FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$
$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

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Section A (50 marks)
Answer **ALL** questions in this section and write your answers in the spaces provided in this Question-Answer Book.

1. Find the coefficient of x^5 in the expansion of $(2-x)^9$.

(4 marks)

2. Consider the following system of linear equations in x, y, z

$$\begin{cases} x - 7y + 7z = 0 \\ x - ky + 3z = 0 \\ 2x + y + kz = 0 \end{cases}$$, where k is a real number.

If the system has non-trivial solutions, find the two possible values of k.

(4 marks)

Answers written in the margins will not be marked.

Page Total

3. Prove by mