$

$$t = 6 \text{ years}$$

$$n = tm = 6(12) = 72 \text{ periods}$$

Find: F

$$\begin{aligned}
 F &= R \frac{(1+j)^n - 1}{j} \\
 &= 200 \frac{(1+0.000208\bar{3})^{72} - 1}{0.000208\bar{3}} \\
 &= 14,507.02
 \end{aligned}$$

Hence, Marie will be able to save ₱14,507.02 for her graduation. □

- (i) Discuss how to compute for the present value of an ordinary annuity (annuity immediate).

EXAMPLE 3. (Recall the problem in Example 1.) Suppose Mrs. Remoto would like to know the present value of her monthly deposit of ₱3,000 when interest is 9% compounded monthly. How much is the present value of her savings at the end of 6 months?

Solution.

Given: periodic payment $R = 3,000$

term $t = 6$ months

interest rate per annum $i^{(12)} = 0.09$

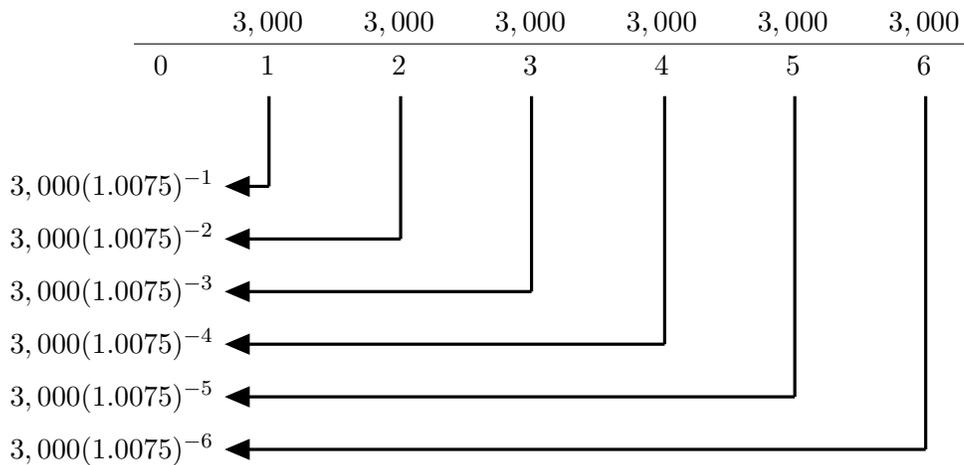
number of conversions per year $m = 12$

interest rate per period $j = \frac{0.09}{12} = 0.0075$

Find: Present Value P

- (1) Discount the payment of each period to the beginning of the term—that is, find the present value of each payment. Recall the formula

$$P = \frac{F}{\left(1 + \frac{i^{(m)}}{m}\right)^{mt}} = \frac{3,000}{1.0075^t} = 3,000(1.0075)^{-t}.$$



- (2) Add the discounted payments to get the present value.

$$(3,000)(1.0075)^{-1} = 2,977.667$$

$$(3,000)(1.0075)^{-2} = 2,955.501$$

$$(3,000)(1.0075)^{-3} = 2,933.50$$

$$(3,000)(1.0075)^{-4} = 2,911.663$$

$$(3,000)(1.0075)^{-5} = 2,889.988$$

$$(3,000)(1.0075)^{-6} = 2,868.474$$

$$P = 17,536.79$$

Thus, the cost of the TV set at the beginning of the term is ₱17,536.79. □

Alternate Solution to Example 3:

Since we already know from Example 1 that the accumulated amount at the end of 6 months is ₱18,340.89, then we can simply get the present value of this amount using the formula

$$P = \frac{F}{(1+j)^n} = \frac{F}{\left(1 + \frac{i^{(m)}}{m}\right)^{tm}} = \frac{18,340.89}{\left(1 + \frac{.09}{12}\right)^6} = 17,536.79.$$


Teaching Tip

The present and future values of an annuity are also related by

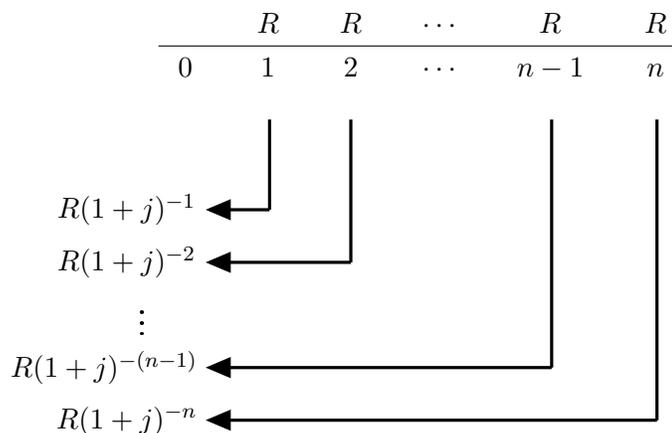
$$F = P(1+j)^n \quad \text{and} \quad P = \frac{F}{(1+j)^n}$$

- (j) Explain how the formula for finding the present value P of an ordinary annuity is derived.

The derivation of the formula in finding the present value P of an annuity immediate is similar to the solution of Example 3.

Discount or get the value of each payment at the beginning of the term and then add to get the present value of an ordinary annuity. Use the formula

$$P = \frac{F}{(1+j)^n} = F(1+j)^{-n}.$$



$$P = R(1+j)^{-1} + R(1+j)^{-2} + \dots + R(1+j)^{-(n-1)} + R(1+j)^{-n}$$

$$P = \frac{R}{(1+j)^1} + \frac{R}{(1+j)^2} + \frac{R}{(1+j)^3} + \dots + \frac{R}{(1+j)^{n-1}} + \frac{R}{(1+j)^n} \quad (7.3)$$

Multiply both sides by $\frac{1}{1+j}$ to get

$$\frac{P}{1+j} = \frac{R}{(1+j)^2} + \frac{R}{(1+j)^3} + \cdots + \frac{R}{(1+j)^n} + \frac{R}{(1+j)^{n+1}} \quad (7.4)$$

From Equation (7.3), subtract Equation (7.4) to produce

$$\begin{aligned} P - P\frac{1}{1+j} &= \frac{R}{1+j} - \frac{R}{(1+j)^n} \\ P\left(1 - \frac{1}{1+j}\right) &= \frac{R}{1+j} \left(1 - \frac{1}{(1+j)^n}\right) \\ &= P\left(\frac{1+j-1}{1+j}\right) = \frac{R}{1+j} (1 - (1+j)^{-n}) \\ P\left(\frac{j}{1+j}\right) &= \frac{R}{1+j} (1 - (1+j)^{-n}) \\ Pj &= R(1 - (1+j)^{-n}) \\ P &= R\frac{1 - (1+j)^{-n}}{j} \end{aligned}$$

The expression $\frac{1 - (1+j)^{-n}}{j}$ is usually denoted by the symbol $a_{\overline{n}|j}$ (this is read as ‘ a angle n ’).

Hence, the present value P of an ordinary annuity can be written as

$$P = Ra_{\overline{n}|j} = R\frac{1 - (1+j)^{-n}}{j}.$$

Alternate Derivation

The future value of an ordinary annuity was given earlier by

$$F = R\frac{(1+j)^n - 1}{j}.$$

To get the present value of this amount, we use the formula $P = \frac{F}{(1+j)^n}$ and obtain

$$P = \frac{F}{(1+j)^n} = \frac{R\frac{(1+j)^n - 1}{j}}{(1+j)^n} = R\frac{(1+j)^n - 1}{j(1+j)^n} = R\frac{1 - (1+j)^{-n}}{j}.$$

Present Value of an Ordinary Annuity (Annuity-Immediate)

	R	R	R	R	R	\dots	R
0	1	2	3	4	5	\dots	n

The present value of an annuity-immediate is given by

$$P = R_{\overline{n}|j} = R \frac{1 - (1 + j)^{-n}}{j},$$

where R is the regular payment,
 j is the interest rate per period,
 n is the number of payments

- (k) Let the student solve Example 3 using the derived formula for P .

$$\begin{aligned} P &= R \frac{1 - (1 + j)^{-n}}{j} \\ &= 3,000 \frac{1 - (1 + 0.0075)^{-6}}{0.0075} \\ &= 17,536.79 \end{aligned}$$

- (l) Introduce to the students the concept of cash value or cash price.

Definition.

The **cash value** or **cash price** is equal to the down payment (if there is any) plus the present value of the installment payments.

EXAMPLE 4. Mr. Ribaya paid ₱200,000 as down payment for a car. The remaining amount is to be settled by paying ₱16,200 at the end of each month for 5 years. If interest is 10.5% compounded monthly, what is the cash price of his car?

Solution.

Given: down payment = 200,000

$$R = 16,200$$

$$i^{(12)} = 0.105$$

$$j = \frac{0.105}{12} = 0.00875$$

$$t = 5 \text{ years}$$

$$n = mt = 12(5) = 60 \text{ periods}$$

Find: cash value or cash price of the car

The time diagram for the installment payments is given by:

$$\begin{array}{cccccc}
 P = ? & & & & & \\
 & 16,200 & 16,200 & 16,200 & \cdots & 16,200 \\
 \hline
 & 0 & 1 & 2 & 3 & \cdots & 60
 \end{array}$$

The present value of this ordinary annuity is given by

$$\begin{aligned}
 P &= R \frac{1 - (1 + j)^{-n}}{j} \\
 &= 16,200 \frac{1 - (1 + 0.00875)^{-60}}{0.00875} \\
 &= 753,702.20
 \end{aligned}$$

$$\begin{aligned}
 \text{Cash Value} &= \text{Down Payment} + \text{Present Value} \\
 &= 200,000 + 753,702.20 \\
 &= \text{P}953,702.20
 \end{aligned}$$

The cash price of the car is **₱953,702.20**. □

EXAMPLE 5. Paolo borrowed ₱100 000. He agrees to pay the principal plus interest by paying an equal amount of money each year for 3 years. What should be his annual payment if interest is 8% compounded annually?

Solution.

Given: $P = 100,000$

$$i^{(1)} = 0.08$$

$$m = 1$$

$$j = 0.08$$

$$t = 3 \text{ years}$$

$$n = mt = 1(3) = 3 \text{ periods}$$

Find: periodic payment R

The cash flow of this annuity is illustrated in the time diagram given below.

$$\begin{array}{cccc}
 P = 10,000 & & & \\
 & R = ? & R = ? & R = ? \\
 \hline
 & 0 & 1 & 2 & 3
 \end{array}$$

Since $P = R \frac{1 - (1 + j)^{-n}}{j}$, then

$$\begin{aligned} R &= \frac{P}{\frac{1 - (1 + j)^{-n}}{j}} \\ &= \frac{100,000}{\frac{1 - (1 + 0.08)^{-3}}{0.08}} \\ &= 38,803.35 \end{aligned}$$

Thus, the man should pay ₱38,803.35 every year for 3 years. □

Formulas

Future Value of an Ordinary Annuity: $F = Rs_{\overline{n}|} = R \frac{(1 + j)^n - 1}{j}$

Present Value of an Ordinary Annuity: $P = Ra_{\overline{n}|} = R \frac{1 - (1 + j)^{-n}}{j}$

Remember that an ordinary annuity is an annuity where payments are made at the end of each period.

(C) Seatwork

Seatwork 1. Find the present value P and amount F of the following ordinary annuities.

- a. Quarterly payments of ₱2,000 for 5 years with interest rate of 8% compounded quarterly

Answer: $F = ₱48,594.74$, $P = ₱32,702.87$

- b. Semi-annual payments of ₱8,000 for 12 years with interest rate of 12% compounded semi-annually

Answer: $F = ₱406,524.60$, $P = ₱100,402.90$

- c. Daily payments of ₱50 for 30 days with interest rate of 20% compounded daily

Answer: $F = ₱1,511.98$, $P = ₱1,487.33$

Note

The interest rate per conversion period is $\frac{0.2}{365}$, and there are 30 (not 30×365) conversion periods.

Seatwork 2. Answer the following problems.

- a. Peter started to deposit ₱5,000 quarterly in a fund that pays 1% compounded quarterly. How much will be in the fund after 6 years? **Answer:** $F = ₱123,514.09$

- b. The buyer of a lot pays ₱50,000 cash and ₱10,000 every month for 10 years. If money is 8% compounded monthly, how much is the cash value of the lot?

Answer: $CV = 50,000 + 824,214.81 = ₱874,214.81$

- c. How much should be invested in a fund each year paying 2% compounded annually to accumulate ₱100,000 in 5 years? **Answer:** $R = ₱19,215.84$

- d. A ₱50,000 loan is payable in 3 years. To repay the loan, the debtor must pay an amount every 6 months with an interest rate of 6% compounded semi-annually. How much should he pay every 6 months? **Answer:** $R = ₱9,229.88$
- e. An appliance is for sale at either (a) ₱15,999 cash or (b) on terms, ₱1,499 each month for the next 12 months. Money is 9% compounded monthly. Which is lower, the cash price or the present value of the installment terms? Explain.

Answer: Cash price since cash price (₱15,999) is less than the present value of all monthly payments (₱17,140.93)

(D) Evaluation

(a) Fill in the blanks.

a. A sequence of payments made at equal time periods is a/an _____. **Answer:** Annuity

b. A simple annuity in which the payments are made at the end of each period is a/an _____.
Answer: Ordinary annuity or annuity immediate

c. An annuity where the payment interval is not the same as the interest period is a/an _____.
Answer: General annuity

d. An annuity where the payment interval is the same as the interest period is a/an _____.
Answer: Simple annuity

e. An annuity in which payments begin and end at definite times is a/an _____.
Answer: Annuity certain

(b) Solve the following problems.

a. Find the present value and the amount (future value) of an ordinary annuity of ₱5,000 payable semi-annually for 10 years if money is worth 6% compounded semi-annually.

Answer: $P = ₱74,387.37$, $F = ₱134,351.87$

b. To pay for his debt at 12% compounded quarterly, Ruben committed for 8 quarterly payments of ₱28,491.28 each. How much did he borrow? **Answer:** ₱200,000

c. A high school student would like to save ₱50,000 for his graduation. How much should he deposit in a savings account every month for 5.5 years if interest is at 0.25% compounded monthly? **Answer:** ₱752.46

d. The buyer of a car pays ₱169,000 cash and ₱12,000 every month for 5 years. If money is 10% compounded monthly, how much is the cash price of the car?

Answer: Cash price = $169\,000 + 564784.43 = ₱733,784.43$

e. A television (TV) set is for sale at ₱13,499 in cash or on installment terms, ₱2,500 each month for the next 6 months at 9% compounded monthly. If you were the buyer, what would you prefer, cash or installment?

Answer: If you have cash on hand, it is better to buy the TV set in cash since the cash price (₱13,499) is lower than the cash value (₱14,613.99)

LESSON 29: General Annuity

TIME FRAME: 180 minutes

CODE: M11GM0II c-d-1 and d-2

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to find the future and present values of general annuities, and calculate the fair market value of a cash flow stream that includes an annuity.

LESSON OUTLINE:

1. Future value of a general Annuity
2. Present value of a general Annuity
3. Cash value of a cash flow

PREREQUISITE SKILLS: Knowledge of Simple Annuity

DEVELOPMENT OF THE LESSON

(A) Introduction/Motivation

Group the students. Allow them to recall how to find equivalent rates through a game. Some suggested games are as follows.

- Game 1

Write on strips of papers interest rates that they will be converting to the specified equivalent rates.

12% compounded monthly = ____ compounded annually

10% compounded quarterly = ____ compounded semi-annually

Put these strips of papers inside a box. Let each group draw 3 papers. For a given time period, let them compute for equivalent rates. The group with the most number of correct answers is the winner. You may also consider the fastest group with the highest number of correct answers.

- Game 2

Group the students. Let the students think of equivalent interest rates. Randomly pick a group who will call/tag another group to convert an interest rate to its equivalent rate at a given time span (one or two minutes). For example, the group will call the next group as ‘convert 12% compounded monthly to an interest rate compounded annually!’ If a group gets the correct answer, this group has the right to call/tag another group to convert equivalent rates. If the group did not get the correct equivalent rate, the group is out of the game. The remaining/last group in the game is the winner.

(B) Lesson Proper

(a) Let the students recall the definition of general annuity.

- **General Annuity** - an annuity where the payment interval is not the same as the interest compounding period
- **General Ordinary Annuity** - a general annuity in which the periodic payment is made at the end of the payment interval


Teaching Tip

The terms used in simple annuity have the same meanings in general annuity. The lesson will focus on general ordinary annuity only.

(b) Allow the students to provide real life examples of general annuity.

Examples of Annuity: monthly installment payment of a car, lot or house with an interest rate that is compounded annually; paying a debt semi-annually when the interest is compounded monthly

(c) Allow the students to illustrate the cash flow of a general annuity in a time diagram.

Annuity							
	R	R	R	R	R	\dots	R
0	1	2	3	4	5	\dots	n


Teaching Tip

Let the students observe that the time diagram is similar to a simple ordinary annuity. The main difference is that the payment interval is not the same as the interest compounding period.

- (d) Discuss how to compute the amount (future value) of a general ordinary annuity.

 **Teaching Tip**

The formula for the future value is just the same as that for a simple ordinary annuity. The **extra step** occurs in finding j : the given interest rate per period must be converted to an equivalent rate per payment interval.

Amount (Future) Value of a General Ordinary Annuity

	R	R	R	R	R	\dots	F
0	1	2	3	4	5	\dots	n

The amount or future value of general ordinary annuity is given by

$$F = R \frac{(1 + j)^n - 1}{j},$$

where R is the regular payment,
 j is the equivalent interest rate per payment interval, and
 n is the number of payments

EXAMPLE 1. Mel started to deposit ₱1,000 monthly in a fund that pays 6% compounded quarterly. How much will be in the fund after 15 years?

Solution.

Given: $R = 1,000$

$$n = 12(15) = 180 \text{ payments}$$

$$i^{(4)} = 0.06$$

$$m = 4$$

Find: F

The cash flow for this problem is shown in the diagram below.

	1,000	1,000	1,000	\dots	1,000	F
0	1	2	3	\dots	179	180

- (1) Since payments are monthly, the interest rate of 6% compounded quarterly must be converted to its equivalent interest rate that is compounded monthly.

Recall the procedure shown in Lesson 28:

$$\begin{aligned}
 F_1 &= F_2 \\
 P \left(1 + \frac{i^{(12)}}{12} \right)^{12t} &= P \left(1 + \frac{i^{(4)}}{4} \right)^{4t} \\
 \left(1 + \frac{i^{(12)}}{12} \right)^{12t} &= \left(1 + \frac{i^{(4)}}{4} \right)^{4t} \\
 \left(1 + \frac{i^{(12)}}{12} \right)^{12} &= (1.015)^4 \\
 1 + \frac{i^{(12)}}{12} &= [(1.015)^4]^{1/12} \\
 \frac{i^{(12)}}{12} &= (1.015)^{1/3} - 1 \\
 \frac{i^{(12)}}{12} &= 0.004975 = j
 \end{aligned}$$

Thus, the interest rate per monthly payment interval is 0.004975 or 0.4975%.

- (2) Apply the formula in finding the future value of an ordinary annuity using the computed equivalent rate

$$\begin{aligned}
 F &= R \frac{(1+j)^n - 1}{j} \\
 &= 1,000 \frac{(1+0.004975)^{180} - 1}{0.004975} \\
 &= 290,076.28
 \end{aligned}$$

Thus, Mel will have ₱290,076.28 in the fund after 20 years.

□

EXAMPLE 2. A teacher saves ₱5,000 every 6 months in a bank that pays 0.25% compounded monthly. How much will be her savings after 10 years?

Solution.

Given: $R = ₱5,000$

$n = 2(10) = 20$ payments

$i^{(12)} = 0.25\% = 0.0025$

$m = 12$

Find: F

The cash flow for this problem is shown in the diagram below.

0	5,000	5,000	5,000	...	5,000	5,000
1	2	3	...	19	20	

- (1) Convert 0.25% compounded monthly to its equivalent interest rate for each semi-annual payment interval.

$$\begin{aligned}
 F_1 &= F_2 \\
 P \left(1 + \frac{i^{(2)}}{2}\right)^{2t} &= P \left(1 + \frac{i^{(12)}}{12}\right)^{12t} \\
 \left(1 + \frac{i^{(2)}}{2}\right)^2 &= \left(1 + \frac{i^{(12)}}{12}\right)^{12} \\
 \left(1 + \frac{i^{(2)}}{2}\right)^2 &= (1.00020833)^{12} \\
 1 + \frac{i^{(2)}}{2} &= [(1.00020833)^{12}]^{1/2} \\
 \frac{i^{(2)}}{2} &= (1.00020833)^6 - 1 \\
 \frac{i^{(2)}}{2} &= 0.00125063 = j
 \end{aligned}$$

Thus, the interest rate per semi-annual payment interval is 0.00125063 or 0.125%.

- (2) Apply the formula in finding the present value of an ordinary annuity using the computed equivalent rate

$$\begin{aligned}
 F &= R \frac{(1+j)^n - 1}{j} \\
 &= 5,000 \frac{(1+0.00125063)^{20} - 1}{0.00125063} \\
 &= 101,197.06
 \end{aligned}$$

□

- (e) Discuss how to compute the present value of a general ordinary annuity

Teaching Tip

As with the future value formula, the formula for the present value of a general annual immediate involves an **extra step**: find an equivalent rate j with the same compounding period as the payment interval.

Present Value of an Ordinary Annuity

$$\begin{array}{cccccccc}
 & & P & & & & & \\
 & & R & R & R & R & \dots & R \\
 \hline
 & 0 & 1 & 2 & 3 & 4 & 5 & \dots & n
 \end{array}$$

The present value of an ordinary annuity is given by

$$P = R \frac{1 - (1 + j)^{-n}}{j},$$

where R is the regular payment,
 j is the equivalent interest rate per payment interval
 n is the number of payments

EXAMPLE 3. Ken borrowed an amount of money from Kat. He agrees to pay the principal plus interest by paying ₱38,973.76 each year for 3 years. How much money did he borrow if interest is 8% compounded quarterly?

Solution.

Given: $R = \text{₱}38,973.76$

$$i^{(4)} = 0.08$$

$$m = 4$$

$$n = 3 \text{ payments}$$

Find: present value P

The cash flow for this problem is shown in the diagram below.

$$\begin{array}{cccc}
 P = ? & & & \\
 & R = 38,973.76 & R = 38,973.76 & R = 38,973.76 \\
 \hline
 & 0 & 1 & 2 & 3
 \end{array}$$

(1) Convert 8% compounded quarterly to its equivalent interest rate for each payment interval.

Again recall the procedure shown in Lesson 28.

$$\begin{aligned}
 F_1 &= F_2 \\
 P \left(1 + \frac{i^{(1)}}{1} \right)^{(1)t} &= P \left(1 + \frac{i^{(4)}}{4} \right)^{4t} \\
 \left(1 + \frac{i^{(1)}}{1} \right) &= \left(1 + \frac{0.08}{4} \right)^4 \\
 \frac{i^{(1)}}{1} &= (1.02)^4 - 1 \\
 \frac{i^{(1)}}{1} &= j = 0.082432 = 8.24\%
 \end{aligned}$$

Thus, the interest rate per payment interval is 0.082432 or 8.2432%.

- (2) Apply the formula in finding the present value of an ordinary annuity using the computed equivalent rate $j = 0.082432$.

$$\begin{aligned}
 P &= R \left[\frac{1 - (1 + j)^{-n}}{j} \right] \\
 &= 38,973.76 \left[\frac{1 - (1 + 0.082432)^{-3}}{0.082432} \right] \\
 &= 100,000.00
 \end{aligned}$$

Hence, Ken borrowed ₱100,000 from Kat.

□

EXAMPLE 4. Mrs. Remoto would like to buy a television (TV) set payable monthly for 6 months starting at the end of the month. How much is the cost of the TV set if her monthly payment is ₱3,000 and interest is 9% compounded semi-annually?

Solution.

Given: $R = 3,000$

$$i^{(2)} = 0.09$$

$$m = 2$$

$$n = 6 \text{ payments}$$

Find: cost (present value) at the beginning of the term P

- (1) Convert the given to its equivalent interest rate for each payment interval. Again recall the procedure shown in Lesson 28.

$$\begin{aligned}
 F_1 &= F_2 \\
 P \left(1 + \frac{i^{(12)}}{12} \right)^{(12)t} &= P \left(1 + \frac{i^{(2)}}{2} \right)^{(2)t} \\
 \left(1 + \frac{i^{(12)}}{12} \right)^{12} &= \left(1 + \frac{0.09}{2} \right)^2 \\
 \left(1 + \frac{i^{(12)}}{12} \right)^{12} &= (1 + 0.045)^2 \\
 1 + \frac{i^{(12)}}{12} &= [(1.045)^2]^{1/12} \\
 \frac{i^{(12)}}{12} &= (1.045)^{1/6} - 1 \\
 \frac{i^{(12)}}{12} &= 0.00736312 = j
 \end{aligned}$$

Thus, the interest rate per monthly payment interval is 0.00736312 or 0.736312

- (2) Apply the formula in finding the present value of an ordinary annuity using the computed equivalent rate $j = 0.00736312$.

$$\begin{aligned}
 P &= R \left(\frac{1 - (1 + j)^{-n}}{j} \right) \\
 &= 3,000 \left(\frac{1 - (1 + 0.00736312)^{-6}}{0.00736312} \right) \\
 &= 17,545.08
 \end{aligned}$$

Thus, the cost of the TV set is ₱17,545.08

□

We will now apply the concepts of present and future values to solve problems about cash flows.

A **cash flow** is a term that refers to payments received (cash inflows) or payments or deposits made (cash outflows). Cash inflows can be represented by positive numbers and cash outflows can be represented by negative numbers.

The **fair market value** or **economic value** of a cash flow (payment stream) on a particular date refers to a *single amount* that is equivalent to the value of the payment stream at that date. This particular date is called the **focal date**.

EXAMPLE 5. Mr. Ribaya received two offers on a lot that he wants to sell. Mr. Ocampo

has offered ₱50 000 and ₱1 million lump sum payment 5 years from now. Mr. Cruz has offered ₱50,000 plus ₱40,000 every quarter for five years. Compare the fair market values of the two offers if money can earn 5% compounded annually. Which offer has a higher market value?

Solution.

Given:

Mr. Ocampo's offer	Mr. Cruz's offer
₱50,000 down payment	₱50,000 down payment
₱1,000,000 after 5 years	₱40,000 every quarter for 5 years

Find: fair market value of each offer

(1) Illustrate the cash flows of the two offers using time diagrams.

Mr. Ocampo's offer:

50,000						1,000,000
0	1	2	3	...		20

Mr. Cruz's offer:

50,000	40,000	40,000	40,000	...	40,000
0	1	2	3	...	20

(2) Choose a **focal date** and determine the values of the two offers at that focal date.

 **Teaching Tip**

It is usually convenient to choose focal dates to either be at the start or at the end of the term.

SOLUTION 1. Choose the focal date to be the start of the term. Since the focal date is at $t = 0$, compute for the present value of each offer.

Mr. Ocampo's offer:

Since ₱50,000 is offered today, then its present value is still ₱50,000.

The present value of ₱1,000,000 offered five years from now is:

$$\begin{aligned}
 P &= F(1 + j)^{-n} \\
 &= 1,000,000(1 + 0.05)^{-5} \\
 &= ₱783,526.17
 \end{aligned}$$

$$\begin{aligned}
 \text{Fair Market Value (FMV)} &= 50,000 + 783,526.17 \\
 &= ₱833,526.17
 \end{aligned}$$

Mr. Cruz's offer:

Compute for the present value of a general annuity with quarterly payments but with annual compounding at 5%.

Solve the equivalent rate of 5% compounded annually.

$$\begin{aligned}
 F_1 &= F_2 \\
 P \left(1 + \frac{i^{(4)}}{4} \right)^{(4)(t)} &= P \left(1 + \frac{i^{(1)}}{1} \right)^{(1)(t)} \\
 \left(1 + \frac{i^{(4)}}{4} \right)^4 &= \left(1 + \frac{0.05}{1} \right)^1 \\
 1 + \frac{i^{(4)}}{4} &= (1.05)^{1/4} \\
 \frac{i^{(4)}}{4} &= (1.05)^{1/4} - 1 \\
 i^{(4)} &= 0.012272
 \end{aligned}$$

The present value of an annuity is given by

$$\begin{aligned}
 P &= R \frac{1 - (1 + j)^{-n}}{j} \\
 &= 40,000 \frac{1 - (1 + 0.012272)^{-20}}{0.012272} \\
 &= \text{P}705,572.68
 \end{aligned}$$

$$\begin{aligned}
 \text{Fair Market Value (FMV)} &= (\text{Downpayment}) + (\text{Present Value}) \\
 &= 50,000 + 705,572.68 \\
 &= \text{P}755,572.68
 \end{aligned}$$

Hence, Mr. Ocampo's offer has a higher market value. The difference between the market values of the two offers at the start of the term is

$$833,526.17 - 755,572.68 = \text{P}77,953.49$$

SOLUTION 2. Choose the focal date to be the end of the term.

Mr. Ocampo's offer:

At the end of the term, ₱1,000,000 is valued as such (because this is the value at $t = 5$).

The future value of ₱50,000 at the end of the term at 5% compounded annually is given by

$$\begin{aligned} F &= P(1 + j)^n \\ &= 50,000(1 + 0.05)^5 \\ &= \text{₱}63,814.08 \end{aligned}$$

$$\begin{aligned} \text{Fair Market Value} &= 63,814.08 + 1,000,000 \\ &= \text{₱}1,063,814.08 \end{aligned}$$

Mr. Cruz's offer:

The future value of this ordinary general annuity is given by:

$$\begin{aligned} F &= R \frac{(1 + j)^n - 1}{j} \\ &= 40,000 \frac{(1 + 0.012272)^{20} - 1}{0.012272} \\ &= \text{₱}900,509.40 \end{aligned}$$

The future value of ₱50,000 at the end of the term is ₱63,814.08, which was already determined earlier.

$$\begin{aligned} \text{Fair Market Value} &= 63,814.08 + 900,509.40 \\ &= \text{₱}964,323.48 \end{aligned}$$

As expected, Mr. Ocampo's offer still has a higher market value, even if we the focal date to be the end of the term. The difference between the market values of the two offers at the end of the term is

$$1,063,814.08 - 964,323.48 = \text{₱}99,490.60.$$

You can also check that the present value of the difference is the same as the difference computed when the focal date was the start of the term:

$$P = 99,490.60(1 + 0.05)^{-5} = \text{₱}77,953.49$$

□

EXAMPLE 6. Company A offers ₱150,000 at the end of 3 years plus ₱300,000 at the end of 5 years. Company B offers ₱25,000 at the end of each quarter for the next 5 years. Assume that money is worth 8% compounded annually. Which offer has a better market value?

Solution.

Given:

Company A	Company B
₱150,000 at the end of 3 years	₱25,000 at the end of each
₱300,000 at the end of 5 years	quarter for the next 5 years

Find: fair market value of each offer

- i. Illustrate the cash flows of the two offers using time diagrams.

Company A Offer:

			150,000		300,000
0	1	2	3	4	5

Company B Offer:

	25,000	25,000	25,000	...	25,000
0	1	2	3	...	20

- ii. Choose a focal date and determine the values of the two offers at that focal date. Suppose that selected focal date is the start of the term.
- iii. Since the focal date is the start of the term, compute for the present value of each offer.

Company A Offer:

The present value of ₱150,000 three years from now is

$$\begin{aligned}
 P_1 &= F(1 + j)^{-n} \\
 &= 150,000(1 + 0.04)^{-6} \\
 &= \text{₱}118,547.18
 \end{aligned}$$

The present value of ₱300,000 five years from now is

$$\begin{aligned}
 P_2 &= F(1 + j)^{-n} \\
 &= 300,000(1 + 0.04)^{-10} \\
 &= \text{₱}202,669.25
 \end{aligned}$$

$$\begin{aligned}
 \text{Fair Market Value (FMV)} &= P_1 + P_2 \\
 &= 118,547.18 + 202,669.25 \\
 &= \text{₱}321,216.43
 \end{aligned}$$

Company B Offer:

Compute for the present value of a general annuity with quarterly payments but with semi-annual compounding at 8%.

Solve the equivalent rate, compounded quarterly, of 8% compounded semi-annually.

$$\begin{aligned}
 F_1 &= F_2 \\
 P \left(1 + \frac{i^{(4)}}{4}\right)^{4(5)} &= P \left(1 + \frac{i^{(2)}}{2}\right)^{2(5)} \\
 \left(1 + \frac{i^{(4)}}{4}\right)^{20} &= \left(1 + \frac{0.08}{2}\right)^{10} \\
 1 + \frac{i^{(4)}}{4} &= (1.04)^{1/2} \\
 \frac{i^{(4)}}{4} &= (1.04)^{1/2} - 1 \\
 \frac{i^{(4)}}{4} &= 0.019803903
 \end{aligned}$$

The present value of an annuity is given by

$$\begin{aligned}
 P &= R \frac{1 - (1 + j)^{-n}}{j} \\
 &= 25,000 \frac{1 - (1 + 0.019803903)^{-20}}{0.019803903} \\
 &= \text{P}409,560.4726
 \end{aligned}$$

Therefore, Company B's offer is preferable since its market value is larger.

□

(C) Seatwork

Seatwork 1. Find the present value P and amount F of the following.

- (a) Semi-annual payments of P500 at the end of each term for 10 years with interest rate of 5% compounded semi-annually **Answer:** P7,794.58; P12,772.33 (use $j = 0.025$, $n=20$)
- (b) Annual payments of P1,000 at the end of each term for 8 years with interest rate of 6% compounded quarterly **Answer:** P6,176.42; P9,946.04 (use $j = 0.061364$, $n=8$)
- (c) Quarterly payments of P2,000 at the end of each term for 5 years with interest rate of 8% compounded annually **Answer:** P32,884.42; P48,318.22 (use $j = 0.019427$, $n=20$)
- (d) Semi-annual payments of P8,000 at the end of each term for 12 years with interest rate of 12% compounded quarterly **Answer:** P99,573.23; P411,461.66 (use $j = 0.0609$, $n=24$)

Seatwork 2. Answer the following problems.

- (a) On a girl's 10th birthday, her father started to deposit ₱5,000 quarterly at the end of each term in a fund that pays 1% compounded monthly. How much will be in the fund on his daughter's 17th birthday? **Answer:** ₱144,832.94 (use $j = 0.002502$, $n=28$)
- (b) The buyer of a lot pays ₱10,000 every month for 10 years. If money is 8% compounded annually, how much is the cash value of the lot? **Answer:** ₱834,325.22 (use $j = 0.006434$, $n=120$)
- (c) In order to save for her high school graduation, Kathrina decided to save ₱200 at the end of every other month, starting the end of the second month. If the bank pays 0.250% compounded monthly, how much will be her money at the end of 5 years? **Answer:** ₱6,036.42 (use $j = 0.000417$, $n=30$)
- (d) Mr. Bajada paid ₱200,000 as down payment for a farm. The remaining amount (liability) is to be settled by paying ₱16,200 at the end of each month for 5 years. If interest is 5% compounded semi-annually, what is the cash price of his farm? **Answer:** ₱1,059,519.92 (use $j = 0.0041239$, $n=60$)

(D) Evaluation

Solve the following problems.

- (a) Find the present value and amount of an ordinary annuity of ₱5,000 payable semi-annually for 10 years if money is worth 9% compounded annually. **Answer:** ₱65,589.25; ₱155,274.64 (use $j = 0.044031$, $n=20$)
- (b) To pay for his debt at 12% compounded semi-annually, Ruben committed for 8 quarterly payments of ₱24,491.28 each. How much did he borrow? **Answer:** ₱172,238.61 (use $j = 0.029563$, $n=8$)
- (c) The buyer of a car pays ₱169,000 cash and ₱12,000 every month for 5 years. If money is 10% compounded annually, how much is the cash price of the car? **Answer:** ₱739,464.24 (use $j = 0.007974$, $n=60$)
- (d) A television (TV) set is for sale at ₱13,499 in cash or on installment terms, ₱2,500 each month for the next 6 months at 9% compounded annually. If you were the buyer, what would you prefer, cash or installment?
Answer: Present value of installment is ₱14,628.79 (use $j = 0.007207$, $n=6$); better to pay ₱13,499 if cash is available, or wait until you can save enough cash than to pay by installment

Performance Task

Find situations involving annuities in your community. For example:

- Go to an appliance store, and ask how much a certain appliance costs if it is (a) paid in full, or (b) paid by installment.
- If you know someone borrowing from a five-six moneylender, you can ask how much you will be charged if you want to loan ₱1,000, payable in 1 year.

For the situation you chose, determine the interest rate for the period and the annual interest rate.

For example, if an appliance that costs ₱15,000 can be paid in 8 monthly payments of ₱2,000, then the formula $15,000 = 2,000 \frac{1 - (1 + j)^{-8}}{j}$ must be satisfied. Experiment with different values for j to determine the interest rate for the period and the annual interest rate.

Based on the interest rates you computed, do you think it is a good idea to loan?

[Possible answers: If I can wait before buying a certain item, it may be better to just save until I have enough money, rather than pay now with a large interest. However, if the need is urgent, then I might need to borrow with interest.]

LESSON 30: Deferred Annuity

TIME FRAME: 2 hours

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to calculate the present value and period of deferral of a deferred annuity.

LESSON OUTLINE:

1. Deferred annuity
2. Present value of a deferred annuity
3. Period of deferral of a deferred annuity

PREREQUISITE SKILLS: Knowledge of Exponential Functions

DEVELOPMENT OF THE LESSON

(A) Introduction/Motivation

Ask the students what they would like to do on their 18th birthday. Then describe to them the following situation.

Iya, who is celebrating her 17th birthday today, does not want an extravagant party for her 18th birthday. Instead, she asks her parents if she could receive ₱500 per month until her 21st birthday. Iya's mom decided to save today so that she can provide extra allowance every month after Iya's 18th birthday. But how much should the mother save today so that she will have ₱200 every month for 3 years which is due exactly one year from now?

Tell the students that this is an example of a deferred annuity. Tell them that deferred annuities are series of payments, as what they have already learned in the past lessons on annuities, but will start on a later date.

Discuss some examples of this type of annuity in real life.

- A credit card company offering its clients to purchase today but to start paying monthly with their choice of term after 3 months.
- A real estate agent is urging a condominium unit buyer to purchase now and start paying after 3 years when the condominium is ready for occupancy.
- A worker who has gained extra income now and wants to save his money so that he can withdraw his money monthly starting on the day of his retirement from work.

(B) Lesson Proper

(a) Review definition of terms.

- **Annuity** - a sequence of payments made at equal(fixed) intervals or periods of time
- **Annuity Immediate or Ordinary Annuity** - a type of annuity in which the payments are made at the end of each period

(b) Present definition of terms.

- **Deferred Annuity** - an annuity that does not begin until a given time interval has passed
- **Period of Deferral** - time between the purchase of an annuity and the start of the payments for the deferred annuity

(c) Review how to calculate the present value of **an annuity immediate or ordinary annuity** and give an example.

Present Value of an Ordinary Annuity

P								
	R	R	R	R	R	\dots	R	
0	1	2	3	4	5	\dots	n	

The present value of an ordinary annuity is given by

$$P = Ra_{\overline{n}|j} = R \frac{1 - (1 + j)^{-n}}{j},$$

where R is the regular payment,
 j is the interest rate per period, and
 n is the number of payments

Review. Suppose Mr. Gran wants to purchase a cellular phone. He decided to pay monthly for 1 year starting at the end of the month. How much is the cost of the cellular phone if his monthly payment is ₱2,500 and interest is at 9% compounded monthly?

Solution.

Given: $R = 2,500$

$$i^{(12)} = 0.09$$

$$t = 1$$

$$m = 12$$

Find: P

	$2,500$	$2,500$	\dots	R
0	1	2	\dots	n

The interest per period is $j = \frac{i^{(12)}}{m} = \frac{0.09}{12} = 0.0075$.

The number of payments is $n = mt = 12(1) = 12$.

The present value of the annuity is

$$P = Ra_{\overline{n}|j} = 2,500 \frac{1 - (1 + 0.0075)^{-12}}{0.0075} = 28,587.28.$$

Thus, the cost of the cellular phone now is ₱28,587.28. □

- (d) Define deferred annuity and illustrate. Give the formula for finding the present value.

- a. Discuss the following problem as an introduction to deferred annuity. Emphasize that for deferred annuities, the start of the payments is on a later date.

Recall the previous example. What if Mr. Gran is considering another cellular phone that has a different payment scheme? In this scheme, he has to pay ₱2,500 for 1 year starting at the end of the fourth month. If the interest rate is also 9% converted monthly, how much is the cash value of the cellular phone?

Note that the two payment schemes have the same number of payments n and the same interest rate per period j . Their main difference is the start of the payments. The first scheme started at the end of the first interval which makes it an **ordinary annuity**. The second scheme started on a later date. This annuity is called **deferred annuity**.

In this example, Mr. Gran pays starting at **the end of the 4th month to the end of the 15th month**. The time diagram for this option is given by:

$$\begin{array}{cccccccc}
 P = ? & & & & & & & \\
 \hline
 & & & & 2,500 & 2,500 & \cdots & R \\
 0 & 1 & 2 & 3 & 4 & 5 & \cdots & 15
 \end{array}$$

Now, how do we get the present value of this annuity?

Step 1: We first assume payments are also being made during the period of deferral; in other words, there are no skipped payments. The associated time diagram is:

$$\begin{array}{cccccccc}
 P' = ? & & & & & & & \\
 \hline
 & 2,500 & 2,500 & 2,500 & 2,500 & 2,500 & \cdots & R \\
 0 & 1 & 2 & 3 & 4 & 5 & \cdots & 15
 \end{array}$$

From the previous lesson, the present value P' of this ordinary annuity is given by

$$P' = Ra_{\overline{n}|j} = R \frac{1 - (1 + j)^{-n}}{j} = 2,500 \frac{1 - (1 + 0.0075)^{-15}}{0.0075} = 35,342.49$$

Step 2: We now subtract the present value of the payments made during the period of deferral. Again, based on the previous lesson, the present value P^* of the 3 payments during the period of deferral is

$$P^* = Ra_{\overline{n}|j} = R \frac{1 - (1 + j)^{-n}}{j} = 2,500 \frac{1 - (1 + 0.0075)^{-3}}{0.0075} = 7,388.89$$

Step 3: Since the payments in the period of deferral are **artificial payments**, we subtract the present value of these payments. We obtain:

$$P = P' - P^* = 35,342.49 - 7,388.89 = 27,953.60$$

Thus, the present value of the cellular phone is ₱27,953.60.

Comparing the present values of the two schemes, the present value in the second scheme is lower than the present value in the first because the payments in the second scheme will be received on a later date.

- b. Derive the formula for calculating the present value of a deferred annuity by generalizing the procedure from the previous example.

Consider the following time diagram where k artificial payments of R^* are placed in the period of deferral.

	R^*	R^*	\dots	R^*	R	R	\dots	R
0	1	2	\dots	k	$k + 1$	$k + 2$	\dots	$k + n$

Present Value of a Deferred Annuity

P								
					R	R	\dots	R
0	1	2	\dots	k	$k + 1$	$k + 2$	\dots	$k + n$

The present value of a deferred annuity is given by

$$P = Ra_{\overline{k+n}|j} - R_{\overline{k}|j} = R \frac{1 - (1 + j)^{-(k+n)}}{j} - R \frac{1 - (1 + j)^{-k}}{j},$$

where R is the regular payment,
 i is the interest rate per period,
 n is the number of payments, and
 k is the number of conversion periods in the period of deferral
 (or number of artificial payments)

EXAMPLE 1. On his 40th birthday, Mr. Ramos decided to buy a pension plan for himself. This plan will allow him to claim ₱10,000 quarterly for 5 years starting 3 months after his 60th birthday. What one-time payment should he make on his 40th birthday to pay off this pension plan, if the interest rate is 8% compounded quarterly?

Solution.Given: $R = 10,000$

$$i^{(4)} = 0.08$$

$$t = 5$$

$$m = 4$$

Find: P

The annuity is deferred for 20 years and it will go on for 5 years. The first payment is due three months (one quarter) after his 60th birthday, or at the end of the 81st conversion period. Thus, there are 80 artificial payments.

Number of artificial payments: $k = mt = 4(20) = 80$ Number of actual payments: $n = mt = 4(5) = 20$ Interest rate per period: $j = \frac{i^{(4)}}{m} = \frac{0.08}{4} = 0.02$

If you assume that there are payments in the period of deferral, there would be a total of payments.

Time Diagram:

P									
						10,000	10,000	...	10,000
0	1	2	...	80	81	82	100

Thus, the present value of the deferred annuity can be solved as

$$\begin{aligned}
 P &= Ra_{\overline{k+n}|} - Ra_{\overline{k}|} = R \frac{1 - (1+j)^{-(k+n)}}{j} - R \frac{1 - (1+j)^{-k}}{j} \\
 &= 10,000 \frac{1 - (1+0.02)^{-100}}{0.02} - 10,000 \frac{1 - (1+0.02)^{-80}}{0.02} \\
 &= 33,538.38
 \end{aligned}$$

Therefore, the present value of these monthly pensions is ₱33,538.38. □

EXAMPLE 2. A credit card company offers a deferred payment option for the purchase of any appliance. Rose plans to buy a smart television set with monthly payments of ₱4,000 for 2 years. The payments will start at the end of 3 months. How much is the cash price of the TV set if the interest rate is 10% compounded monthly?


Teaching Tip

The first payment is due at the end of 3 months, or at the end of the 3rd conversion period. Thus, there are 2 artificial payments.

Solution.

Given: $R = 4,000$

$$i^{(12)} = 0.10$$

$$t = 2$$

$$m = 12$$

Find: P

The annuity is deferred for 2 months, and payments will go on for 2 years. Number of artificial payments

Number of artificial payments: $k = 2$

Number of actual payments: $n = mt = 12(2) = 24$

Interest rate per period: $j = \frac{i^{(12)}}{m} = \frac{0.10}{12} = 0.00833$

If you assume that there are payments in the period of deferral, there would be a total of $k + n = 2 + 24 = 26$ payments.

Time Diagram:

P						
			4,000	4,000	...	4,000
0	1	2	3	4	...	26

Thus, the present value of the deferred annuity can be solved as

$$\begin{aligned}
 P &= R \frac{1 - (1 + j)^{-(k+n)}}{j} - R \frac{1 - (1 + j)^{-k}}{j} \\
 &= 4,000 \frac{1 - (1 + 0.00833)^{-26}}{0.00833} - 4,000 \frac{1 - (1 + 0.00833)^{-2}}{0.00833} \\
 &= 85,260.53
 \end{aligned}$$

Therefore, the cash price of the TV set is **₱85,260.53**. □

(C) Seatwork

Seatwork 1. Find the period of deferral in each of the following deferred annuity problem (one way to find the period of deferral is to count the number of artificial payments)

- (a) Monthly payments of ₱2,000 for 5 years that will start 7 months from now

Solution. The first payment is at time 7. The period of deferral will be from time 0 to 6, which is equivalent to **6 periods or 6 months.**

- (b) Annual payments of ₱8,000 for 12 years that will start 5 years from now

Solution. Five years from now is at time 5. The period of deferral will be from time 0 to time 4. Thus, the period of deferral is **4 periods or 4 years.**

- (c) Quarterly payments of ₱ 5,000 for 8 years that will start two years from now.

Solution. Two years from now will be at time 8 if one quarter is considered as one period. Thus, the period of deferral is from time 0 to time 7, which is equivalent to **7 quarters or 7 periods.**

- (d) Semi-annual payments of ₱60,000 for 3 years that will start 5 years from now

Solution. The first payment is due five years from now which is equivalent to time 10 if payments are made semi-annually. The period of deferral will be from time 0 to time 9, which is equivalent to **9 semi-annual intervals or 9 periods.**

- (e) Payments of ₱3,000 every 2 years for 10 years starting at the end of 6 years

Solution. The first payment is due at the end of 6 years which is at time 3 if payments are made every 2 years. The period of deferral is from time 0 to time 2, which is equivalent to **2 periods or 2 two-year intervals.**

Seatwork 2. Answer the following problems completely.

- (a) Emma availed of a cash loan that gave her an option to pay ₱10,000 monthly for 1 year. The first payment is due after 6 months. How much is the present value of the loan if the interest rate is 12% converted monthly? **Answer: ₱107,088.20**
- (b) Adrian purchased a laptop through the credit cooperative of their company. The cooperative provides an option for a deferred payment. Adrian decided to pay after 4 months of purchase. His monthly payment is computed as ₱3,500 payable in 12 months. How much is the cash value of the laptop if the interest rate is 8% convertible monthly? **Answer: ₱39,441.14**
- (c) Mr. and Mrs. Mercado decided to sell their house and to deposit the fund in a bank. After computing the interest, they found out that they may withdraw ₱350,000 yearly for 4 years starting at the end of 7 years when their child will be in college. How much is the fund deposited if the interest rate is 3% converted annually? **Answer: ₱1,089,533.99**

- (d) A group of employees decided to invest a portion of their bonus. After 3 months from today, they want to withdraw from this fund ₱5,000 monthly for 12 months to fund their gathering that they decide to do every month. How much is the total deposit now if the interest rate is 5% converted monthly? **Answer: ₱57,922.41**
- (e) Anna converted her loan to light payments which gives her an option to pay ₱1,500 every month for 2 years. The first payment is due 3 months from now. How much is the amount of the loan if the interest rate is 9% converted monthly? **Answer: ₱32,346.70**

(D) Evaluation

- (a) Read the following annuity problem. Fill in the blanks in the statements that follow.
A loan of ₱30,000 is to be repaid monthly for 5 years that will start at the end of 4 years. If interest rate is 12% converted monthly, how much is the monthly payment?
- The type of annuity illustrated in the problem is a _____. **Answer: deferred annuity**
 - The total number of payments is _____. **Answer: 60**
 - The number of conversion periods in the period of deferral is _____. **Answer: 47**
 - The interest rate per period is _____. **Answer: 0.01**
 - The present value of the loan is _____. **Answer: ₱30,000**
- (b) Find the period of deferral in the following problems.
- Monthly payments of ₱10,000 for 8 years that will start 6 months from now **Answer: 5 months or 5 periods**
 - Semi-annual payments of ₱15,000 for 10 years that will start 5 years from now **Answer: 9 periods or 9 half-year intervals**
 - Payments of ₱5,000 every 4 months for 10 years that will start five years from now. **Answer: 14 periods or 14 4-month intervals**
 - Annual payments of ₱600 for 20 years that will start 10 years from now
Answer: 9 years or 9 periods
 - Payments of ₱3,000 every 3 years for 12 years starting at the end of 9 years
Answer: 2 periods or 2 three-year intervals
- (c) Solve the following problems completely.
- A loan is to be repaid quarterly for 5 years that will start at the end of 2 years. If interest rate is 6% converted quarterly, how much is the loan if the quarterly payment is ₱10,000?
Answer: ₱154,694.03
 - A car is to be purchased in monthly payments of ₱17,000 for 4 years starting at the end of 4 months. How much is the cash value of the car if the interest rate used is 12% converted monthly?
Answer: ₱626,571.56

- c. A mother's savings may allow her to withdraw ₱50,000 semi-annually for 5 years starting at the end of 5 years. How much is the mother's savings if the interest rate is 8% converted semi-annually? **Answer: ₱284,930.39**
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Chapter 8

Basic Concepts of Stocks and Bonds

LESSON 31: Stocks and Bonds

TIME FRAME: 120 minutes

CODE: M11GM-IIe-1 and e-2

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to 1) illustrate stocks and bonds and 2) distinguish between stocks and bonds.

LESSON OUTLINE:

1. Definition of terms related to stocks
 2. Definition of terms related to bonds
 3. Examples
-

DEVELOPMENT OF THE LESSON

(A) Introduction/Motivation

- Stocks

Ask the students who among them want to be part owners of the big companies in the Philippines. Tell the students that it is not impossible to be part owners of these companies. Introduce to the students how the stock market works. Tell them that if they buy stocks of a certain company, they become one of the many owners. Being one of the owners, they are entitled to the earnings of the company. They may get their earnings through dividends or they may opt to sell their stocks at a higher price the moment the market value has increased.

Discuss the following examples:

- (a) Five years ago, Ms. Salceda bought 500 shares of stocks in a certain corporation worth ₱48.00 each. Now, each share is worth ₱60.50.
- (b) Mr. Tagle bought 1,000 shares of stocks in a corporation that had issued 100,000 shares. This means Mr. Tagle acquired 1% of the total shares.
- (c) A certain corporation declared to give ₱100,000,000 dividend to the common stockholders. If there are 1,000,000 shares, then there will be ₱100 dividend per share.

- Bonds

Now ask students if they ever thought they could fund big companies or even the government. Tell the students that big companies or the government often need large amounts of money for their projects. To raise money, they can issue bonds. Investors who purchase bonds are essentially 'lenders' to the issuer. However, the investors should be compensated for the lending their money. Aside from being paid the loan at the end of a fixed amount of time, the investor also receives regular payments (called coupons), usually every six months.

Discuss the following examples:

- (a) Ms. Ante bought a 10% bond for ₱100,000. After 10 years, she receives ₱100,000 back. She also receives $\text{₱}100,000(0.10)/2 = \text{₱}5,000$ every six months for 10 years.
- (b) Mr. de la Cruz is offered an 8% bond for ₱50,000. The bond has a face value of ₱50,000 with maturity date exactly 5 years from now. He receives $\text{₱}50,000(0.08)/2 = \text{₱}2,000$ every six months for 5 years.

(B) Lesson Proper

1. Illustrate stocks and bonds.

- Stocks¹

Some corporations may raise money for their expansion by issuing stocks. Stocks are shares in the ownership of the company. Owners of stocks may be considered as part owners of the company. There are two types of stocks: common stock and preferred stock. Both will receive dividends or share of earnings of the company. Dividends are paid first to preferred shareholders.

Stocks can be bought or sold at its current price called the market value. When a person buys some shares, the person receives a certificate with the corporation's name, owner's name, number of shares and par value per share.

- Bonds²

¹Stock Basics Tutorial, Accessed from <http://www.investopedia.com/university/stocks/>

²Bonds Basics Tutorial, Accessed from <http://www.investopedia.com/university/bonds/>

Bonds are interest bearing security which promises to pay amount of money on a certain maturity date as stated in the bond certificate. Unlike the stockholders, bondholders are lenders to the institution which may be a government or private company. Some bond issuers are the national government, government agencies, government owned and controlled corporations, non-bank corporations, banks and multilateral agencies.

Bondholders do not vote in the institution's annual meeting but the first to claim in the institution's earnings. On the maturity date, the bondholders will receive the face amount of the bond. Aside from the face amount due on the maturity date, the bondholders may receive coupons (payments/interests), usually done semi-annually, depending on the coupon rate stated in the bond certificate.

2. Distinguish between stocks and bonds.

Stocks	Bonds
A form of equity financing or raising money by allowing investors to be part owners of the company.	A form of debt financing, or raising money by borrowing from investors
Stock prices vary every day. These prices are reported in various media (newspaper, TV, internet, etc).	Investors are guaranteed interest payments and a return of their money at the maturity date
Investing in stock involves some uncertainty. Investors can earn if the stock prices increase, but they can lose money if the stock prices decrease or worse, if the company goes bankrupt.	Uncertainty comes from the ability of the bond issuer to pay the bondholders. Bonds issued by the government pose less risk than those by companies because the government has guaranteed funding (taxes) from which it can pay its loans.
Higher risk but with possibility of higher returns	Lower risk but lower yield
Can be appropriate if the investment is for the long term (10 years or more). This can allow investors to wait for stock prices to increase if ever they go low.	Can be appropriate for retirees (because of the guaranteed fixed income) or for those who need the money soon (because they cannot afford to take a chance at the stock market)

3. Definition of terms in relation to stocks

- **Stocks** - share in the ownership of a company
- **Dividend** - share in the company's profit
- **Dividend Per Share** - ratio of the dividends to the number of shares

- **Stock Market** - a place where stocks can be bought or sold. The stock market in the Philippines is governed by the Philippine Stock Exchange (PSE)
- **Market Value** - the current price of a stock at which it can be sold
- **Stock Yield Ratio** - ratio of the annual dividend per share and the market value per share. Also called current stock yield.
- **Par Value** - the per share amount as stated on the company certificate. Unlike market value, it is determined by the company and remains stable over time

4. Give examples to illustrate these terms.

EXAMPLE 1. A certain financial institution declared a ₱30,000,000 dividend for the common stocks. If there are a total of 700,000 shares of common stock, how much is the dividend per share?

Given: Total Dividend = ₱30,000,000

Total Shares = 700,000

Find: Dividend per Share

$$\begin{aligned} \text{Dividend per Share} &= \frac{\text{Total Dividend}}{\text{Total Shares}} \\ &= \frac{30,000,000}{700,000} \\ &= 42.86 \end{aligned}$$

Therefore, the dividend per share is ₱42.86.

EXAMPLE 2. A certain corporation declared a 3% dividend on a stock with a par value of ₱500. Mrs Lingan owns 200 shares of stock with a par value of ₱500. How much is the dividend she received?

Solution.

Given: Dividend Percentage = 3%

Par Value = ₱500

Number of Shares = 200

Find: Dividend

The dividend per share is $\text{₱}500 \times 0.03 = \text{₱}15$. Since there are 300 shares, the total dividend is $\text{₱}15/\text{share} \times 200 \text{ shares} = \text{₱}3,000$.

In summary,

$$\begin{aligned}\text{Dividend} &= \text{Dividend Percentage} \times \text{Par Value} \times \text{Number of Shares} \\ &= 0.03(500)(200) \\ &= 3,000\end{aligned}$$

Thus, the dividend is ₱3,000. □

EXAMPLE 3. Corporation A, with a current market value of ₱52, gave a dividend of ₱8 per share for its common stock. Corporation B, with a current market value of ₱95, gave a dividend of ₱12 per share. Use the **stock yield ratio** to measure how much dividends shareholders are getting in relation to the amount invested.

Solution. Corporation A:

Given: Dividend per Share = ₱8

Market Value = ₱52

Find: Stock Yield Ratio

$$\begin{aligned}\text{Stock Yield Ratio} &= \frac{\text{Dividend per Share}}{\text{Market Value}} \\ &= \frac{8}{52} \\ &= 0.1538 = 15.38\%\end{aligned}$$

Corporation B:

Given: Dividend per Share = ₱12

Market Value = ₱95

Find: Stock Yield Ratio

$$\begin{aligned}\text{Stock Yield Ratio} &= \frac{\text{Dividend per Share}}{\text{Market Value}} \\ &= \frac{12}{95} \\ &= 0.1263 = 12.63\%\end{aligned}$$

Corporation A has a higher stock yield ratio than Corporation B. Thus, each peso would earn you more if you invest in Corporation A than in Corporation B. If all other things are equal, then it is wiser to invest in Corporation A. □

As Example 3 shows, the stock yield ratio can be used to compare two or more investments.

5. Definition of terms in relation to bonds

- **Bond** - interest-bearing security which promises to pay
 - (1) a stated amount of money on the maturity date, and
 - (2) regular interest payments called **coupons**.
- **Coupon** - periodic interest payment that the bondholder receives during the time between purchase date and maturity date; usually received semi-annually
- **Coupon Rate** - the rate per coupon payment period; denoted by r
- **Price of a Bond** - the price of the bond at purchase time; denoted by P

- **Par Value or Face Value** - the amount payable on the maturity date; denoted by F

If $P = F$, the bond is **purchased at par**.

If $P < F$, the bond is **purchased at a discount**.

If $P > F$, the bond is **purchased at premium**.

- **Term (or Tenor) of a Bond** - fixed period of time (in years) at which the bond is redeemable as stated in the bond certificate; number of years from time of purchase to maturity date
- **Fair Price of a Bond** - present value of all cash inflows to the bondholder

6. Give examples to illustrate these terms.

EXAMPLE 4. Determine the amount of the semi-annual coupon for a bond with a face value of ₱300,000 that pays 10%, payable semi-annually for its coupons.

Given: Face Value $F = ₱300,000$

Coupon Rate $r = 10\%$

Find: Amount of the Semi-annual Coupon

Annual coupon amount: $300,000(0.10) = 30,000$

Semi-annual coupon amount: $30,000 \left(\frac{1}{2}\right) = 15,000$

Thus, the amount of the semi-annual coupon is ₱15,000.

The **coupon rate** is used only for computing the coupon amount, usually paid semi-annually. It is not the rate at which money grows. Instead current market conditions are reflected by the **market rate**, and is used to compute the present value of future payments.

EXAMPLE 5. Suppose that a bond has a face value of ₱100,000 and its maturity date is 10 years from now. The coupon rate is 5% payable semi-annually. Find the fair price of this bond, assuming that the annual market rate is 4%.

Given: Face Value $F = ₱100,000$

Coupon Rate $r = 5\%$

Time to Maturity = 10 years

Number of Periods = $2(10) = 20$

Market Rate = 4%

Find: Fair Price of the Bond

Amount of semi-annual coupon: $100,000 \left(\frac{0.05}{2}\right) = 2,500$

The bondholder receives 20 payments of ₱2,500 each, and ₱100,000 at $t = 10$.

Present value of ₱100,000:

$$P = \frac{F}{(1+j)^n} = \frac{100,000}{(1+0.04)^{10}} = 67,556.42$$

Present value of ₱100,000:

Convert 4% to equivalent semi-annual rate:

$$(1+0.04)^1 = \left(1 + \frac{i^{(2)}}{2}\right)^2$$

$$\frac{i^{(2)}}{2} = 0.019804$$

Thus,

$$P = R \frac{1 - (1+j)^{-n}}{j} = 2,500 \frac{1 - (1+0.019804)^{-20}}{0.019804} = 40,956.01, \text{ and}$$

$$\text{Price} = 67,556.42 + 40,956.01 = 108,512.43.$$

Thus, a price of ₱108,512.14 is equivalent to all future payments, assuming an annual market rate of 4%.

(C) Seatwork

- (a) The table below shows the data on 5 stockholders given the par value, the dividend percentage and the number of shares of stock they have with a certain corporation. Find the dividend of the 5 stockholders.

Stockholder	Par Value (in Pesos)	Dividend (%)	Number of Shares
A	50	3%	100
B	48	2.75%	150
C	35	2.5%	300
D	42	3.12%	400
E	58	3.5%	500

Answers: (Use Dividend = Dividend Percentage \times Par Value \times Number of Shares)

A: 150, B: 198, C: 262.5, D: 524.15, E: 1,015

- (b) Answer the following problems completely.

- a. A land developer declared a dividend of ₱10,000,000 for its common stock. Suppose there are 600,000 shares of common stock, how much is the dividend per share? **Answer: ₱16.67**

- b. A certain company gave out ₱25 dividend per share for its common stock. The market value of the stock is ₱92. Determine the stock yield ratio. **Answer:** 27.17%
- c. A property holdings declared a dividend of ₱9 per share for the common stock. If the common stock closes at ₱76, how large is the stock yield ratio on this investment? **Answer:** 11.84%
- d. Find the amount of the semi-annual coupon for a ₱250,000 bond which pays 7% convertible semi-annually for its coupons. **Answer:** ₱8,750

(D) Evaluation

- (a) Tell whether the following is a characteristic of stocks or bonds.
- a. A form of equity financing or raising money by allowing investors to be part owners of the company. **Answer:** Stock
- b. A form of debt financing, or raising money by borrowing from investors. **Answer:** Bond
- c. Investors are guaranteed interest payments and a return of their money at the maturity date. **Answer:** Bond
- d. Investors can earn if the security prices increase, but they can lose money if the security prices decrease or worse, if the company goes bankrupt. **Answer:** Stock
- e. It can be appropriate for retirees (because of the guaranteed fixed income) or for those who need the money soon. **Answer:** Bond
- (b) Answer the following problems completely.
- a. A food corporation declared a dividend of ₱25,000,000 for its common stock. Suppose there are 180,000 shares of common stock, how much is the dividend per share? **Answer:** ₱138.89
- b. A certain financial institution declared ₱57 dividend per share for its common stock. The market value of the stock is ₱198. Determine the stock yield ratio. **Answer:** 28.79%
- c. A certain land developer declared a dividend of ₱28 per share for the common stock. If the common stock closes at ₱99, how large is the stock yield ratio on this investment? **Answer:** 28.28%
- d. Determine the amount of a semi-annual coupon paid for a 3% bond with a face value of ₱80,000 which matures after 15 years. **Answer:** ₱1,200
- e. A ₱450,000 bond is redeemable at ₱550,000 after 5 years. Coupons are given at 5% convertible semi-annually. Find the amount of the semi-annual coupon. **Answer:** ₱11,250

LESSON 32: Market Indices for Stocks and Bonds

TIME FRAME: 60 minutes

CODE: M11GM-IIe-3

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to 1) describe the different markets for stocks and bonds and 2) analyze the different market indices for stocks and bonds.

LESSON OUTLINE:

1. Stock market indices
2. Stock market tables
3. Practical information about stocks
4. Bond market index
5. Practical information about bonds

PREREQUISITE SKILLS: Knowledge of Compound Interest

DEVELOPMENT OF THE LESSON

(A) Introduction/Motivation

Ask students to bring some newspaper clippings that show information about stocks and bonds. Ask students the words they see in the clippings, and to think about what these terms might mean.

(Note: if newspaper clippings are not available, just provide sample clippings, such as those in the lesson, below).

(B) Lesson Proper

(a) Discuss the stock market.

A **stock market index** is a measure of a portion of the stock market. One example is the PSE Composite Index or PSEi. It is composed of 30 companies³ carefully selected to represent the general movement of market prices. The up or down movement in percent change over time can indicate how the index is performing.

Other indices are sector indices, each representing a particular sector (e.g., financial institutions, industrial corporations, holding firms, service corporations, mining/oil, property)⁴.

The stock index can be a standard by which investors can compare the performance of their stocks. A financial institution may want to compare its performance with those of others. This can be done by comparing with the “financials” index.

³<http://www.pse.com.ph/stockMarket/home.html>

⁴Ibid.

(b) Discuss terms in stock market tables

Stock indices are reported in the business section of magazines or newspapers, as well as online (<http://www.pse.com.ph/stockMarket/home.html>). The following table shows how a list of index values is typically presented (values are hypothetical).

Index	Val	Chg	%Chg
PSEi	7,523.93	-14.20	-0.19
Financials	4,037.83	6.58	0.16
Holding Firms	6,513.37	2.42	0.037
Industrial	11,741.55	125.08	1.07
Property	2,973.52	-9.85	-0.33
Services	1,622.64	-16.27	-1.00
Mining and Oil	11,914.73	28.91	0.24

In the table above, the terms mean the following:

- **Val** - value of the index
- **Chg** - change of the index value from the previous trading day (i.e., value today minus value yesterday)
- **%Chg** - ratio of Chg to Val (i.e., Chg divided by Val)

Stock Tables

Newspapers or magazines may also report on stock prices of individual companies. The following table shows how information about stocks can be presented (values are hypothetical).

52-WK HI	52-WK LO	STOCK	HI	LO	DIV	VOL (100s)	CLOSE	NETCHG
94	44	AAA	60	35.5	.70	2050	57.29	0.10
88	25	BBB	45	32.7	.28	10700	45.70	-0.2

In the table above, the terms mean the following:

- **52-WK HI/LO** - highest/lowest selling price of the stock in the past 52 weeks
- **HI/LO** - highest/lowest selling price of the stock in the last trading day
- **STOCK** - three-letter symbol the company is using for trading
- **DIV** - dividend per share last year
- **VOL (100s)** - number of shares (in hundreds) traded in the last trading day (In this case, stock AAA sold 2,050 shares of 100 which is equal to 20,500 shares.)
- **CLOSE** - closing price on the last trading day

- **NETCHG** - net change between the two last trading days (In the case of AAA, the net change is 0.10. The closing price the day before the last trading day is ₱57.29 – ₱0.10 = ₱57.19.)

Such a table may be seen in newspapers or websites which summarize the price movement on selected stocks for the previous trading day.

Buying or Selling Stocks

To buy or sell stocks, one may go to the PSE personally. However, most transactions nowadays are done by making a phone call to a registered broker or by logging on to a reputable online trading platform. Those with accounts in online trading platforms may often encounter a table like the following:

Bid			Ask/Offer		
Size		Price	Price	Size	
122	354,100	21.6000	21.8000	20,000	1
9	81,700	21.5500	21.9000	183,500	4
42	456,500	21.5000	22.1500	5,100	1
2	12,500	21.4500	22.2500	11,800	4
9	14,200	21.4000	22.3000	23,400	6

In the table above, the terms mean the following:

- **Bid Size** - the number of individual buy orders and the total number of shares they wish to buy
- **Bid Price** - the price these buyers are willing to pay for the stock
- **Ask Price** - the price the sellers of the stock are willing to sell the stock.
- **Ask Size** - how many individual sell orders have been placed in the online platform and the total number of shares these sellers wish to sell.

For example, the first row under Bid means that there are a total of 122 traders who wish to buy a total of 354,100 shares at ₱21.60 per share. On the other hand, the first row under Ask means that just 1 trader is willing to sell his 20,000 shares at a price of ₱21.80 per share.

(c) Discuss the bond market.

Bond Market Indices

The main platform for bonds or fixed income securities in the Philippines is the Philippine Dealing and Exchange Corporation (or PDEX). Unlike stock indices which are associated with virtually every stock market in the world, bond market indices are far less common. In fact, other than certain regional bond indices which have subindices covering the Philippines, our bond market does not typically compute a bond market index. Instead, the market rates produced from the bond market are interest rates which may be used as benchmarks for other financial instruments.

The Bond Market and Government Bonds

Government bonds are auctioned out to banks and other brokers and dealers every Monday by the Bureau of the Treasury. Depending on their tenors, these bonds are also called treasury bills (t-bonds), treasury notes (t-notes), or treasury bonds (t-bonds). The resulting coupon rates and the total amount sold for these bonds are usually reported by news agencies on the day right after the auction. Since these bond transactions involve large amounts, these bonds are usually limited to banks, insurance firms, and other major financial institutions. The banks may then re-sell these bonds to its clients.

Although the coupon rate for bonds is fixed, bond prices fluctuate because they are traded among investors in what is called the secondary market. These prices are determined by supply and demand, the prevailing interest rates, as well as other market forces. As the price of the bond may increase or decrease, some investors may choose sell back to banks the bonds they acquired before their maturity to cash in their gains even before maturity.

Despite the fact that investing in bonds is a relatively safer investment than investing in stocks, it is important to note that there are still risks involved when investing in bonds. While unlikely, the most extreme scenario is that of a default by the issuer. In this case, the investor stands to lose not only the coupons, but even the money invested in the bond. As such, before investing in bonds, one must be aware not only of the financial condition of the issuer of the bond, but also the prevailing market conditions.

LESSON 33: Theory of Efficient Markets

TIME FRAME: 60 minutes

CODE: M11GM-IIe-4

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to interpret the theory of efficient markets.

LESSON OUTLINE:

1. The Efficient Market Hypothesis
2. Different forms of the Efficient Market Hypothesis

DEVELOPMENT OF THE LESSON

(A) Introduction/Motivation

As an introduction, ask the students the question “Can you beat the market?”. Let the students give their own idea on how they can have profits on stocks.

Discuss what some investors do: If some investors think that a certain stock is undervalued (suggesting that the price will go down in the future), they buy the stocks at a lower price now and sell it at a higher price in the future. By contrast, if a certain stock is overvalued, investors would want to sell the stocks at a higher price now rather than later.

However this situation runs under the assumption that a stock can be overvalued or undervalued. The Theory of Efficient Markets counters this assumption.

(B) Lesson Proper

(a) Definition of terms (These definitions are mainly for reference. More in-depth discussion is provided later.)

- **Fundamental Analysis** - analysis of various public information (e.g., sales, profits) about a stock
- **Technical Analysis** - analysis of patterns in historical prices of a stock
- **Weak Form of Efficient Market Theory** - asserts that stock prices already incorporate all past market trading data and information (historical price information) *only*
- **Semistrong Form of Efficient Market Theory** - asserts that stock prices already incorporate all publicly available information *only*
- **Strong Form of Efficient Market Theory** - asserts that stock prices already incorporate all information (public and private)

(b) Discuss the efficient market hypothesis.

The theory of efficient markets was developed by Eugene Fama in the 1970s. It says that stock prices already reflect all the available information about the stock⁵. This means that stock prices are ‘accurate’ —they already give a correct measure of the value of a stock precisely because the prices are already based on all information and expectation about the stock.

The slogan “**Trust market prices!**” can sum up the theory. One can trust market prices because they give an accurate measure of all possible information about the stock.

Since all stocks are ‘correctly priced’ (because they are based on all available information), then there is no such thing as discovering undervalued or overvalued stocks from which to gain profits. Thus, the theory implies that **investors cannot beat the market even if they do a lot of research**. In the end, investors will just find out that the correct price is what is already published.

⁵<http://www.investopedia.com/terms/e/efficientmarkethypothesis.asp>

There are three form of efficient market, as discussed by Clarke, Jandik, and Mandelker⁶.

For the **weak form of the theory**, stock prices already reflect all past market trading data and historical information only. Thus, knowing past data will not give investors an edge. If the weak form of the theory is true, then a **technical analysis** (an analysis of past prices) will not yield new information and hence will not lead to systematic profits.

For the **semistrong form of the theory**, stock prices already reflect all publicly available data, including those involving the product, management team, financial statement, competitors and industry. If the semistrong form of the theory is true, then doing a **fundamental analysis** (gathering all public data) will still not lead to systematic profits.

For the **strong form of the theory**, all information (public and private) are incorporated in the price. If the strong form of the theory is true, then investors still cannot gain systematic profits even if they gather information that is not yet publicly known.

Is the theory true? The theory is often tested against various markets in the world. For example, in the Philippines, some investors might argue that there *is* a pattern to stock prices. If this is true, then analyzing historical prices can lead to gain, thereby discrediting the weak form of the theory. By contrast, in some other countries, investors might argue that stock prices quickly reflect all historical data. If this were true, then analyzing patterns will not lead to systematic profits. So, is the theory true? This is an open question, and various arguments have been proposed for or against it.

SUMMARY. The theory of efficient market says all available information are already incorporated in the stock price. Thus, **investors cannot generate systematic profits (except by chance)**. It assumes that, on the average, investors receive a return that pays off the time value of money and the risks associated with the investment. However, **emphasize that this is only a theory.**

(c) Problem-Based Approach and Collaborative Learning

Divide the class into two groups. Form debate teams and let each group be assigned to one of the following arguments:

- a. We can beat the market.
- b. We cannot beat the market.

Use the concepts mentioned in this lesson, and do additional research if possible.

Sample arguments that can arise from the debate are as follows.

⁶Clarke, J., Jandik, T., & Mandelker, G. (n.d.) The efficient markets hypothesis. <http://m.e-m-h.org/CIJM.pdf>

We can beat the market	We cannot beat the market
Knowing more information about stocks can help us determine the best place to invest.	Much information about stocks are public. We cannot profit from what everyone else knows.
It takes time before stock prices can respond to new information. If I invest quickly enough, then I can gain some profit.	Stock prices already reflect all the given information about the stocks. Since all stocks are "correctly" priced, we cannot buy under-valued stock or sell over-valued stocks.
Computers are now making it faster for stock prices to model new information.	There are human errors involved in computing the correct value of a stock.
If I know relevant information that is not publicly available about a certain stock, then I might gain profits.	In order to access all the information, I need time and resources. My profits may actually just be equal to all my costs in analyzing the information.
Several people have gained millions (or even billions) from stock trading.	Everything gained is by chance. As in lotteries, it is very unlikely to win, but someone is bound to win.

(C) Evaluation

(a) Fill in the blanks.

- a. _____ are shares in the ownership of a company. **Answer:** Stocks
- b. The _____ states that prices of investments reflect all available information. **Answer:** Theory of efficient markets
- c. The weak form of the theory of efficient markets asserts that all _____ price information are incorporated in the price of stocks. **Answer:** Historical (or past)
- d. The _____ form of the theory of efficient markets asserts that all information (public and private) are incorporated in the price **Answer:** Strong
- e. The _____ form of the theory of efficient markets asserts that only all publicly available information is incorporated in the price. **Answer:** Semi-strong
-
-

Chapter 9

Basic Concepts of Loans

LESSON 34: Business Loans and Consumer Loans

TIME FRAME: 60 minutes

CODE:M11GM-II-f-1 and f-2

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to illustrate business and consumer loans and distinguish between business and consumer loans.

LESSON OUTLINE:

1. Definition of terms
2. Examples of business and consumer loans

PREREQUISITE SKILLS: Knowledge of Simple Interest, Compound Interest and Ordinary Annuity

DEVELOPMENT OF THE LESSON

(A) Introduction

Discuss with the students the cost of a college education. Let the students tell some success stories of how some college students were able to graduate because of the parents' sacrifices, or the students' dedication by resorting to having part-time jobs. Tell them that some families choose to borrow money from the banks or other lending institutions. Ask the students if they know how to borrow money from banks or other lending institutions. Tell the students that this will be the topic of discussion for this meeting.

Discuss the following examples.

- (a) Ms. Capili and her family wants to open a small cafeteria near the high school in their community. She has a good place for the cafeteria because she will just use the ground floor of their two-storey house. But, the family lacks the capital for starting the business. They decided to borrow money from the bank.
- (b) Mr. Melwin Canlapan is a 25-year old bachelor who works in an insurance company. He wants to buy a brand new car. He decided to apply for an auto loan.
- (c) Mr. De La Cruz is a 35-year old family man. He owns an auto-repair shop. He wants to have another branch for this business. He decided to borrow money from the bank for the expansion of his business.

(B) Lesson Proper

- (a) Definition of terms
 - **Business Loan** - money lent specifically for a business purpose. It may be used to start a business or to have a business expansion
 - **Consumer Loan** - money lent to an individual for personal or family purpose
 - **Collateral** - assets used to secure the loan. It may be real-estate or other investments
 - **Term of the Loan** - time to pay the entire loan
- (b) Discuss the basics of consumer loans and business loans to the students

Consumer loans are loans given to individuals for personal or family purpose while business loans are loans for businesses. Consumer loans and business loans may require a collateral. The collateral for both loans may be real estate or other investments. For business loans, they can use equipment, fixtures or furniture as collateral. Consumer loans do not usually require a guarantor. Business loans require the business owners to sign as guarantors. For consumer loans, the bank or the lending institution may require a credit report, bank statements, and an income tax return, and if the lendee is employed, a certificate of employment and employee pay slips. For business loans, the lendee has to submit a credit report, income tax returns and company's financial statement. The term of the business loan is generally shorter than the consumer loan. The interest rate for the business loan is usually higher than that for the consumer loan.

- (c) Ask the students to answer the following questions.

In Examples 1-5, identify whether the following is a consumer or business loan.

EXAMPLE 1. Mr. Agustin plans to have a barbershop. He wants to borrow some money from the bank in order for him to buy the equipment and furniture for the barbershop.

Answer: Business loan

EXAMPLE 2. Mr and Mrs Craig wants to borrow money from the bank to finance the college education of their son.

Answer: Consumer loan

EXAMPLE 3. Mr. Alonzo wants to have some improvements on their 10-year old house. He wants to build a new room for their 13-year old daughter. He will borrow some money from the bank to finance this plan.

Answer: Consumer loan

EXAMPLE 4. Mr. Samson owns a siomai food cart business. He wants to put another food cart on a new mall in the other city. He decided to have a loan to establish the new business.

Answer: Business loan

EXAMPLE 5. Roan has a computer shop. She owns 6 computers. She decided to borrow some money from the bank to buy 10 more computers.

Answer: Business loan

(C) Evaluation

Identify the following whether it is a business loan or a consumer loan.

- (a) Mr. Lim wants to have another branch for his cellphone repair shop. He decided to apply for a loan that he can use to pay for the rentals of the new branch.

Answer: business loan

- (b) Trillas runs a trucking business. He wants to buy three more trucks for expansion of his business. He applied for a loan in a bank.

Answer: business loan

- (c) Mrs. Alonzo decided to take her family for a vacation. To cover the expenses, she decided to apply for a loan.

Answer: consumer loan

- (d) Glenn decided to purchase a condominium unit near his workplace. He got a loan worth ₱2,000,000.

Answer: consumer loan

- (e) Mr. Galang purchased renovated her house ₱80,000. This was made possible because of an approved loan worth ₱75,000.

Answer: consumer loan

LESSON 35: Solving Problems on Business and Consumer Loans (Amortization and Mortgage)

TIME FRAME: 120 minutes

CODE: M11GM-III-3

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to solve problems on business and consumer loans (amortization and mortgage).

LESSON OUTLINE:

1. Definition of terms
2. Loan Repayment
3. Interest Amount
4. Mortgage and Amortization
5. Outstanding Balance

PREREQUISITE SKILLS: Knowledge of Simple Interest, Compound Interest and Ordinary Annuity

DEVELOPMENT OF THE LESSON

(A) Introduction

Discuss with the students past lesson on paying your loan with regular payments on regular intervals of time. Give some examples on real life situation where they can apply this in business loans and consumer loans like car loan, housing loan, SSS and GSIS salary loan, credit card cash loan and others. Tell the students that once they get a loan, they should fulfill their obligation by paying regularly.

Discuss the following examples.

- (a) Mr. Apolto decided to buy a house and lot for his family. The loan amounts to ₱2,500,000. He amortized the loan by paying ₱23,000 monthly for 20 years.
- (b) Ms. Salceda bought a car for ₱800,000. After deducting the downpayment, her total loan amounts to ₱750,000. She amortized the loan by paying ₱18,000 monthly for 5 years.
- (c) Mr. Francisco applied for a salary loan from his social security insurance. He got a ₱20,000 loan. To pay for this loan, he has to pay ₱1,100 monthly for 2 years.

(B) Lesson Proper

(a) Definition of terms

- **Amortization Method** - method of paying a loan (principal and interest) on installment basis, usually of equal amounts at regular intervals
- **Mortgage** - a loan, secured by a collateral, that the borrower is obliged to pay at specified terms.
- **Chattel Mortgage** - a mortgage on a movable property
- **Collateral** - assets used to secure the loan. It may be a real-estate or other investments
- **Outstanding Balance** - any remaining debt at a specified time

(b) Discuss the basics of a mortgage loan to the students.

A **mortgage** is a business loan or a consumer loan that is secured with a collateral. **Collaterals** are assets that can secure a loan. If a borrower cannot pay the loan, the lender has a right to the collateral. The most common collaterals are real estate property. For business loans, equipment, furniture and vehicles may also be used as collaterals. Usually, the loan is secured by the property bought. For example, if a house and lot is purchased, the purchased house and lot will be used as a mortgaged property or a collateral. During the term of the loan, the **mortgagor**, the borrower in a mortgage, still has the right to possess and use the mortgaged property. In the event that the mortgagor does not make regular payments on the mortgage, the **mortgagee** or the lender in a mortgage can repossess the mortgaged property. The most common type of mortgage is the **fixed-rate mortgage** wherein the interest remains constant throughout the term of the loan.

(c) Solve problems involving business or consumer loans

EXAMPLE 1. Mr. Garcia borrowed ₱1,000,000 for the expansion of his business. The effective rate of interest is 7%. The loan is to be repaid in full after one year. How much is to be paid after one year?

Solution.

Given: $P = 1,000,000$

$$j = 0.07$$

$$n = 1$$

Find: F

$$F = P(1 + j)^n = 1,000,000(1 + 0.07) = 1,070,000$$

An amount of ₱1,070,000 must be paid after one year. □

EXAMPLE 2. (Chattel mortgage). A person borrowed ₱1,200,000 for the purchase of a car. If his monthly payment is ₱31,000 on a 5-year mortgage, find the total amount of interest.

Solution.

Given: $P = 1,200,000$

Monthly payment = 31,000

Find: Total Interest

The total amount paid is given by

$$\begin{aligned} \text{Total Amount} &= (31,000)(12 \text{ months})(5 \text{ years}) \\ &= 1,860,000 \end{aligned}$$

Thus, the total interest is the difference between the total amount paid and the amount of the mortgage:

$$\begin{aligned} \text{Total Interest} &= 1,860,000 - 1,200,000 \\ &= 660,000 \end{aligned}$$

The mortgage amount is ₱2,400,000. □

EXAMPLE 3. If a house is sold for ₱3,000,000 and the bank requires 20% down payment, find the amount of the mortgage.

Solution.

$$\begin{aligned}
 \text{Down payment} &= (\text{down payment rate}) \times (\text{cash price}) \\
 &= 0.20(3,000,000) \\
 &= 600,000
 \end{aligned}$$

$$\begin{aligned}
 \text{Amount of the Loan} &= (\text{cash price}) - (\text{down payment}) \\
 &= 3,000,000 - 600,000 \\
 &= 2,400,000
 \end{aligned}$$

The mortgage amount is ₱2,400,000. □

Note:

In this case, the house itself is used as the mortgaged property. Also please take note that the other way to solve this is to directly compute the mortgaged amount by multiplying the cash value of the property by the percentage of the financed amount, which in this case, $100\% - 20\% = 80\%$. Thus, the amount of the loan is given by

$$(0.80)(\text{₱}3,000,000) = \text{₱}2,400,000.$$

EXAMPLE 4. Ms Rosal bought a car. After paying the downpayment, the amount of the loan is ₱400,000 with an interest rate of 9% compounded monthly. The term of the loan is 3 years. How much is the monthly payment?

Solution.

Given: $P = 400,000$

$$i^{(12)} = 0.09$$

$$j = \frac{i^{(12)}}{12} = \frac{0.09}{12} = 0.0075$$

$$n = 36$$

Find: the regular payment R

$$R = \frac{P}{\left[\frac{1 - (1 + j)^{-n}}{j} \right]} = \frac{400,000}{\left[\frac{1 - (1 + 0.0075)^{-36}}{0.0075} \right]} = 12,719.89$$

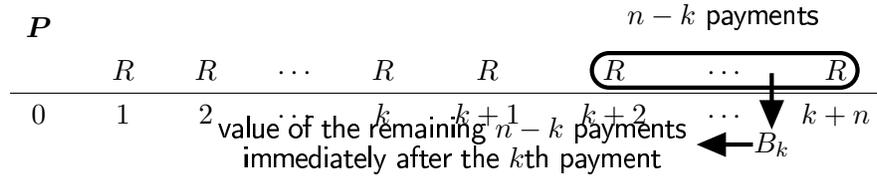
The regular payment is ₱12,719.89. □

(d) Discuss the outstanding balance of a loan.

Outstanding Balance

Recall that the outstanding balance of a loan is the amount of the loan at this time. One method to compute the outstanding balance is to get the present value of all remaining payments. This method is called the **prospective method**.

Time Diagram:



We use the symbol B_k to denote the outstanding balance after k payments. In other books, ${}_pOB_k$ is used (the "p" stands for "prospective").

EXAMPLE 5. Mrs. Se borrowed some money from a bank that offers an interest rate of 12% compounded monthly. His monthly amortization for 5 years is ₱11,122.22. How much is the outstanding balance after the 12th payment?

Solution.

Given: $R = 11,122.22$

$$i^{(12)} = 0.12$$

$$j = \frac{i^{(12)}}{12} = \frac{0.12}{12} = 0.01$$

$k = 12$ number of payments paid

$n - k = 48$ since only 48 payments remain

Find: present value of 48 future payments (since there are 48 payments left)

$$B_k = R \left[\frac{1 - (1 + j)^{-(n-k)}}{j} \right] = 11,122.22 \left[\frac{1 - (1.01)^{-48}}{0.01} \right] = 422,354.73$$

The outstanding balance is ₱422,354.73. □

- (e) Discuss the basics of loan amortization (compute the interest and principal components of a loan.)

EXAMPLE 6. Mr. and Mrs. Banal purchased a house and lot worth ₱4,000,000. They paid a down payment of ₱800,000. They plan to amortize the loan of ₱3,200,000 by paying monthly for 20 years. The interest rate is 12% convertible monthly.

- How much is the monthly payment?
- What is the total interest paid?
- What are the principal and interest components of the 51st payment?

Solution.

a. Given: $P = 3,200,000$

$$i^{(12)} = 0.12$$

$$j = \frac{i^{(12)}}{12} = \frac{0.12}{12} = 0.01$$

$$n = mt = (12)(20) = 240$$

Find: regular payment R

Using the formula $P = R \left[\frac{1 - (1 + j)^{-n}}{j} \right]$,

then $R = \frac{P}{\left[\frac{1 - (1 + j)^{-n}}{j} \right]} = \frac{3,200,000}{\left[\frac{1 - (1 + 0.01)^{-240}}{0.01} \right]} = 35,234.76$

Therefore, the monthly payment is ₱35,234.76.

b. Given: $P = 3,200,000$

$$R = 35,234.76$$

$$n = 240$$

Find: total interest paid

There are 240 payments of ₱35,234.76. The total payment is $240 \times ₱35,234.76 = ₱8,456,342.40$.

The principal is only ₱3,200,000.

$$\begin{aligned} \text{Interest Amount} &= (\text{Total Payments}) - (\text{Principal}) \\ &= 8,456,342.40 - 3,200,000 = 5,256,342.40 \end{aligned}$$

The interest amount is ₱5,256,342.40

Note:

Students may be surprised to learn that much of what is being paid is for the interest. This is particularly true if a loan is being paid over a long period of time.

c. Given: $P = 3,200,000$

$$i^{(12)} = 0.12$$

$$j = \frac{i^{(12)}}{12} = \frac{0.12}{12} = 0.01$$

$$n = mt = (12)(20) = 240$$

$$R = 35,234.76$$

Find: principal and interest components of the 51st payment

The 51st payment of ₱35,234.76 is partly used to pay for the principal, and partly to pay for the interest.

Step 1: Get the outstanding balance after the 50th payment (the balance after the 50th payment is what the 51st payment will be for).

Since 50 payments have been paid already, there will be 190 remaining payments.

The outstanding balance after the 50th payment is:

$$B_{50} = R \left[\frac{1 - (1 + j)^{-190}}{j} \right] = 35,234.76 \left[\frac{1 - (1 + 0.01)^{-190}}{0.01} \right] = 2,991,477.63$$

Step 2: After the 50th payment, the outstanding balance is ₱2,991,477.63.

Since the interest rate per period is $j = 0.01$, then the remaining balance of ₱2,991,477.63 will be charged an interest of ₱29,914.78. The 51st payment of ₱35,234.76 will be used to pay for this interest. Thus, the interest component of the 51st payment is ₱29,914.78.

The remaining portion of the 51st payment is the principal component, denoted by PR_{51} , is:

$$PR_{51} = R - I_{50} = 35,234.76 - 29,914.78 = 5,319.98$$

Thus, for the 51st payment, the part that goes to pay the interest is ₱29,914.78 and the part that goes to pay the principal is ₱5,319.98. \square

(f) Problem-based Approach and Collaborative Learning

Group the students into 5. Interview a real estate agent, vehicle dealer or someone who sells by installment. Get a sample of the quotation for what they are selling. Make sure the following details will be reported to class:

- i. Cash Value of the property
- ii. Downpayment
- iii. Mortgaged Amount
- iv. Term of the Loan
- v. Monthly Payment
- vi. Total Interest

(C) Seatwork

Seatwork 1. A loan of ₱200,000 is to be repaid in full after 3 years. If the interest rate is 8% per annum. How much should be paid after 3 years?

$$\text{Answer: } F = (1 + j)^n = 100,000, (1 + 0.08)^3 = 251,942.40$$

Seatwork 2. For a purchase of a house and lot worth ₱3,800,000, the bank requires 20% down payment, find the mortgaged amount.

$$\text{Answer: } ₱3,040,000$$

Seatwork 3. A car dealer offers a 15% down payment for the purchase a car. How much is the mortgaged amount if the cash value of the car is ₱1,500,000?

$$\text{Answer: } ₱1,275,000$$

Seatwork 4. If a condominium is purchased for ₱5,700,000 and the bank requires 30% down payment, how much is the mortgaged amount?

$$\text{Answer: } ₱3,990,000$$

Seatwork 5. A family obtained a ₱1,000,000 mortgage. If the monthly payment is ₱38,000 for four years, how much is the total interest paid?

$$\text{Answer: } ₱824,000$$

Seatwork 6. Mr. Canlapan obtained a 10-year mortgage for ₱4,200,000. If his monthly payment is ₱39,500 , how much is the total interest?

$$\text{Answer: } ₱540,000$$

Seatwork 7. Mr. Ramos is considering to pay his outstanding balance after 3 years of payment. The original amount of the loan is ₱100,000 payable annually in 5 years. If the interest rate is 10% per annum and the regular payment is ₱26,379.75 annually, how much is the outstanding balance after the 3rd payment?

$$\text{Answer: } B_3 = R \left[\frac{1 - (1 + j)^{-2}}{j} \right] = 26,379.75 \left[\frac{1 - (1 + 0.10)^{-2}}{0.10} \right] = 45,783.04$$

Seatwork 8. Ms. Saya got a business loan worth ₱900,000. She promised to pay the loan semi-annually in 3 years. The semi-annual payment is ₱183,026.37 if money is worth 12% converted semi-annually. How much is the outstanding balance after the first payment?

$$\text{Answer: } B_1 = R \left[\frac{1 - (1 + j)^{-5}}{j} \right] = 183,026.37 \left[\frac{1 - (1.06)^{-5}}{0.06} \right] = 770,973.65$$

Seatwork 9. A consumer loan worth ₱30,000 is to be repaid in 12 months at 9% convertible monthly. How much is the monthly payment?

$$\text{Answer: } R = \frac{P}{\left[\frac{1 - (1 + j)^{-n}}{j} \right]} = \frac{30,000}{\left[\frac{1 - (1 + 0.0075)^{-12}}{0.0075} \right]} = 2,623.54$$

Seatwork 10. A business loan worth ₱750,000 is to be repaid in quarterly installment in 2 years. How much is the quarterly payment if money is worth 8% converted quarterly?

$$\text{Answer: } R = \frac{P}{\left[\frac{1 - (1 + j)^{-n}}{j} \right]} = \frac{750,000}{\left[\frac{1 - (1 + 0.02)^{-8}}{0.02} \right]} = 102,382.35$$

Seatwork 11. A newly married couple decided to buy a brand new car. The net amount of the loan is ₱500,000. They plan to amortize the loan in monthly installments for 3 years. If money is worth 12% convertible monthly,

(a) how much is the monthly installment?

$$\text{Answer: } R = \frac{P}{\left[\frac{1 - (1 + j)^{-n}}{j} \right]} = \frac{500,000}{\left[\frac{1 - (1 + 0.01)^{-36}}{0.01} \right]} = ₱16,607.15$$

(b) How much is the outstanding balance after the second payment?

$$\text{Answer: } B_2 = R \left[\frac{1 - (1 + j)^{-34}}{j} \right] = 16,607.15 \left[\frac{1 - (1 + 0.01)^{-34}}{0.01} \right] = ₱476,669.63$$

(c) How much of the third payment goes to pay the interest and the principal?

$$\text{Answer: } I_3 = i(P_2) = (0.01)(476,669.63) = ₱4,766.00 \text{ for interest}$$

$$PR_{51} = R - I_{50} = 16,607.15 - 4,766.7 = ₱11,840.45 \text{ for principal}$$

(D) Evaluation

(a) Fill in the blank.

a. A _____ is a loan, secured by a collateral, that the borrower is obliged to pay at specified terms. **Answer:** mortgage

b. _____ is a mortgage on a movable property. **Answer:** chattel mortgage

c. A _____ is a mortgage with a fixed interest rate for its entire term.

Answer: fixed-rate mortgage

d. The _____ is the lender in a mortgage. **Answer:** mortgagee

e. The _____ is the borrower in a mortgage. **Answer:** mortgagor

(b) Solve the following problems completely.

- i. A business loan of 800,000 is to be repaid in full after 2 years. What is the amount to be paid if the effective rate of interest is 8%?

$$\text{Answer: } F = P(1 + j)^n = 800,000(1 + 0.08)^2 = 933,120$$

- ii. For the purchase of a farm worth ₱2,800,000, the bank requires 30% down payment, find the mortgaged amount.

$$\text{Answer: } ₱1,960,000$$

- iii. If a condominium is purchased for ₱1,700,000 and the bank requires 25% down payment, how much is the mortgaged amount?

$$\text{Answer: } ₱1,275,000$$

- iv. Mr. Sia got a ₱1,100,000 mortgage. If the monthly payment is ₱33,000 for five years, how much is the total interest paid?

$$\text{Answer: } ₱880,000$$

- v. Mr. Yuson obtained a 20-year mortgage for ₱2,200,000. If his monthly payment is ₱18,500, how much is the total interest?

$$\text{Answer: } ₱2,240,000$$

(c) Answer the following problems completely.

- a. Mr. Abaya got a ₱700,000 loan for the expansion of his business payable monthly in 4 years. How much is the monthly amortization if the interest rate is 12% compounded monthly?

$$\text{Answer: } R = \frac{P}{\left[\frac{1 - (1 + j)^{-n}}{j} \right]} = \frac{700,000}{\left[\frac{1 - (1 + 0.01)^{-48}}{0.01} \right]} = 18,433.68$$

- b. A consumer loan of ₱300,000 is to be repaid quarterly for 5 years. If the interest rate is 10% converted quarterly, How much is the quarterly payment?

$$\text{Answer: } R = \frac{P}{\left[\frac{1 - (1 + j)^{-n}}{j} \right]} = \frac{300,000}{\left[\frac{1 - (1 + 0.025)^{-20}}{0.025} \right]} = 19,244.14$$

- c. A business loan of ₱650,000 is to be settled by paying ₱29,994.20 monthly in 2 years. If the interest rate is 10% converted monthly, how much is the outstanding balance after 6 payments?

$$\text{Answer: } B_6 = R \left[\frac{1 - (1 + j)^{-18}}{j} \right] = 29,994.20 \left[\frac{1 - \left(1 + \frac{0.10}{12}\right)^{-18}}{\frac{0.10}{12}} \right] = 499,428.21$$

- d. Ms. Sena got a ₱500,000 loan to be repaid quarterly in 5 years. The interest rate applied is 10% convertible quarterly. The quarterly payment is solved as ₱32,073.56. How much is the outstanding balance after 2 years?

$$\text{Answer: } B_8 = R \left[\frac{1 - (1 + j)^{-12}}{j} \right] = 32,073.56 \left[\frac{1 - (1.025)^{-12}}{0.025} \right] = 329,003.03$$

- e. A business loan of ₱45,000 is given to Ms. Alfonso. The monthly payment for the loan amounts to ₱11,485.35 for 4 months. The interest rate used is 10% convertible monthly. How much is the outstanding balance after the first payment?

$$\text{Answer: } P_1 = R \left[\frac{1 - (1 + j)^{-3}}{j} \right] = 11,485.35 \left[\frac{1 - \left(1 + \frac{0.10}{12}\right)^{-3}}{\frac{0.10}{12}} \right] = \text{₱}33,889.65$$

- f. Mr. Lachica applied for a ₱100,000 loan. She plans to pay for this loan annually at 8% per annum for 5 years. The annual payment is computed as ₱25,045.65.

A. How much is the outstanding balance after the fourth payment?

B. How much of the 5th payment goes to pay the interest and principal?

$$\begin{aligned} \text{Answer: a. } P_4 &= R \left[\frac{1 - (1 + j)^{-1}}{j} \right] = 25,045.65 \left[\frac{1 - (1 + 0.08)^{-1}}{0.08} \right] = \text{₱}23,190.42 \\ \text{b. } I_5 &= j(P_4) = (0.08)(23,190.42) = 1,855.23 \\ PR_5 &= R - I_5 = 25,045.65 - 1,855.23 = 23,190.42 \end{aligned}$$

Chapter 10

Logic

LESSON 36: Propositions

TIME FRAME: 120 minutes

CODE: M11GM-IIg-1, g-2, and g-3

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to illustrate a proposition, symbolize propositions, and distinguish between simple and compound propositions.

LESSON OUTLINE:

1. Propositions
 2. Simple and compound propositions
-

DEVELOPMENT OF THE LESSON

(A) Introduction, motivation.

Divide the class into groups of three to five students. Ask each group to make a convincing argument for or against a statement. Each group should write their argument on manila paper. You may provide the following example. Keep the manila paper for the groupwork at the end of the lesson.

Statement: Everyone should study logic.

Argument for the statement: Logic is used in engineering, law, philosophy, computer science, mathematics, and many other disciplines. Logic makes our electronic gadgets, such as cellphones, work. Logic is fun and interesting. If logic is useful, fun and interesting, then everyone should study it. Therefore, everyone should study logic.


Teaching tip

You may give this activity as homework. In such case, just present your example in class when you give the homework.
Require each group to provide at least five supporting statements.

Ask each group to present their arguments in class. Remark that this is one of the many ways we use logic in everyday life—to convince other people that our beliefs or opinions are correct. Ask the class whether the arguments are convincing. Remark further that, at the end of the chapter, the students will know how to determine whether a given argument is valid or not.

(B) Lesson Proper

1. Define a proposition.

Definition

A **proposition** is a declarative sentence that is either true or false, but not both. If a proposition is true, then its truth value is true which is denoted by T ; otherwise, its truth value is false and is denoted by F .

Remark that propositions are usually denoted by small letters as shown in the next example. For example,

$$p: \text{Everyone should study logic.}$$

may be read as

$$p \text{ is the proposition "Everyone should study logic."}$$

If a sequence of propositions is considered, we denote the propositions by p_1, p_2, \dots

EXAMPLE 1. Determine whether each of the following statements is a proposition or not. If a proposition, give its truth value.

p : Mindanao is an island in the Philippines.

q : Find a number which divides your age.

r : My seatmate will get a perfect score in the Logic exam.

s : Welcome to the Philippines!

t : $3 + 2 = 5$

u : $f(x) = \frac{\sqrt{x}}{x+1}$ is a rational function.

v : What is the domain of the function?

w : I am lying.

p_1 : It is not the case that $\sqrt{2}$ is a rational number.

- p_2 : Either logic is fun and interesting, or it is boring.
- p_3 : If you are a Grade 11 student, then you are a Filipino.
- p_4 : If you are more than 60 years old, then you are entitled to a Senior Citizen's card, and if you are entitled to a Senior Citizen's card, then you are more than 60 years old.

Solution. Recall that for a statement to be a proposition it has to be a declarative sentence, and it should have a truth value of either true or false, but not both true and false at the same time.

- p . This is a declarative sentence, and Mindanao is an island in the Philippines. Hence, p is a true proposition.
- q . This is an imperative sentence, and so it is not a proposition.
- r . The statement is a declarative sentence. Although the truth value will only be known after the Logic exam, we know that it can only be either true (my seatmate gets a perfect score) or false (she has some mistakes), but not both. Hence, r is a proposition.

Remark that for a declarative sentence to be a proposition, it is not necessary that its true value is immediately known.

- s . Statement s is an exclamatory sentence, and so it is not a proposition.
- t . Obviously, $3 + 2 = 5$ is a true mathematical sentence. But, is it a declarative sentence? Yes! To see this, note that you may read it as "The sum of three and two is five," which is clearly a declarative sentence. Therefore, t is a true proposition.

Remark that the propositions that we consider in mathematics are of this form. More examples are given below.

Proposition in symbolic form	Proposition in English
$2 \in \mathbb{N}$	Two is a natural number.
$-100 \in \mathbb{Z}$	Negative 100 is an integer.
$\pi \notin \mathbb{Q}$	The constant π is an irrational number.
$\pi \in \mathbb{R}$	The constant π is a real number.
$\sqrt{2} < 2$	The square root of two is less than two.
$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$	The set of all natural numbers is a subset of the set of all integers, the set of all integers is a subset of the set of all rational numbers, and the set of all rational numbers is a subset of the set of all real numbers.

- u . This is a declarative sentence. Since the numerator of the function is not a polynomial, the function f is not rational and so the statement is false. Therefore, u is a false proposition. You can remark that false mathematical sentences are still propositions.
- v . It is an interrogative sentence. Hence, it is not a proposition.
- w . Although w is a declarative sentence, it is not a proposition because it can neither be true nor false. To see this, suppose one assumes that w is true—that is, it is the case that I am lying.

Since I am lying, my statement is not true, and so w must also be false. Similarly, assuming that w is false would also lead to the conclusion that it is true as well.

- p_1 . This is a declarative sentence and it will be shown later that $\sqrt{2}$ cannot be expressed as a quotient of two integers. Thus, p_1 is a proposition.
- p_2 . It is a declarative sentence, which is true for anyone because each of us finds logic either fun or boring. Hence, p_2 is a true proposition.
- p_3 . This is a declarative sentence, but it is not true. There are also Grade 11 students of other nationalities.
- p_4 . We know that it is a true proposition. Furthermore, we can express the whole sentence as "You are more than 60 years old *if and only if* you are entitled to a Senior Citizen's card." (You may remark that this is an example of a biconditional statement, which will be discussed in the next lesson.)

□

Remark

Consider the following mathematical sentences.

(a) $2x = 1$

(b) $(x + y)^2 = x^2 + y^2$

Note that both are declarative sentences, which may be read as "Twice a number x is 1," and "The square of a sum is the sum of the squares," respectively. But, are they true or false? We cannot tell because both sentences involve some unknowns x and y . Thus, the given statements are not propositions.

2. Define simple and compound proposition.

Definition

A **compound proposition** is a proposition formed from simpler propositions using **logical connectors** or some combination of logical connectors. Some logical connectors involving propositions p and/or q may be expressed as follows:

$$\begin{aligned} & \textit{not } p \\ & p \textit{ and } q \\ & p \textit{ or } q \\ & \textit{if } p \textit{ then } q \end{aligned}$$

where $\langle \cdot \rangle$ stands for some proposition.

A proposition is **simple** if it cannot be broken down any further into other component propositions.

Remark that simple propositions are also called atomic propositions because they are the building blocks of propositional logic as atoms are considered building blocks in chemistry. Consequently, any compound proposition is composed of simple propositions.

EXAMPLE 2. For each of the propositions in Example 1, determine whether it is a simple or compound proposition. If it is a compound proposition, identify the simple components.

Solution. The propositions p , r , t , and u are all simple propositions. On the other hand, the following are compound propositions.

p_1 : **It is not the case** that $\sqrt{2}$ is a rational number.

p_2 : Either logic is fun and interesting, **or** it is boring.

p_3 : **If** you study hard, **then** you will get good grades.

p_4 : **If** you are more than 60 years old, **then** you are entitled to a Senior Citizen's card, **and if** you are entitled to a Senior Citizen's card, **then** you are more than 60 years old.

Furthermore, we can determine the simple propositions that make up the propositions p_1 , p_2 , p_3 , and p_4 . We do so in the following table.

Proposition	Simple Components
p_1 .	r : $\sqrt{2}$ is a rational number.
p_2 .	f : Logic is fun. i : Logic is interesting. b : Logic is boring.
p_3 .	h : You study hard. g : You get good grades.
p_4 .	d : You can drive. l : You have a driver's license.

Then the compound propositions may be expressed as follows:

p_1 : **not** r .

p_2 : f **and** i , **or** b .

p_3 : **if** h , **then** g .

p_4 : **(if** d , **then** l) **and** **(if** l , **then** d)

□

Remark that compound propositions will be revisited more closely in the next meeting.

3. Seatwork

Seatwork 1. Determine whether each of the following statements is a proposition or not. If a proposition, give its truth value.

(a) Mabuhay!

- (b) Jose Rizal is our National Hero.
- (c) Who is the first president of the republic?
- (d) Ferdinand Magellan did not arrive the Philippines in 1521.
- (e) 2.5 is an integer.
- (f) Our Logic teacher is either pretty or handsome.
- (g) Smile at your seatmate.
- (h) The last kilometer marker up north is in Sta. Ana, Cagayan, and the Palau Island is also found there. [*Teachers can replace this with a statement involving the students' locality.*]
- (i) 2 is even and prime.
- (j) Is 2 a square of some number?
- (k) If an integer is even, then its square is also even.

Answer: (b), (d), (e), (f), (h), (i), (k) are propositions. The rest are not. Of these, (b), (h), (i), and (k) are true; the truth value of (f) can be discussed and agreed upon by the class

Seatwork 2. For each proposition in item 1, identify whether it is a simple or compound proposition. If it is a compound proposition, identify its primitive components.

Answer: Only (b) and (e) are simple propositions.

(C) Groupwork.

In the motivation activity, each group was asked to provide an argument *for* or *against* a statement of their choice. Ask each group to determine whether each sentence in their argument is a proposition. They should also identify whether it is simple or compound. If it is a compound proposition, they should also identify its simple components.

LESSON 37: Logical Operators

TIME FRAME: 120 minutes

CODE: M11GM-IIg-4

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to perform different types of operations on propositions.

LESSON OUTLINE:

1. Truth tables
2. Negation, conjunction, disjunction, conditional, and biconditional.

DEVELOPMENT OF THE LESSON

(A) Introduction.

In the previous meeting, we learned about compound propositions as a combination of simple propositions using logical connectors. In this lesson, we will introduce the symbolic counterpart of the connectors ‘*not*’, ‘*and*’, ‘*or*’, ‘*if... then...*’, and ‘*...if and only if...*’, which we call logical operators. To do that, we need to talk first about an important tool in logic, the truth table.

(B) Lesson Proper

1. Introduce how to construct a truth table.

Given a proposition, its **truth table** shows all its possible truth values.

EXAMPLE 1. Since a proposition has two possible truth values, a proposition p would have the following truth table.

p
T
F

The truth table is useful because we can use it to display all the possible truth value combinations of two or more propositions. For example, suppose p and q are propositions. We can construct a truth table displaying the relationship between the possible truth values of p and the truth values of q . The rows of the table will correspond to each of the possible truth value combination of p and q , and so there will be $2^2 = 4$ rows. Thus, for propositions p and q , we have the following table.

p	q
T	T
T	F
F	T
F	F

Similarly, suppose p , q , and r are propositions. Then a truth table involving the given propositions has $2^3 = 8$ rows as shown below.

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

In general, a truth table involving n propositions has 2^n rows.

As mentioned previously, we will use truth tables to define the logical operators. If a given proposition is a compound proposition, the truth table is used to exhibit the relationship between the truth values of its simple components and the truth values of the given compound proposition.

2. Define *negation*.

The simplest logical operator is the negation operator, which is denoted by \sim .

Definition.

The **negation** of a proposition p is denoted by

$$\sim p : (\text{read as 'not' } p,)$$

and is defined through its truth table:

p	$\sim p$
T	F
F	T

The truth table tells us that when p is true, its negation $\sim p$ is false. On the other hand, when p is false, the negation $\sim p$ is true.

EXAMPLE 2. State the negation of the following propositions.

n_1 : $p(x) = \frac{x-1}{x+2}$ is a polynomial function.

n_2 : 2 is an odd number.

n_3 : The *tinikling* is the most difficult dance.

n_4 : Everyone in Visayas speaks Cebuano.

Solution. The negation of the given propositions are given below.

- $\sim n_1$: ‘It is not true that $p(x) = \frac{x-1}{x+2}$ is a polynomial function’, or, more simply, ‘ $p(x) = \frac{x-1}{x+2}$ is *not* a polynomial function’.
- $\sim n_2$: ‘It is not true that 2 is an odd number’, or ‘2 is an even number.’
- $\sim n_3$: ‘The *tinikling* is not the most difficult dance.’
- $\sim n_4$: ‘Not everyone in Visayas speaks Cebuano.’

□

Teaching tip

It is a common mistake to assume that the proposition $\sim p$ is automatically false, just because it involves a negation. However, $\sim p$ can be true—this happens when p is false. For example, $\sim n_2$ is true because n_2 is false.

Also note that negation $\sim p$ is not necessarily the complete opposite of p . For example, the negation of the statement ‘Everyone in Visayas speaks Cebuano’ merely states that *there are* Visayans who do not speak Cebuano. The negation does not say that all Visayans do *not* speak Cebuano.

3. Define *conjunction*.

The next logical operator allows us to state an expression for two propositions to be true. It is called the conjunction operator and is denoted by \wedge .

Definition.

The **conjunction** of propositions p and q is denoted by

$$p \wedge q : (p \text{ and } q,)$$

and is defined through its truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

The propositions p and q are called **conjuncts**.

The conjunction $p \wedge q$ is true only when both conjuncts p and q are true as shown in its truth table.

EXAMPLE 3. Let p and q be the following propositions.

p : Angels exist.

q : $\pi > 3$

Express the following conjunctions as English sentences or in symbols, as the case may be.

- (a) $p \wedge q$
- (b) $p \wedge (\sim q)$
- (c) ‘Angels do not exist and $\pi \leq 3$.’
- (d) ‘While angels do not exist, $\pi > 3$.’

Solution. The corresponding English sentences are given below.

- (a) $p \wedge q$: ‘Angels exist and $\pi > 3$.’
- (b) $p \wedge (\sim q)$: ‘Angels exist and $\pi \leq 3$ ’, or ‘Angels exist, yet $\pi \leq 3$.’
- (c) In symbols, we have $(\sim p) \wedge (\sim q)$
- (d) In logic, the statement is a conjunction and so, in symbols, $(\sim p) \wedge q$

□

In the example above, since proposition q is true, the conjunction $p \wedge q$ is true when p is proven to be true. On the other hand, both conjunctions $p \wedge (\sim q)$ and $(\sim p) \wedge (\sim q)$ are false because one of the conjuncts, namely $\sim q$, is false.

Teaching tip

Emphasize that conjunctions do not always use the word ‘and’. Words like ‘but’, ‘even though’, ‘yet’, and ‘while’ are also used to flag conjunctions.

Consider the following sentences.

- Carlo is competitive and hardworking.
- Carlo is competitive but hardworking.
- Carlo is competitive yet hardworking.
- Although Carlo is competitive, he is hardworking.
- While Carlo is competitive, he is hardworking.

In ordinary language, these sentences have subtle differences. However, in logic, we are only interested in whether the proposition is true or not, and on how the truth value of the compound proposition (a conjunction, in this case) depends on the truth value of its simple components. Hence, in logic, all these statements can be represented by the conjunction $p \wedge q$, where p : ‘Carlo is competitive’ and q : ‘Carlo is hardworking.’

4. Define *disjunction*.

The third logical operator is called the disjunction operator and is denoted by \vee . A disjunction allows us to express alternatives.

Definition.

The **disjunction** of propositions p and q is denoted by

$$p \vee q : (p \text{ or } q,)$$

and is defined through its truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The propositions p and q are called **disjuncts**.

The truth table above tells us that the disjunction $p \vee q$ is false only when both disjuncts p and q are false.

 **The meaning of ‘or’**

Note that ‘or’ has several meanings in ordinary language. In our case, we use what mathematicians call the *inclusive or*. That is, ‘ $p \vee q$ ’ means that p is true *or* q is true *or* BOTH are true.

EXAMPLE 4. Let p , q and r be the following propositions.

p : Victor has a date with Liza.

q : Janree is sleeping.

r : Eumir is eating.

Express the following propositions in English sentences or in symbols, as the case may be.

(a) $p \vee q$

(b) $q \vee (\sim r)$

(c) $p \vee (q \vee r)$

(d) ‘Either Victor has a date with Liza or Janree is sleeping, or Eumir is eating.’

(e) ‘Either Victor has a date with Liza and Janree is sleeping, or Eumir is eating.’

(f) ‘Either Victor has a date with Liza, or Janree is sleeping and Eumir is eating.’

- (g) ‘Either Victor has a date with Liza and Janree is sleeping, or Victor has a date with Liza and Eumir is eating.’

Solution. The corresponding English expressions or symbols are given below.

- (a) $p \vee q$: ‘Victor has a date with Liza or Janree is sleeping.’
 (b) $q \vee (\sim r)$: ‘Either Janree is sleeping, or Eumir is not eating.’
 (c) $p \vee (q \vee r)$: ‘Either Victor has a date with Liza, or Janree is sleeping or Eumir is eating.’
 (d) $(p \vee q) \vee r$
 (e) $(p \wedge q) \vee r$
 (f) $p \vee (q \wedge r)$
 (g) $(p \wedge q) \vee (p \wedge r)$

□

The propositions

‘Either Victor has a date with Liza, or Janree is sleeping or Eumir is eating.’

‘Either Victor has a date with Liza or Janree is sleeping, or Eumir is eating.’

are represented in symbols as $p \vee (q \vee r)$ and $(p \vee q) \vee r$, respectively. Notice that even as English sentences they are very similar. In the next meeting, we will show that in logic they are actually equivalent; that is, the compound propositions $p \vee (q \vee r)$ and $(p \vee q) \vee r$ are logically equivalent, and so we can get rid of the parentheses and write them as $p \vee q \vee r$.

In a similar manner, we will also see later that $p \wedge (q \wedge r)$ and $(p \wedge q) \wedge r$ are logically equivalent. Again, we can get rid of the parentheses and write $p \wedge q \wedge r$.

EXAMPLE 5. Let p , q and r be propositions as in the previous example.

Consider the following scenario. One Friday night, Victor and Janree are busy studying for their Logic exam. Meanwhile, Eumir just tweeted a picture of himself eating crispy *pata* and *sisig*!

What is the truth value of the proposition $(\sim p) \vee (q \wedge r)$?

Solution. From Eumir’s tweet, we can conclude that he is eating and so proposition r is true. Since Janree is studying, proposition q is false. This implies that conjunction $q \wedge r$ is false, since one of the conjuncts is false. The proposition p is also false because Victor is studying, which means that $(\sim p)$ is true. Hence, the disjunction $(\sim p) \vee (q \wedge r)$ is true as one of the disjuncts is true.

This discussion may be summarized in a table as in the following:

p	q	r	$\sim p$	$q \wedge r$	$(\sim p) \vee (q \wedge r)$
F	F	T	T	F	T



5. Define a conditional.

Conditionals are very important in mathematics since most mathematical statements are expressible in this form.

Definition.

The **conditional** of propositions p and q is denoted by

$$p \rightarrow q : (\text{If } p, \text{ then } q,)$$

and is defined through its truth table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The conditional $p \rightarrow q$ may also be read as ‘ p implies q ’. The proposition p is called the **hypothesis**, while the proposition q is called the **conclusion**.

The following example illustrates that one must be careful in interpreting conditionals.

EXAMPLE 6. Suppose that Geebee is a Grade 11 student. Consider the following conditionals:

p_1 : If Geebee is in Grade 11, then she is a senior high school student.

p_2 : If Geebee is in Grade 11, then she is working as a lawyer.

p_3 : If Geebee has a degree in Computer Science, then she believes in true love.

Analyze the truth values of these conditionals.

Solution.

p_1 : ‘Geebee is in Grade 11’ and ‘Geebee is a senior high school student’ are both true. Since the hypothesis and conclusion are both true, then **p_1 is true**, as the first row of the truth table asserts.

p_2 : ‘Geebee is in Grade 11’ is true and ‘Geebee is working as a lawyer’ is false because a Grade 11 student is not yet qualified to be a lawyer. Since the hypothesis is true but the conclusion is false, then **p_2 is false**, as the second row of the truth table indicates.

p_3 : ‘Geebee has a degree in Computer Science’ is false because Geebee is still in Grade 11, and so cannot not possibly have a university degree yet. On the other hand, we do not know the truth value of ‘Geebee believes in true love’. However, according to the last two rows of the truth table, **p_3 is true** regardless of the truth value of its conclusion.

□

Another way to understand the truth value of conditional proposition $p \rightarrow q$ is to think of it as a promise or a contract. The conditional $p \rightarrow q$ is false or, equivalently, the promise is broken when the hypothesis p is true, while the conclusion q is false.

EXAMPLE 7.

One day, Richard tweeted to the utter delight of his friends: ‘If I get promoted, then I will stop posting selfies on Facebook.’ Let p be the statement ‘Richard gets promoted’ and let q be the statement, ‘Richard stops posting selfies on Facebook.’ Determine whether the conditional $p \rightarrow q$ is true in each of the following scenarios.

Scenario A: Richard got the promotion and promptly stopped posting selfies on Facebook.

Scenario B: Richard got promoted, but then he realized that posting selfies has become a habit that he cannot break easily.

Scenario C: Richard stopped posting selfies, but he did not get the promotion.

Solution.

Scenario A: In this scenario both the hypothesis and the conclusion are true. This means that Richard kept his promise. According to the first row of the truth table, the conditional $p \rightarrow q$ is true.

Scenario B: In this scenario, Richard broke his promise. According to the second row of the truth table, the conditional $p \rightarrow q$ is false.

Scenario C: In this scenario, did Richard break his promise? Of course not. He did not say that his promotion is the only way to make him stop posting selfies. Perhaps Richard read an article which claims that people who are fond of posting selfies have some psychological disorder! In any case he did not break his promise, and according to the third and fourth rows of the truth table, the conditional $p \rightarrow q$ is true.

In fact, the truth table asserts that when the hypothesis p is false, the conditional $p \rightarrow q$ is true regardless of the truth value of the conclusion q . One way to look at this is that there is no way that Richard can break his promise, when he fails to get the promotion.

□

Let us look at one last example.

EXAMPLE 8. Determine the truth values of the following propositions.

- (a) If $2 > 0$, then there are more than 100 million Filipinos.
- (b) If $2 > 0$, then there are only 5 languages spoken in the Philippines.

(c) If $2 < 0$, then it is more fun in the Philippines.

Solution. The number 2 is a positive number, and so the proposition ' $2 > 0$ ' is true, while ' $2 < 0$ ' is false.

- (a) The hypothesis and the conclusion are both true. Hence the conditional is true.
- (b) The hypothesis is true, but the conclusion is wrong because there are more than 5 languages in the Philippines! In fact there are more than 100 languages in the country. Thus, the conditional is false.
- (c) Because the hypothesis is false, the conditional is true whether it is indeed more fun in the Philippines or not.

□



Teaching tip

Remark that the in the previous example the hypotheses ' $2 > 0$ ' and ' $2 < 0$ ' have nothing to do with the conclusions. For example, we are not saying that there are more than 100 million Filipinos *because* $2 > 0$. In logic, the hypothesis need not *cause* the conclusion.

6. Define biconditional proposition.

We now define another compound proposition which we can derive from two conditional propositions.

Definition.

The **biconditional** of propositions p and q is denoted by

$$p \leftrightarrow q : (p \text{ if and only if } q,)$$

and is defined through its truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

The proposition may also be written as ' p iff q '. The propositions p and q are the **components** of the biconditional.

The truth table of a biconditional tells us that its truth value is true when the truth values of p and q are the same.

Let us revisit the scenario in our previous example.

EXAMPLE 9. Suppose that Geebee is a Grade 11 student. Let us now consider the following biconditionals:

p_1 : Geebee is in Grade 11 if and only if she is a senior high school student.

p_2 : Geebee is in Grade 11 if and only if she is working as a lawyer.

p_3 : Geebee has a degree in Computer Science if and only if she believes in true love.

Analyze the truth values of the given biconditionals.

Solution.

p_1 : Again, both simple components of p_1 are true. Hence, the biconditional is true, according to the first row of the truth table.

p_2 : Since it is true that Geebee is in Grade 11 but it is not true that Geebee is working as a lawyer, the biconditional is false as the second row of the truth table indicates.

p_3 : The truth value of the biconditional p_3 depends on whether Geebee believes in true love or not. If Geebee does not believe in true love, then both components of the biconditional are false which makes the biconditional true according to the truth table. On the other hand, supposing Geebee believes in true love, the truth value of biconditional is false.

□

7. Seatwork

Seatwork 1. Express the following propositions in symbols, where p , q and r are defined as follows.

p : Neil is a big eater.

q : Len has a big voice.

r : Jeric likes to travel.

s : Lemy likes violet.

- (a) ‘While Len has a big voice, Neil is not a big eater.’
- (b) ‘Jeric likes to travel or he does not.’
- (c) ‘It is not true that Neil is a big eater and Lemy does not like violet.’
- (d) ‘It may or may not be the case that Lemy likes violet.’
- (e) ‘Either Neil is a big eater or Len has a big voice, yet Lemy likes violet.’
- (f) ‘If Neil is a big eater or Len has a big voice, then Lemy likes violet.’
- (g) ‘Neil is a big eater or Len has a big voice if and only if Lemy likes violet and Jeric likes to travel.’

Seatwork 2. Express the following propositions in words, using the given propositions in Seatwork 1.

- (a) $p \wedge (\sim q)$
- (b) $p \wedge (\sim p)$
- (c) $\sim (q \vee r)$
- (d) $((\sim p) \vee q) \wedge r$
- (e) $(p \wedge (\sim q)) \vee (r \wedge s)$
- (f) $\sim (q \rightarrow r)$
- (g) $(p \wedge (\sim q)) \rightarrow (r \wedge s)$
- (h) $p \leftrightarrow (\sim p)$
- (i) $((\sim p) \rightarrow q) \leftrightarrow r$

Seatwork 3. Find the truth values of the propositions in Seatwork 1 and 2 for the given scenario.

<i>p</i>	<i>q</i>	<i>r</i>	<i>s</i>
<i>T</i>	<i>F</i>	<i>T</i>	<i>T</i>

Seatwork 4. Determine the truth value of the biconditional ‘A polygon is a square if and only if it is a rectangle.’

(C) Class Activity.

In this activity we look at the Instagram world of four girls: Janella, Julia, Kathryn and Liza. We summarize their instagram dynamics—who follows who—in a table such as the following.

	Janella	Julia	Kathryn	Liza
Janella				
Julia				
Kathryn				
Liza				

A check in a cell of table means that the girl named at the beginning of the row follows on instagram the girl at the head of the column.

Instruction: Make the following propositions true by checking the appropriate cell.

Liza follows Kathryn, but does not follow Janella.

Either Julia follows Kathryn, or Julia follows Liza.

While Janella follows everyone that Julia follows, Janella does not follow Liza.

Kathryn follows everyone who follows her.

Nobody follows herself.

There are several ways that this can be done.

Solution to the activity.

These are some of the Instagram ‘worlds’ where the given propositions are all true.

	Janella	Julia	Kathryn	Liza
Janella			✓	
Julia			✓	
Kathryn	✓	✓		✓
Liza			✓	

	Janella	Julia	Kathryn	Liza
Janella		✓	✓	
Julia			✓	
Kathryn	✓	✓		✓
Liza			✓	

	Janella	Julia	Kathryn	Liza
Janella			✓	
Julia			✓	
Kathryn	✓	✓		✓
Liza		✓	✓	

	Janella	Julia	Kathryn	Liza
Janella		✓	✓	
Julia			✓	
Kathryn	✓	✓		✓
Liza		✓	✓	

LESSON 38: Constructing Truth Tables

TIME FRAME: 60 minutes

CODE: M11GM-IIg-3, g-4, and h-1

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to determine the possible truth values of propositions.

LESSON OUTLINE:

1. Constructing truth tables
 2. Tautology and contradiction.
-

DEVELOPMENT OF THE LESSON**(A) Introduction.**

In previous sessions, we identified the following statements as compound propositions and also defined their corresponding logical operators:

Logical Connectors	Logical Operators
<i>not p</i>	$\sim p$
<i>p and q</i>	$p \wedge q$
<i>p or q</i>	$p \vee q$
<i>if p then q</i>	$p \rightarrow q$
<i>p if and only if q</i>	$p \leftrightarrow q$

where p and q are propositions.

By now, you should have mastered the truth table of the logical operators because we are going to use them to determine all the possible truth values of more complex compound propositions using truth tables.

(B) Lesson Proper

1. Constructing truth tables.

We discuss the construction via some examples.

EXAMPLE 1. Let p and q be propositions. Construct the truth table of the compound proposition $(p \rightarrow q) \wedge (q \rightarrow p)$.

Solution. As previously discussed, since there are two primitive propositions p and q involved, the truth table should have four rows which consist of all possible truth values combination of p and q .

p	q
T	T
T	F
F	T
F	F

The given proposition is a conjunction of the conditionals $(p \rightarrow q)$ and $(q \rightarrow p)$ as the conjuncts. In the next two columns, we encode the truth values of these conditionals using the definition discussed in the previous meeting.

p	q	$p \rightarrow q$	$q \rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

In the final column, we encode the truth values of the conjunction $(p \rightarrow q) \wedge (q \rightarrow p)$ using the third and fourth columns.

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

□

Teaching tip

You may remark that the conjunction $(p \rightarrow q) \wedge (q \rightarrow p)$ and biconditional $p \leftrightarrow q$ have the same truth tables.

p	q	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

This makes it natural to use the symbol \leftrightarrow for a biconditional.

EXAMPLE 2. Consider the compound proposition $[(p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow [(p \vee q) \rightarrow r]$. Construct its truth table.

Solution. The given proposition is a conditional with the propositions $(p \rightarrow r) \wedge (q \rightarrow r)$ and $(p \vee q) \rightarrow r$ as hypothesis and conclusion, respectively. As there are three primitive propositions p, q and r involved, its corresponding truth table should have $2^3 = 8$ rows. The first 3 columns will be for the propositions p, q and r , and on these columns we encode all the possible truth value combination of p, q and r .

First, we consider the hypothesis $(p \rightarrow r) \wedge (q \rightarrow r)$. In the next three columns, we encode the truth values of the compound propositions $p \rightarrow r$, $q \rightarrow r$, and their conjunction $(p \rightarrow r) \wedge (q \rightarrow r)$, using the truth values of conditional and conjunction.

p	q	r	$p \rightarrow r$	$q \rightarrow r$	$(p \rightarrow r) \wedge (q \rightarrow r)$
T	T	T	T	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

Next, we consider the conclusion $(p \vee q) \rightarrow r$. We encode the truth values of $p \vee q$ and the conditional $(p \vee q) \rightarrow r$ in the next two columns.

p	q	r	$p \rightarrow r$	$q \rightarrow r$	$(p \rightarrow r) \wedge (q \rightarrow r)$	$p \vee q$	$(p \vee q) \rightarrow r$
T	T	T	T	T	T	T	T
T	T	F	F	F	F	T	F
T	F	T	T	T	T	T	T
T	F	F	F	T	F	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	F
F	F	T	T	T	T	F	T
F	F	F	T	T	T	F	T

On the final column, we encode the truth value of the given compound conditional $s : [(p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow [(p \vee q) \rightarrow r]$.

p	q	r	$p \rightarrow r$	$q \rightarrow r$	$(p \rightarrow r) \wedge (q \rightarrow r)$	$p \vee q$	$(p \vee q) \rightarrow r$	s
T	T	T	T	T	T	T	T	T
T	T	F	F	F	F	T	F	T
T	F	T	T	T	T	T	T	T
T	F	F	F	T	F	T	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	F	F	T	F	T
F	F	T	T	T	T	F	T	T
F	F	F	T	T	T	F	T	T

Notice that the last column of the truth table consists entirely of T . This means that the proposition $[(p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow [(p \vee q) \rightarrow r]$ is *always true* for all possible combinations of the truth values of $p, q,$ and r . Such propositions are called **tautologies**.

□

2. Define a tautology and contradiction.

Definition.

A proposition that is always true is called a **tautology**, while a proposition that is always false is called a **contradiction**. A tautology is denoted by τ and a contradiction by ϕ .

EXAMPLE 3. Let p and q be propositions. Using truth tables, show the following:

- i. $p \vee \tau$ is a tautology,
- ii. $p \wedge \phi$ is a contradiction,
- iii. $p \rightarrow (p \vee q)$ is a tautology, and
- iv. $(p \wedge (\sim q)) \wedge (p \wedge q)$ is a contradiction.

Solution. We give the truth table for each proposition.

- i. Note that τ is a proposition which is always true, so the second column consists of T . The proposition $p \vee \tau$ is true for any truth value of p ; hence, it is a tautology.

p	τ	$p \vee \tau$
T	T	T
F	T	T

- ii. Since ϕ is a proposition which is always false, the second column is all F . The truth table of $p \wedge \phi$ says that it is false regardless of the truth value of p , and so $p \wedge \phi$ is a contradiction.

p	ϕ	$p \wedge \phi$
T	F	F
F	F	F

- iii. The proposition $p \rightarrow (p \vee q)$ is a tautology as its truth table shows that it is always true for any truth value combinations of p and q .

p	q	$p \vee q$	$p \rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

- iv. Since $(p \wedge \sim q) \wedge (p \wedge q)$ is always false for any truth value combinations of p and q , it is a contradiction.

p	q	$\sim q$	$p \wedge (\sim q)$	$p \wedge q$	$(p \wedge (\sim q)) \wedge (p \wedge q)$
T	T	F	F	T	F
T	F	T	T	F	F
F	T	F	F	F	F
F	F	T	F	F	F

□

3. Seatwork.

Seatwork 1. Construct the truth table of the following propositions.

- (a.) $((p \rightarrow q) \wedge q) \rightarrow p$
- (b.) $((p \rightarrow q) \wedge (\sim p)) \rightarrow \sim q$
- (c.) $((p \vee q) \wedge p) \rightarrow (\sim q)$
- (d.) $(p \rightarrow q) \rightarrow (q \rightarrow p)$
- (e.) $(\sim (p \wedge q) \wedge (\sim p)) \rightarrow q$
- (f.) $(p \rightarrow q) \rightarrow ((\sim p) \rightarrow (\sim q))$

Seatwork 2. Show that the following are tautologies.

- (a.) $(p \wedge q) \rightarrow p$
- (b.) $p \rightarrow (p \vee q)$
- (c.) $(p \wedge q) \rightarrow (p \wedge q)$
- (d.) $((p \rightarrow q) \wedge p) \rightarrow q$
- (e.) $((p \rightarrow q) \wedge (\sim q)) \rightarrow \sim p$
- (f.) $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$
- (g.) $((p \vee q) \wedge (\sim p)) \rightarrow q$
- (h.) $((\sim p) \rightarrow \phi) \rightarrow p$
- (i.) $((p \rightarrow r) \wedge (q \rightarrow r)) \rightarrow ((p \vee q) \rightarrow r)$

LESSON 39: Logical Equivalence and Forms of Conditional Propositions

TIME FRAME: 180 minutes

CODE: M11GM-II h-2

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to illustrate different forms of conditional propositions.

LESSON OUTLINE:

1. Logically equivalent propositions

2. Different forms of conditional proposition

DEVELOPMENT OF THE LESSON

(A) Introduction.

In the previous lesson, we constructed the truth tables of the conjunction $(p \rightarrow q) \wedge (q \rightarrow p)$ and biconditional $p \leftrightarrow q$, which are shown in a combined table below.

p	q	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

Observe that the third and fourth columns are exactly the same, which means that these two propositions have the same truth values! We call such pair of propositions *logically equivalent*.

(B) Lesson Proper

1. Define logical equivalence.

Definition.

Two propositions p and q are **logically equivalent**, denoted by $p \Leftrightarrow q$, if they have the same truth values for all possible truth values of their simple components.

The logical equivalence between propositions is analogous to the concept of equality between numbers.

 **Teaching tip**

Note that, by definition, $p \Leftrightarrow q$ implies that p and q have the same truth values, which is the same as saying that $p \leftrightarrow q$ is always true (and so is a tautology).

EXAMPLE 1. Show that $(p \rightarrow q) \Leftrightarrow [(\sim p) \vee q]$. We shall call this logical equivalence the *Switcheroo law*.¹

¹Waner, C. & Costenoble, S.R. (2001). *Supplementary chapters to accompany Finite Mathematics, 2nd ed.* Brooks/Cole. (<http://www.zweigmedia.com/RealWorld/logic/logicintro.html>)

Solution. We show that $(p \rightarrow q)$ and $(\sim p) \vee q$ have the same truth tables.

p	q	$p \rightarrow q$	$\sim p$	$(\sim p) \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

The third and fifth columns show that $(p \rightarrow q)$ and $(\sim p) \vee q$ have the same truth tables. Hence, $(p \rightarrow q) \Leftrightarrow [(\sim p) \vee q]$.

Alternate solution. We can also show that $(p \rightarrow q) \leftrightarrow [(\sim p) \vee q]$ is a tautology.

p	q	$p \rightarrow q$	$\sim p$	$(\sim p) \vee q$	$(p \rightarrow q) \leftrightarrow [(\sim p) \vee q]$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

The final column shows that that $(p \rightarrow q) \leftrightarrow [(\sim p) \vee q]$ is a tautology. □

The following table presents logical equivalences which are commonly used in logical manipulations.

Table of Logical Equivalences		
Let p , q , and r be propositions. One can verify using truth tables that the following are indeed logical equivalences.		
Identity Laws	$(p \wedge \tau) \Leftrightarrow p$	$(p \vee \phi) \Leftrightarrow p$
Domination Laws	$(p \vee \tau) \Leftrightarrow \tau$	$(p \wedge \phi) \Leftrightarrow \phi$
Idempotent Laws	$(p \vee p) \Leftrightarrow p$	$(p \wedge p) \Leftrightarrow p$
Inverse Laws	$(p \vee [\sim p]) \Leftrightarrow \tau$	$(p \wedge [\sim p]) \Leftrightarrow \phi$
Double Negation	$\sim(\sim p) \Leftrightarrow p$	
Associative Laws	$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$	$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$
Commutative Laws	$p \vee q \Leftrightarrow q \vee p$	$p \wedge q \Leftrightarrow q \wedge p$
Distributive Laws	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
De Morgan's Laws	$\sim(p \vee q) \Leftrightarrow (\sim p) \wedge (\sim q)$	$\sim(p \wedge q) \Leftrightarrow (\sim p) \vee (\sim q)$
Absorption Laws	$p \vee (p \wedge q) \Leftrightarrow p$	$p \wedge (p \vee q) \Leftrightarrow p$


Teaching tip

If two propositions are logically equivalent, then we can *substitute* the occurrence of one with the other, as the next example shows.

EXAMPLE 2. Show that $\sim(p \rightarrow q) \Leftrightarrow [p \wedge (\sim q)]$.

Solution. One way to do this is to show that $\sim(p \rightarrow q)$ and $p \wedge (\sim q)$ have the same truth tables. However, in this example, we use local equivalences to transform $\sim(p \rightarrow q)$ into $p \wedge (\sim q)$:

	Reason
$\sim(p \rightarrow q)$	
$\Leftrightarrow \sim((\sim p) \vee q)$	Switcheroo
$\Leftrightarrow \sim(\sim p) \wedge (\sim q)$	De Morgan's Laws
$\Leftrightarrow p \wedge (\sim q)$	Double Negation

□

EXAMPLE 3. Let p and q be propositions. Construct the truth tables of the following conditionals: $p \rightarrow q$, $q \rightarrow p$, $\sim p \rightarrow \sim q$, and $\sim q \rightarrow \sim p$.

Solution. The combined truth table is given below.

p	q	$p \rightarrow q$	$q \rightarrow p$	$\sim p$	$\sim q$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Since the third and the last columns are exactly the same, we have shown that $(p \rightarrow q) \Leftrightarrow \sim q \rightarrow \sim p$.

Similarly, since the fourth and the seventh columns are identical, we have that $(q \rightarrow p) \Leftrightarrow (\sim p \rightarrow \sim q)$. □

2. Define different forms of conditional proposition.

The conditionals we considered in the previous example are the different forms of conditional propositions.

Definition.

Given propositions p and q . There are three propositions that we can derive from the conditional $p \rightarrow q$, namely, its

- (a) **converse:** $q \rightarrow p$
- (b) **contrapositive:** $\sim q \rightarrow \sim p$
- (c) **inverse:** $\sim p \rightarrow \sim q$

Logical equivalences of conditionals

In the previous example, we have shown a proposition and its contrapositive are logically equivalent. Also a proposition's converse and inverse are logically equivalent.

Another way to show these logical equivalences is via the Switcheroo law and the table of logical equivalences.

To show that $(p \rightarrow q) \Leftrightarrow \sim q \rightarrow \sim p$:

	Reason
$\sim q \rightarrow \sim p$	
$\Leftrightarrow \sim(\sim q) \vee \sim p$	Double Switcheroo
$\Leftrightarrow q \vee \sim p$	Double Negation
$\Leftrightarrow \sim p \vee q$	Commutative Law
$\Leftrightarrow p \rightarrow q$	Switcheroo

Similarly, to show that $(q \rightarrow p) \Leftrightarrow (\sim p \rightarrow \sim q)$:

	Reason
$\sim p \rightarrow \sim q$	
$\Leftrightarrow \sim(\sim p) \vee (\sim q)$	Switcheroo
$\Leftrightarrow p \wedge (\sim q)$	Double Negation
$\Leftrightarrow (\sim q) \wedge p$	Commutative Law
$\Leftrightarrow q \rightarrow p$	Switcheroo

Usually, when you show logical equivalence without using the truth table, start with the more complicated proposition and then use some known logical equivalences to arrive at the other proposition.

EXAMPLE 4. Consider the following true conditional:

$p \rightarrow q$: "If Geebee is in Grade 11, then she is a senior high school student."

State its (a) converse, (b) contrapositive, and (c) inverse, and determine whether each statement is also true.

Solution.

(a) *converse*

$q \rightarrow p$: "If Geebee is a senior high school student, then she is in Grade 11." This is **not necessarily true**, because Geebee may be a Grade 12 student.

(b) *contrapositive*

$\sim q \rightarrow \sim p$: "If Geebee is not a senior high school student, then she is not in Grade 11." This is **true**.

(c) *inverse*

$\sim p \rightarrow \sim q$: "If Geebee is not in Grade 11, then she is not a senior high school student." This is **not necessarily true**, because she may be in Grade 10, which is not part of senior high school.

□

A conditional and its contrapositive have the same truth values. This is (intuitively) clear from the given example. If Geebee is not a senior high school student, then she cannot possibly be in Grade 11 ($\sim q \rightarrow \sim p$).

A conditional and its converse do not necessarily have the same truth values. In the previous example, we saw that if Geebee is a senior high school student then we cannot assume that she is in Grade 11 (she could be in Grade 12).

 **Teaching tip**

Remark that this example illustrates that in order to show that two propositions are not logically equivalent, it is sufficient to show that they do not have the same truth value for a particular scenario. In this example, we used the scenario that Geebee is in Grade 12.

EXAMPLE 5. Let $p \rightarrow q$ be the political slogan: 'If there are no corrupt people, then there are no poor people.' (This is more popularly known in the vernacular: 'Kung walang kurap, walang mahirap.') State the converse, contrapositive, and inverse of $p \rightarrow q$.

Solution. Note that the hypothesis and conclusion are p : "there are no corrupt people" and q : "there are no poor people", and their negations are $\sim p$: "there are corrupt people" and $\sim q$: "there are poor people".

(a) converse: If there are no poor people, then there are no corrupt people.

- (b) contrapositive: If there are poor people, then there are corrupt people.
 (c) inverse: If there are corrupt people, then there are poor people.

□

3. Seatwork

Seatwork 1. Verify each proposition in the Table of Logical Equivalences.

Seatwork 2. Show the following logical equivalences (a) using truth table and (b) using the known logical equivalences:

(a.) $\sim (p \vee ((\sim p) \wedge q)) \Leftrightarrow ((\sim p) \wedge (\sim q))$

(b.) $((p \wedge q) \rightarrow (p \vee q)) \Leftrightarrow \tau$

Seatwork 3. Show that $\sim (p \rightarrow (\sim q)) \Leftrightarrow (p \wedge q)$ by completing the table below.

	Reason
$\sim (p \rightarrow (\sim q))$	
$\Leftrightarrow \sim (\sim p \vee \sim q)$?
$\Leftrightarrow \sim (\sim p) \wedge \sim (\sim q)$?
$\Leftrightarrow p \wedge q$?

[Ans:] Switcheroo, De Morgan's Law, Double Negation

Seatwork 4. Show that $(p \rightarrow (q \wedge r)) \Leftrightarrow ((p \rightarrow q) \wedge (p \rightarrow r))$ by completing the table below.

	Reason
$(p \rightarrow (q \wedge r))$	
$\Leftrightarrow \sim p \vee (q \wedge r)$?
$\Leftrightarrow (\sim p \vee q) \wedge (\sim p \vee r)$?
$\Leftrightarrow ((p \rightarrow q) \wedge (p \rightarrow r))$?

[Ans:] Switcheroo, Distributive Law, Switcheroo

Seatwork 5. Show the following logical equivalences.

(a.) $(p \vee q) \Leftrightarrow ((\sim p) \rightarrow q)$

(b.) $\sim (p \rightarrow q) \Leftrightarrow (p \wedge (\sim q))$

(c.) $((p \rightarrow r) \wedge (q \rightarrow r)) \Leftrightarrow ((p \vee q) \rightarrow r)$

(d.) $((p \rightarrow q) \vee (p \rightarrow r)) \Leftrightarrow (p \rightarrow (q \vee r))$

(e.) $((p \rightarrow r) \vee (q \rightarrow r)) \Leftrightarrow ((p \wedge q) \rightarrow r)$

Seatwork 6. State the converse, contrapositive, and inverse of the given conditionals.

(a.) "If today is Tuesday, then it is a weekday."

(b.) "If it rains, then I will not go the beach."

(c.) "If a positive integer is prime, then it has no divisors other than 1 and itself."

LESSON 40: Valid arguments and fallacies

TIME FRAME: 180 minutes

CODE: M11GM-III-1, i-2, and i-3

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to illustrate different types of tautologies and fallacies, determine the validity of categorical syllogisms, and establish the validity and falsity of real-life arguments using logical propositions, syllogisms, and fallacies.

LESSON OUTLINE:

1. Argument
 2. Validity condition
 3. Rules of inference
 4. Fallacies
 5. Enrichment: Truth condition
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DEVELOPMENT OF THE LESSON

(A) Introduction.

Given a set of information, our goal is to *logically* infer from the given some new information.

For example, we know that only a small portion of the earth's water is freshwater (mostly it's saltwater). We know that if there is a limited supply of freshwater, then we should conserve water. Combining these information urges us to conclude that we should conserve water.

In this lesson, we are going to look at this process of inference formally. We call the set of given information as the premise, while the derived new information is referred to as the conclusion. The premises and the conclusion form what we call an argument.

(B) Define argument.

Definition.

An **argument** is a compound proposition of the form

$$(p_1 \wedge p_2 \wedge \dots \wedge p_n) \rightarrow q.$$

The propositions p_1, p_2, \dots, p_n are the **premises** of the argument, and q is the **conclusion**.

Arguments can be written in **propositional form**, as above, or in column or **standard form**:

$$\begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}$$

The premises of an argument are intended to act as reasons to establish the validity or acceptability of the conclusion. We will make this statement more precise later on.

EXAMPLE 1. Explain why the following set of propositions is an argument.

If General Antonio Luna is a national hero, then he died at the hands of the Americans in 1899.
 General Antonio Luna is a national hero.
 Therefore, General Luna died at the hands of the Americans in 1899.

Solution. The set of propositions is an argument. Its premises are the propositions ‘If General Antonio Luna is a national hero, then he died at the hands of the Americans in 1899,’ and ‘General Antonio Luna is a national hero.’ The conclusion, which is flagged by the word ‘therefore’, is the proposition ‘General Luna died at the hands of the Americans in 1899.’ \square

Is the argument valid? Is it sound? At the end of the lesson, you will be able to answer these questions.

EXAMPLE 2. Write the following argument presented in the introduction in propositional form and in standard form.

If there is limited freshwater supply, then we should conserve water.
 There is limited freshwater supply.
 Therefore, we should conserve water.

Solution. The premises of this argument are

p_1 : If there is limited freshwater supply, then we should conserve water.

p_2 : There is limited freshwater supply.

and its conclusion is

q : We should conserve water.

In symbols, we can write the whole argument in propositional form

$$(p_1 \wedge p_2) \rightarrow q,$$

and in standard form

$$\frac{\begin{array}{c} p_1 \\ p_2 \end{array}}{\therefore q}$$

□

The following example shows that two arguments may have *equivalent logical forms*, even if they are different in *content*.

EXAMPLE 3. Consider the arguments A and A' given below:

A	A'
$p \rightarrow q$ If my alarm sounds, then I will wake up.	$p \rightarrow q$ If my alarm sounds, then I will wake up.
p My alarm sounded.	q I woke up.
$\frac{\quad}{\therefore q}$ Therefore, I woke up.	$\frac{\quad}{\therefore p}$ Therefore, my alarm sounded.

Analyze these arguments separately by answering the following question.

Validity Condition:

Is it *logically impossible* for the premises to be true and the conclusion false?

If the answer is YES, we say that the argument satisfies the validity condition. The argument is valid.

Solution.

Argument A

Can $p \rightarrow q$ and p both be true and q be false?

Suppose that the premises $p \rightarrow q$ and p are true. The truth table below shows that both p and $p \rightarrow q$ are true only in the first row. In this row, q is also true.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Hence, YES it is logically impossible for the premises to be true and the conclusion to be false. We say that A satisfies the validity condition, and so it is valid. In practical terms, this means that if my alarm sounded, then I surely woke up.

Argument A'

Can $p \rightarrow q$ and q be true, and p be false?

Suppose that the premises $p \rightarrow q$ and q are true. The truth table below shows that both $p \rightarrow q$ and q are true in the first and third rows. However, in the third row, the conclusion p is false, even if the premises are true.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Argument A' does not satisfy the validity condition. In practical terms, it is possible that I woke up but my alarm did not sound. I could have woken up due to a bad dream, for example.

In summary, we have the following.

$p \rightarrow q$	If my alarm sounds, then I will wake up.	False \rightarrow True
q	I woke up.	True
$\therefore p$	Therefore, my alarm sounded.	\therefore False

□

(C) Define valid argument.

Definition.

A **valid argument** satisfies the validity condition; that is, the conclusion q is true *whenever* the premises p_1, p_2, \dots, p_n are all true.

Put another way, for a valid argument, the conditional

$$(p_1 \wedge p_2 \wedge \dots \wedge p_n) \rightarrow q$$

is a tautology.

EXAMPLE 4. Prove that the argument $((p \rightarrow q) \wedge p) \rightarrow q$ is valid.

This argument is known as **Modus Ponens** (or Rule of Detachment).

Solution. We only need to show that the $((p \rightarrow q) \wedge p) \rightarrow q$ is a tautology. We can do this using the truth table.

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$((p \rightarrow q) \wedge p) \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Since $((p \rightarrow q) \wedge p) \rightarrow q$ is a tautology, then the argument is valid. □

EXAMPLE 5. Let us consider the arguments in the previous examples.

Argument A

If my alarm sounds, then I will wake up.

My alarm sounded.

Therefore, I woke up.

Argument B

If there is limited freshwater supply, then we should conserve water.

There is limited freshwater supply.

Therefore, we should conserve water.

Argument C

If General Antonio Luna is a national hero, then he died at the hands of the Americans in 1899.

General Antonio Luna is a national hero.

Therefore, General Luna died at the hands of the Americans in 1899.

Note that they are all of the form $((p \rightarrow q) \wedge p) \rightarrow q$, or in standard form

$$\frac{p \rightarrow q}{p} \therefore q$$

Hence, by Modus Ponens, arguments A , B , and C are all valid. **However, this does not mean that the conclusions are true.** Asserting that an argument is valid simply means that the conclusion **logically** follows from the premises.

Note to the teacher:

Remark that this example illustrates that the validity of an argument *does not* depend on the content of the argument, but on its form.

In the next table, we display some basic forms of valid arguments in logic. They are also known as **rules of inference**. The associated tautologies of these arguments were already established in Lesson 38, Seatwork 2.

Rules of Inference		
Let p , q , and r be propositions.		
	Propositional Form	Standard Form
Rule of Simplification	$(p \wedge q) \rightarrow p$	$\frac{p \wedge q}{\therefore p}$
Rule of Addition	$p \rightarrow (p \vee q)$	$\frac{p}{\therefore p \vee q}$
Rule of Conjunction	$(p \wedge q) \rightarrow (p \wedge q)$	$\frac{p}{q}$ $\therefore p \wedge q$
Modus Ponens	$((p \rightarrow q) \wedge p) \rightarrow q$	$\frac{p \rightarrow q}{p}$ $\therefore q$
Modus Tollens	$((p \rightarrow q) \wedge (\sim q)) \rightarrow \sim p$	$\frac{p \rightarrow q}{\sim q}$ $\therefore \sim p$
Law of Syllogism	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$	$\frac{p \rightarrow q}{q \rightarrow r}$ $\therefore p \rightarrow r$
Rule of Disjunctive Syllogism	$((p \vee q) \wedge (\sim p)) \rightarrow q$	$\frac{p \vee q}{\sim p}$ $\therefore q$
Rule of Contradiction	$((\sim p) \rightarrow \phi) \rightarrow p$	$\frac{(\sim p) \rightarrow \phi}{\therefore p}$
Rule of Proof by Cases	$((p \rightarrow r) \wedge (q \rightarrow r)) \rightarrow ((p \vee q) \rightarrow r)$	$\frac{p \rightarrow r}{q \rightarrow r}$ $\therefore (p \vee q) \rightarrow r$

EXAMPLE 6. Determine whether the following argument is valid.

If Antonio and Jose are friends, then they are Facebook friends.
 Antonio and Jose are not Facebook friends.
 Therefore, they are not friends.

Solution. Let

p : Antonio and Jose are friends.
 q : Antonio and Jose are Facebook friends.

Then the given argument is of the form

$$\frac{p \rightarrow q \quad \sim q}{\therefore \sim p}$$

Hence, by Modus Tollens, the argument is valid. □

EXAMPLE 7. Determine which rule is the basis of each argument below.

- (a) Antonio Luna and Jose Rizal like Nelly Boustead.
 Therefore, Antonio Luna likes Nelly Boustead.
- (b) Antonio Luna is a scientist.
 Therefore, either Antonio Luna or Jose Rizal is a scientist.
- (c) If the Spaniards imprison Antonio Luna, then he will repent and not join the revolution.
 If Antonio Luna regrets not joining the revolution, then he will go to Belgium to study the art of war.
 Therefore, if the Spaniards imprison Antonio Luna, then he will go to Belgium to study the art of war.

Solution. (a) Let

p : Antonio Luna likes Nelly Boustead.
 q : Jose Rizal likes Nelly Boustead.

Then the given argument is of the form

$$\frac{p \wedge q}{\therefore p}$$

By the Rule of Simplification, the argument is valid.

(b) Let

p : Antonio Luna is a scientist.

q : Jose Rizal is a scientist.

The argument in standard form is

$$\frac{p}{\therefore p \vee q},$$

which is valid by the Rule of Addition.

(c) Let

p : The Spaniards imprison Antonio Luna.

q : Antonio Luna regrets not joining the revolution.

r : Antonio Luna goes to Belgium to study the art of war.

In standard form, we have

$$\frac{\begin{array}{l} p \rightarrow q \\ q \rightarrow r \end{array}}{\therefore p \rightarrow r}$$

Thus, the argument is valid by the Law of Syllogism.

□

(D) Define fallacy.

Definition.

An argument

$$(p_1 \wedge p_2 \wedge \dots \wedge p_n) \rightarrow q,$$

which is not valid is called a **fallacy**.

In a fallacy, it is possible for the premises p_1, p_2, \dots, p_n to be true, while the conclusion q is false.

Equivalently, for this case, the conditional

$$(p_1 \wedge p_2 \wedge \dots \wedge p_n) \rightarrow q$$

is *not* a tautology.

EXAMPLE 8. Prove that the argument $((p \rightarrow q) \wedge q) \rightarrow p$ is a fallacy.

This is known as the **Fallacy of the Converse**.

Solution. We show that $((p \rightarrow q) \wedge q) \rightarrow p$ is not a tautology using truth table.

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge q$	$((p \rightarrow q) \wedge q) \rightarrow p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

We see that on the third row, the premises q and $p \rightarrow q$ are both true but the conclusion p is false. Therefore, the given argument is a fallacy.

Alternate solution: It is enough to find such truth values for p and q that will make the premises simultaneously true but the conclusion false. We call such set of values a **counterexample**. For this fallacy, the counterexample occurs when **p is false and q is true**. These yield the following truth values for the argument:

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge q$	$((p \rightarrow q) \wedge q) \rightarrow p$
F	T	T	T	F

□

EXAMPLE 9. Show that the following arguments are fallacies.

Argument A'

If my alarm sounds, then I will wake up.

I woke up.

Therefore, my alarm sounded.

Argument B'

If there is limited supply of freshwater, then I will conserve water.

I will conserve water.

Therefore, there is limited supply of freshwater.

Solution. Each of the arguments has the form

$$\frac{p \rightarrow q}{q} \therefore p$$

Thus, by the previous example, each argument is a fallacy of the converse.

For Argument A' , it is possible that I woke up because of a bad dream, or because I suddenly remembered that I still have to study for my Logic exam!

For Argument B' , even if I will conserve water, it does not logically imply that there is a limited supply of freshwater. Perhaps, I just want to lower my monthly water bill!

□

The following table lists some common fallacies in logic.

Table of Fallacies		
Let p , q , and r be propositions.		
	Propositional Form	Standard Form
Fallacy of the Converse	$((p \rightarrow q) \wedge q) \rightarrow p$	$\frac{p \rightarrow q}{q} \\ \therefore p$
Fallacy of the Inverse	$((p \rightarrow q) \wedge (\sim p)) \rightarrow (\sim q)$	$\frac{p \rightarrow q}{\sim p} \\ \therefore \sim q$
Affirming the Disjunct	$((p \vee q) \wedge p) \rightarrow (\sim q)$	$\frac{p \vee q}{p} \\ \therefore \sim q$
Fallacy of the Consequent	$(p \rightarrow q) \rightarrow (q \rightarrow p)$	$\frac{p \rightarrow q}{\therefore q \rightarrow p}$
Denying a Conjunct	$(\sim (p \wedge q) \wedge (\sim p)) \rightarrow q$	$\frac{\sim (p \wedge q)}{\sim p} \\ \therefore q$
Improper Transposition	$(p \rightarrow q) \rightarrow ((\sim p) \rightarrow (\sim q))$	$\frac{p \rightarrow q}{\therefore (\sim p) \rightarrow (\sim q)}$

EXAMPLE 10. Determine whether the given is a valid argument or a fallacy.

Either Alvin sings or dances with Nina.

- (a) Alvin sang with Nina.
Therefore, Alvin did not dance with Nina.

Either Alvin sings or dances with Nina.

- (b) Alvin did not dance with Nina.
Therefore, Alvin sang with Nina.

It is not true that Alvin sings and dances with Nina.

(c) Alvin did not sing with Nina.

Therefore, Alvin danced with Nina.

Solution. Let

p : Alvin sings with Nina.

q : Alvin dances with Nina.

(a) Then the given argument is of the form

$$\frac{p \vee q}{p} \\ \therefore \sim q$$

Thus, the argument is the fallacy of Affirming the Disjunct.

Alternate solution. We can show that the propositional form $[(p \vee q) \wedge p] \rightarrow \sim q$ of the argument is not a tautology (i.e., it is false for some combination of truth values for p and q).

We now find values for p and q so that the hypothesis $[(p \vee q) \wedge p]$ is true but the conclusion $\sim q$ must be false. To make $\sim q$ false, then q must be true. Likewise, p must be true for the hypothesis to be true. We now obtain the following table.

p	q	$\sim q$	$p \vee q$	$(p \vee q) \wedge p$	$[(p \vee q) \wedge p] \rightarrow \sim q$
T	T	F	T	T	F

(b) In symbols, the given argument has of the form

$$\frac{p \vee q}{\sim q} \\ \therefore p$$

By commutativity law $((p \wedge q) \Leftrightarrow (q \wedge p))$, we have

$$\frac{q \vee p}{\sim q} \\ \therefore p$$

Thus, by the Rule of Disjunctive Syllogism, the argument is valid.

(c) Transforming the argument in symbols yields the following.

$$\frac{\sim (p \wedge q)}{\sim p} \\ \therefore q$$

This is the fallacy of Denying a Conjunct.

Alternate solution. We can find truth values for p and q so that the propositional form of the argument is false. This happens when both p and q are false, as shown below.

p	$\sim p$	q	$p \wedge q$	$\sim (p \wedge q)$	$\sim (p \wedge q) \wedge \sim p$	$[\sim (p \wedge q) \wedge \sim p] \rightarrow q$
F	T	F	F	T	T	F

□

(E) **ENRICHMENT:** Introduce the difference between a **valid argument** and a **sound argument**.

Recall that earlier we considered the argument:

$p \rightarrow q$ If General Antonio Luna is a national hero, then he died at the hands of the Americans in 1899.

p General Antonio Luna is a national hero.

$\therefore q$ Therefore, General Luna died at the hands of the Americans in 1899.

We already know that the argument is valid by Modus Ponens. The argument satisfies the validity condition. **But is the argument sound?**

In this section, we ask the question:

Truth Condition^a

Are the premises of the argument all generally true?

If the answer is YES, then the argument is said to satisfy the truth condition.

^adeLaplante, K. (2013). *What is a good argument? The truth condition*
<https://www.youtube.com/watch?v=9mk8RWTsFFw>

We know that General Luna was killed by fellow Filipinos at the height of Filipino-American War. Then the premise $p \rightarrow q$ is false because General Luna is a national hero (p is true), but he did not die at the hands of the Americans (q is false).

Hence, **the argument, though it is valid, does not satisfy the truth condition.** We say that this argument is *not sound*. For an argument to be considered sound it has to satisfy **both the validity condition and the truth condition**².

Definition.

A **sound argument** is a valid argument which also satisfies the truth condition.

An argument which does not satisfy either the validity condition or the truth condition is called a **bad argument**.

EXAMPLE 11. The following arguments were already shown to be valid.

- (a) Antonio Luna and Jose Rizal like Nelly Boustead.
Therefore, Antonio Luna likes Nelly Boustead.
- (b) Antonio Luna is a scientist.
Therefore, either Antonio Luna or Jose Rizal is a scientist.

²Ibid.

A simple history verification will show that the premises of both arguments are true. Nelly Boustead was the object of affection of Antonio Luna and Jose Rizal while they were in Spain. They even almost killed each other in a duel because of her! Moreover, Luna is known as a brilliant general but he is also a scientist. He studied chemistry at the University of Sto. Tomas, and then he went to Spain where he obtained his license and doctorate in pharmacy.

Hence, these arguments satisfy both the truth condition and the validity condition, and so they are sound arguments.

EXAMPLE 12. Determine whether each of the following arguments is valid, and if each is sound.

If I was born poor, then I cannot serve my country.

(a) I was born poor.

Therefore, I cannot serve my country.

If I study every day, then I will develop a good work ethic.

(b) I study every day.

Therefore, I will develop a good work ethic.

Solution. By Modus Ponens, both arguments are valid. Hence, we just have to determine whether they satisfy the truth condition.

(a) Note that being poor does not prevent one from serving one's country (you can probably think of some examples). Hence, the given argument is a bad argument.

(b) It is accepted as true that if I study everyday, then I will develop a good work ethic. Now, is the statement 'I study everyday' true? You should know! If you do, then this argument satisfies both the validity and truth conditions, and you can rightfully assert that you will develop a good work ethic.

□

(F) Seatwork

Seatwork 1. Use any of the methods in Example 8 to prove that each of the arguments in the Table of Fallacies is *not* a valid argument.

Seatwork 2. Determine whether the argument is valid or not. If it valid, identify the rule of inference which justifies its validity. On the other hand, if it is not valid, construct a counterexample and, if possible, identify what kind of fallacy it is.

$$(p \wedge q) \rightarrow r$$

$$(a.) \frac{p \wedge q}{\therefore r}$$

Answer: Valid (Modus Ponens)

$$(p \wedge q) \rightarrow r$$

$$(b.) \frac{r}{\therefore p \wedge q}$$

Answer: Invalid (Fallacy of the Converse)

$$(c.) \frac{p \rightarrow \sim q}{q} \quad \text{Answer: Valid (Modus Tollens)}$$

$$\hline \therefore \sim p$$

$$(d.) \frac{p \rightarrow \sim q}{\sim p} \quad \text{Answer: Invalid (Fallacy of the Inverse)}$$

$$\hline \therefore q$$

(e.) If Nicanor is a famous author, then he knows how to write. But Nicanor is not a famous author. Hence, Nicanor does not know how to write. *Answer: Invalid (Fallacy of the Inverse)*

(f.) If Liwayway is a famous author, then she knows how to write. Moreover, Liwayway knows how to write. So, she is a famous author. *Answer: Invalid (Fallacy of the Converse)*

Seatwork 3. Determine whether each of the following arguments is (a) valid, and (b) sound.

(a.) If the solution turns blue litmus paper red, then the solution contains acid. The solution does not contain acid. So, the solution does not turn blue litmus paper red. *Answer: Valid (Modus Tollens), and Sound*

(b.) If the solution turns blue litmus paper red, then the solution contains acid. The solution turns blue litmus paper red. So, the solution contains acid. *Answer: Valid (Modus Ponens), and Sound*

(c.) If you study hard, you refine your communication skills and build up your confidence. If you refine your communication skills build up your confidence, then your job opportunities increase. Hence, if you study hard, your job opportunities increase. *Answer: Valid (Syllogism), and Sound*

(d.) Kidnapping is wrong if society disapproves of it. Kidnapping is wrong. So, society disapproves of kidnapping. *Answer: Invalid (Fallacy of the Converse); thus, not sound*

(e.) If overeating causes disease, then it is not healthy. Overeating does not cause disease. So, overeating is healthy. *Answer: Invalid (Fallacy of the Inverse); thus not sound*

(f.) The dinosaurs vanished due to a sudden, extreme drop in temperature. If the dinosaurs vanished due to a sudden, extreme drop in temperature, then earth must have suffered some sort of cataclysm millions of years ago. Therefore, the earth must have suffered some sort of cataclysm millions of years ago. *Answer: Valid (Modus Ponens); Not Sound (there is no generally accepted theory that the first premise is true)*

(g.) If ideas are important, then books are important. If ideas change lives, ideas are important. Hence, if books are important, then ideas change lives. *Answer: Invalid; thus not sound*

Seatwork 4. Discuss why each of the following is a **bad argument**.

(a) If I am not good in math, then I am not intelligent. I am not good in math. Then, I am not intelligent. *Answer: Invalid (Fallacy of the Inverse); thus, bad argument*

- (b) If I have have dark skin, then I am not beautiful. I have have dark skin. Therefore, I am not beautiful. *Answer: Valid (Modus Ponens), but wrong premise: It is not true that people with dark skin are not beautiful; thus bad argument*
- (c) If I drink Gilas power energy drink every day, then I will be good in basketball. I drink Gilas power energy drink every day. Hence, I will be good in basketball. *Answer: Valid (Modus Ponens), but wrong premise: It is not true that people who take a power drink everyday automatically becomes good in basketball; thus bad argument*

Seatwork 5. Construct an argument which

- (a) satisfies the validity condition;
- (b) does not satisfy the validity condition;
- (c) satisfies the validity condition but not the truth condition;
- (d) satisfies both the validity and the truth conditions.

LESSON 41: Methods of Proof

TIME FRAME: 180 minutes

CODE: M11GM-IIj-1 and j-2

LEARNING OUTCOME(S): At the end of the lesson, the learner is able to illustrate the different methods of proof and disproof, and justify mathematical and real-life statements using the different methods of proof and disproof.

LESSON OUTLINE:

1. Proof and proving validity of arguments in propositional form
2. Proof and proving validity of arguments in real-life situations
3. Disproof
4. Indirect proofs
5. Proof and proving validity of arguments in mathematics

DEVELOPMENT OF THE LESSON

(A) Introduction.

Suppose we want to establish the validity of the following argument.

$$\begin{array}{l}
 (p \wedge (\sim q)) \rightarrow r \\
 s \rightarrow p \\
 q \rightarrow (\sim u) \\
 u \wedge s \\
 \hline
 \therefore r
 \end{array}$$

In the previous lesson, we learned that it suffices to show that its corresponding conditional

$$(((p \wedge (\sim q)) \rightarrow r) \wedge (s \rightarrow p) \wedge (q \rightarrow (\sim u)) \wedge (u \wedge s)) \rightarrow r$$

is a tautology. However, the truth table would consist of $2^4 = 16$ rows and about 11 columns. Constructing such a table is very tedious! Also, the truth table is not very enlightening because it does not give us any idea *why* the argument is valid.

In this lesson, we are going to discuss another method to establish the validity of an argument by creating what mathematicians call a *proof*. In addition, we are also going to look at how mathematicians use proof to show that a mathematical proposition is true.

(B) Lesson Proper

1. Proof and proving validity of an argument in propositional form.

For this part, we write our proofs in **two-column format**. However, later on, proofs will be written in **paragraph format**.

Basic idea of proofs:

The **goal** of the proof is to show that the conclusion logically follows from the given propositions (or premises).

As for the **content** of the proof, each proposition must be a *valid* assertion: they must be based on a given statement (i.e., a premise), or they must follow from the premise via **logical equivalences** or **rules of inferences**.

EXAMPLE 1. Prove the validity of the following argument.

$$\begin{array}{l}
 p \rightarrow (r \wedge s) \\
 \sim r \\
 \hline
 \therefore \sim p
 \end{array}$$

Solution. *Thinking process:* We assume that all propositions over the line are true. From these two propositions, the **goal** is to establish a logical sequence of propositions to arrive at the conclusion $\sim p$.

A common strategy is to start with the statement *not* involving a conditional (i.e., start with $\sim r$). Now think, if $\sim r$ is true, **how can we reach $\sim p$** ?

To do that we can use Modus Tollens on $p \rightarrow (r \wedge s)$. But first, we need to establish that $\sim (r \wedge s)$ is true. Since $\sim r$ is true, by Rule of Addition $(\sim r) \vee (\sim s)$. Then we have $[(\sim r) \vee (\sim s)] \Leftrightarrow \sim (r \wedge s)$ by De Morgan's Laws.

Actual proof:

	Proposition	Reason
1	$\sim r$	Premise
2	$(\sim r) \vee (\sim s)$	1, Rule of Addition
3	$\sim (r \wedge s)$	2, De Morgan's Laws
4	$p \rightarrow (r \wedge s)$	Premise
5	$\sim p$	3,4, Modus Tollens

□

Writing proofs

From the previous example, we can see that to arrive at the final neat and concise two-column proof, we had to do a lot of thinking! **Writing the final proof is not always easy—many times, you need to gather your thoughts on a separate sheet of paper.**

EXAMPLE 2. Prove the validity of the following argument.

$$\begin{array}{l}
 (p \wedge r) \rightarrow (\sim q) \\
 (\sim q) \rightarrow r \\
 \sim r \\
 \hline
 \therefore \sim (p \wedge r)
 \end{array}$$

Solution. *Thinking process:* Observe that the Law of Syllogism can be applied to the first two premises: $(p \wedge r) \rightarrow (\sim q)$ and $(\sim q) \rightarrow r$ imply $(p \wedge r) \rightarrow r$ (**this is a new proposition we can assume to be true**) since it follows from the premises.

Also, $\sim r$ is true. Then we can establish that $\sim (p \wedge r)$ is true by Modus Tollens on $(p \wedge r) \rightarrow r$ and $\sim r$.

Actual proof:

	Proposition	Reason
1	$(p \wedge r) \rightarrow (\sim q)$	Premise
2	$(\sim q) \rightarrow r$	Premise
3	$(p \wedge r) \rightarrow r$	1, 2, Law of the Syllogism
4	$\sim r$	Premise
5	$\sim (p \wedge r)$	3,4, Modus Tollens

Alternate solution: *Thinking process:* We can start with the statement not involving a conditional (i.e., $\sim r$). From this, think: what can we conclude? If $\sim r$ is true, then by Modus Tollens on $(\sim q) \rightarrow r$, then $\sim(\sim q)$ must be true. Finally, we apply Modus Tollens on $(p \wedge r) \rightarrow (\sim q)$ to conclude that $\sim(p \wedge r)$.

Actual proof:

	Proposition	Reason
1	$\sim r$	Premise
2	$(\sim q) \rightarrow r$	Premise
3	$\sim(\sim q)$	1,2, Modus Tollens
4	$(p \wedge r) \rightarrow (\sim q)$	Premise
5	$\sim(p \wedge r)$	3,4, Modus Tollens

□

Teaching tip

As the previous example shows, there may be more than one correct way to prove that an argument is valid.

EXAMPLE 3. Prove the validity of the following argument.

$$\begin{array}{l}
 p \vee r \\
 (\sim r) \vee (\sim s) \\
 \underline{s} \\
 \therefore p
 \end{array}$$

Solution. *Thinking process:* We can start with the simple proposition s . What can we conclude? From $\sim r \vee \sim s$, then $\sim s$ must be false since s is true. By Disjunctive Syllogism then $\sim r$ is true. Again apply Disjunctive Syllogism on $p \vee r$ to conclude p .

Actual proof:

	Proposition	Reason
1	s	Premise
2	$\sim(\sim s)$	Double Negation
3	$(\sim r) \vee (\sim s)$	Premise
4	$\sim r$	2,3, Disjunctive Syllogism
5	$p \vee r$	Premise
6	p	4,5 Disjunctive Syllogism

Alternate solution: *Thinking process:* It is also valid to transform the premises $p \vee r$ and $(\sim r) \vee (\sim s)$ to $\sim p \rightarrow r$ and $r \rightarrow \sim s$, respectively, via the Switcheroo law. Then we can use Modus Tollens and Syllogism, as shown below.

Actual proof:

	Proposition	Reason
1	$p \vee r$	Premise
2	$\sim p \rightarrow r$	Switcheroo
3	$(\sim r) \vee (\sim s)$	Premise
4	$r \rightarrow \sim s$	Switcheroo
5	$\sim p \rightarrow \sim s$	Law of Syllogism
6	s	Premise
7	$\sim(\sim s)$	6, Double Negation
8	$\sim(\sim p)$	5,7, Modus Tollens
9	p	8, Double Negation

□

2. Proof and proving validity of an argument in real-life situations.

EXAMPLE 4. Analyze the validity of the following argument.

If you start your own business, then you will earn right away. If you go to college, then you will get a college degree after a few years. However, you either start your own business, or you go to college. Therefore, you either earn right away, or you get a college degree after a few years.

Solution. To analyze this argument, we convert each proposition in symbolic form. Let b, e, c, d be defined as follows:

- b : You start your own business.
- e : You earn right away.
- c : You go to college.
- d : You get a college degree after a few years.

Then, the given argument in standard form is given by

$$\begin{array}{l} b \rightarrow e \\ c \rightarrow d \\ b \vee c \\ \hline \therefore e \vee d \end{array}$$

This is a valid argument as shown in the following proof.

	Proposition	Reason
1	$b \vee c$	Premise
2	$\sim(\sim b) \vee c$	1, Double Negation
3	$(\sim b) \rightarrow c$	2, Switcheroo
4	$c \rightarrow d$	Premise
5	$(\sim b) \rightarrow d$	3,4 Law of the Syllogism
6	$b \rightarrow e$	Premise
7	$(\sim e) \rightarrow (\sim b)$	$(b \rightarrow e) \Leftrightarrow (\sim e) \rightarrow (\sim b)$ (Contrapositive)
8	$(\sim e) \rightarrow d$	7,5 Law of the Syllogism
9	$\sim(\sim e) \vee d$	8, Switcheroo
10	$e \vee d$	9, Double Negation

□

Another rule of inference:

We have shown in the previous example that an argument of the form

$$\begin{array}{c}
 p \rightarrow q \\
 r \rightarrow s \\
 \hline
 p \vee r \\
 \hline
 \therefore q \vee s
 \end{array}$$

is valid. This form is called the **constructed dilemma**. Look at the structure of the argument and try to see why it ‘makes sense.’

In the next example, we illustrate how to prove that an argument is *not* valid.

3. Disproof

EXAMPLE 5. Show that the following argument is invalid.

I would like a career on either teaching or diplomacy. If I teach, then I would want to study abroad. Therefore, if I would like a career on diplomacy then I will study abroad.

Solution. We first write the argument in symbolic form using the following propositions:

- t : I would like a career in teaching.
- d : I would like a career in diplomacy.
- s : I would want to study abroad.

The given argument can be written in the following propositional form.

$$\frac{t \vee d \quad t \rightarrow s}{\therefore d \rightarrow s}$$

To show that an argument is not valid, find truth values for each proposition such that the premises are true but the conclusion is false.

For $d \rightarrow s$ to be false, then d must be true and s must be false. If t is false as well, then the premises $t \vee d$ and $t \rightarrow s$ are both true. Thus, we have found truth values for the propositions (t is false; d is true; s is false) for which the premises are true but the conclusion is not. The associated row in the truth table is:

t	d	s	$t \vee d$	$t \rightarrow s$	$d \rightarrow s$
F	T	F	T	T	F

□

Proving that an argument is invalid

The previous example illustrates that **producing a counterexample** is one way to prove that an argument is invalid.

4. Indirect proofs

So far, in proofs of valid statements, we have only used **direct proofs**. These are proofs where we begin from the premises and use rules of logic to reach the conclusion.

Another method is through the use of an **indirect proofs** or a **proof by contradiction**. In these proofs, we show that the assumption that the premises are true but the conclusion is false leads to a contradiction.

EXAMPLE 6. Prove the following argument using three methods: (a) via rules of inference, (b) via truth tables, and (c) via an indirect proof.

$$\frac{p \vee q \quad \sim q}{\therefore p}$$

Solution. (a) via rules of inference

	Proposition	Reason
1	$p \vee q$	Premise
2	$\sim q$	Premise
3	p	1,2, Disjunctive Syllogism

(b) via truth tables

p	q	$p \vee q$	$\sim q$	$((p \vee q) \wedge \sim q)$	$((p \vee q) \wedge \sim q) \rightarrow p$
T	T	T	F	F	T
T	F	T	T	T	T
F	T	T	F	F	T
F	F	F	T	F	T

Since $((p \vee q) \wedge \sim q) \rightarrow p$ is a tautology, then the argument is valid.

(c) via an indirect proof

We assume the conclusion is false while the premises are true, and show that these lead to a contradiction.

Suppose the conclusion p is false. Based on the premise, $\sim q$ is true (i.e., q is false). Thus, we have shown that both p and q are false, so $p \vee q$ is also false. However, this leads to a contradiction because $p \vee q$ is a premise—it has to be true.

In symbols, $\sim (p \vee q) \wedge (p \vee q) \Leftrightarrow \phi$. □

5. Proof and proving validity of arguments in mathematics

We now apply the rules of logic to prove basic results in mathematics. Before we proceed, we first state the following important idea.

Definitions in mathematics are *always* ‘if and only if’ statements.

Look at the following examples.

Definition	‘If and only if’ form
An even number m is a number that can be written as $m = 2k$ where k is an integer.	A number m is even if and only if it can be written as $m = 2k$, where k is an integer.
A binomial is a polynomial with exactly 2 terms.	A polynomial is a binomial if and only if it has exactly 2 terms.
A parallelogram is a quadrilateral with two pairs of parallel sides.	A quadrilateral is a parallelogram if and only if it has two pairs of parallel sides.

Now let us try to justify mathematical arguments.

EXAMPLE 7. Prove the validity of the following argument.

An even number m is a number can be written as $m = 2k$ where k is an integer. The numbers x and y are even. Therefore $x + y$ is even.

Solution. Remember that the first premise can be written as an ‘if and only if’ statement, since it is a definition.

Since x is even, then the first premise implies that x can be written as $x = 2k_1$ where k_1 is an integer. Similarly, since y is even, it can be written as $y = 2k_2$, where k_2 is an integer.

Now consider the sum $x + y = 2k_1 + 2k_2 = 2(k_1 + k_2)$. Since $k_1 + k_2$ is an integer, then $x + y$ is even. (This is just **Modus Ponens** applied to the first premise.) \square

EXAMPLE 8. Prove that the following argument is valid.

If a quadrilateral has three right angles, then it is a rectangle. In quadrilateral $ABCD$, $m_{\overline{A}} = 90^\circ$, $m_{\overline{B}} = 90^\circ$ and $m_{\overline{C}} = 85^\circ$. Then $ABCD$ is not a rectangle.

Solution. *Thinking process:* The only way for $ABCD$ to be a rectangle is if $\overline{D} = 90^\circ$, so that there would be three right angles. We will prove this is *not* the case.

Proof: The sum of the interior angles in a rectangle is 360° . Therefore,

$$m_{\overline{A}} + m_{\overline{B}} + m_{\overline{C}} + m_{\overline{D}} = 360^\circ.$$

Substituting the given values, we obtain $90^\circ + 90^\circ + 85^\circ + m_{\overline{D}} = 360^\circ$. When simplified, this becomes $m_{\overline{D}} = 95^\circ$. Since $ABCD$ only has two right angles (namely \overline{A} and \overline{B}), then it is not a rectangle (note that the reasoning here is due to **Modus Tollens**).

Alternate solution: Indirect proof

Proof: Assume that the conclusion is true, that $ABCD$ is a rectangle. Then it has three right angles. Since \overline{C} is not a right angle, then the three angles must be \overline{A} , \overline{B} , and \overline{D} .

Since

$$m_{\overline{A}} + m_{\overline{B}} + m_{\overline{C}} + m_{\overline{D}} = 360^\circ,$$

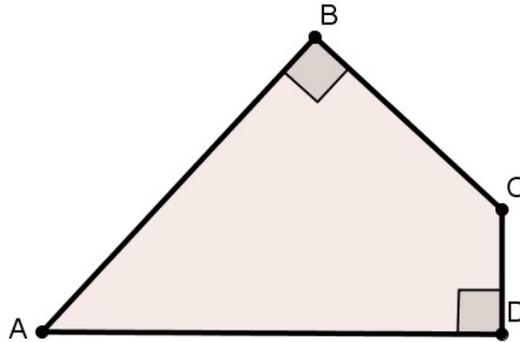
then $90^\circ + 90^\circ + m_{\overline{C}} + 90^\circ = 360^\circ$. When simplified, this becomes $m_{\overline{C}} = 90^\circ$. This is a **contradiction** because the measure of \overline{C} was given to be 85° . \square

Classroom Experiment: In the next example, allow your class to experiment. Let them draw several quadrilaterals and think about the validity of the argument.

EXAMPLE 9. Determine whether the following argument is valid or not.

If a quadrilateral has three right angles, then it is a rectangle. Quadrilateral $ABCD$ has two non-consecutive right angles. Then $ABCD$ is a rectangle.

Solution. The argument is actually *not* valid. One counterexample is shown below.



□

6. Seatwork

Seatwork 1. Establish validity of the following arguments. If the argument is not valid, give a counterexample.

$$(a.) \frac{(p \vee q) \rightarrow r}{q} \quad \text{Answer: Valid (} q \text{ (Premise), then } p \vee q \text{ (Addition), then } r \text{ (Modus Ponens))}$$

$$\therefore r$$

$$(b.) \frac{(p \wedge q) \rightarrow r}{q} \quad \text{Answer: Invalid (Counterexample: } p, r \text{ false; } q \text{ true)}$$

$$\therefore r$$

$$(c.) \frac{p \rightarrow (q \wedge r)}{\therefore (p \rightarrow q) \wedge (p \rightarrow r)}$$

Answer: Valid ($p \rightarrow (q \wedge r)$) (Premise), then $\sim p \vee (q \wedge r)$ (Switcheroo), then $(\sim p \vee q) \wedge (\sim p \vee r)$ (Distributive), then $(p \rightarrow q) \wedge (p \rightarrow r)$ (Switcheroo))

$$(d.) \frac{p \rightarrow (q \vee r)}{r \rightarrow s} \quad \text{Answer: Invalid (Counterexample: } p, q \text{ true; } r, s \text{ false)}$$

$$\therefore p \rightarrow s$$

$$(e.) \frac{p \rightarrow (r \vee t)}{\sim r} \quad \text{Answer: Valid (} \sim r \text{ (Premise), then } \sim r \vee \sim t \text{ (Addition), then } \sim (r \wedge t) \text{ (De Morgan's), then } \sim p \text{ (Modus Tollens))}$$

$$\therefore \sim p$$

$$(f.) \frac{(p \vee q) \rightarrow r}{\sim r} \quad \therefore \sim p$$

Answer: Valid ($\sim r$ (Premise), then $\sim (p \vee q)$ (Modus Tollens), then $\sim p \wedge \sim q$ (De Morgan's), then $\sim p$ (Simplification))

$$(g.) \frac{(p \wedge q) \rightarrow r}{\sim r} \\ \therefore \sim p$$

Answer: Invalid (Counterexample: p true; q, r false)

- (h.) If you love, then you live a full life. You chose to love and not suffer from loneliness. So, you live a full life and you do not have hatred in your heart.

Answer: Valid (Hint: First prove that you live a full life (Modus Ponens), then prove that you do not have hatred in your heart (Modus Tollens))

- (i.) It is not sunny this afternoon and it is colder than yesterday. If we will go swimming, then it is sunny. If we do not go swimming, then we will take a boat trip. If we take a boat trip, then we will be home by sunset. Thus, we will be home by sunset. *Answer: Valid*

- (j.) A number m divisible by 3 can be written as $m = 3k$ where k is an integer. A number n divisible by 6 can be written as $n = 6k$ where k is an integer. Therefore a number divisible by 3 is also divisible by 6. *Answer: Invalid; Counterexample: 9*

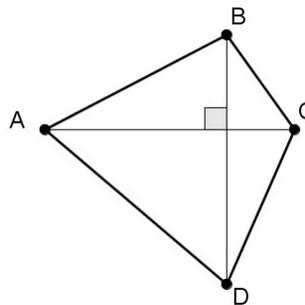
- (k.) A number m divisible by 3 can be written as $m = 3k$ where k is an integer. A number n divisible by 6 can be written as $n = 6k$ where k is an integer. Therefore a number divisible by 6 is also divisible by 3. *Answer: Valid. If $m = 6k$ for an integer k , then $m = 3 \cdot (2k)$. Since $2k$ is an integer, then m is divisible by 3.*

- (l.) In a triangle, two sides are equal if and only if the opposite angles are congruent. In a right triangle with legs a, b and hypotenuse c , then $a^2 + b^2 = c^2$. ABC is a right triangle where one leg measures 6 and the hypotenuse measures $6\sqrt{2}$. Therefore ABC has two equal angles.

Answer: Valid; From $a^2 + 6^2 = (6\sqrt{2})^2$, then $a = 6$; thus, the triangle has two equal sides. By the first premise, then it should also have two equal angles.

- (m.) If a quadrilateral is a rhombus, then its two diagonals are perpendicular and congruent. $ABCD$ is a quadrilateral whose diagonals are perpendicular and congruent. Therefore it is a rhombus.

Answer: Invalid; counterexample below



(C) Performance task

Performance Task 1. In this task, students will use everything they learned about logic to solve a logic puzzle.

Suppose **Lohika**, the president of the Logic Club, has the following information about the attendance of **Goyong**, **Juan**, **Inday**, **Pipay**, and **Kulas** to the annual general assembly of school clubs.

Exactly two of **Goyong**, **Juan**, and **Pipay** attended. If **Kulas** attended, then **Juan** attended. Either both **Juan** and **Kulas** attended, or **Juan** did not attend. If **Juan** attended, then neither **Goyong** nor **Inday** attended. At least one of **Pipay** and **Goyong** attended. **Pipay** attended if and only if **Kulas** was absent. If **Goyong** attended then so did **Inday**.

Help Logika analyze the given information by doing the following.

- (a) Symbolize the given information.
- (b) If exactly two of **Goyong**, **Juan**, and **Pipay** attended, what possibilities are there for them?
Answer: There are 3 possibilities: only Goyong and Juan attended; only Goyong and Pipay attended; only Juan and Pipay attended
- (c) For each case above, determine what possible attendance scenarios will can satisfy *all* the premises. *Answer: To satisfy all the premises, then Goyong, Pipay, and Inday are present, while Juan and Kulas are absent.*

Performance Task 2. Let each group revisit the arguments that they constructed in Lesson 1. Ask them to do the following.

- (a) Identify the premises and conclusion of the argument.
- (b) Assign symbols to the premises and conclusion.
- (c) Determine whether each premise is generally true, or not. If one of the premise is false, then the whole argument cannot be sound.
- (d) Establish the validity of the argument.

In establishing the validity of your argument, you might have to identify some implicit assumptions. For example, in the following argument:

Logic is used in engineering, law, philosophy, computer science, mathematics, and many other disciplines. Logic makes our electronic gadgets, such as cellphones, work. Logic is fun and interesting. If logic is useful, fun and interesting, then everyone should study it. Therefore, everyone should study logic.

An implicit assumption to this argument is *if logic influences other disciplines and technological development, then it is useful.*

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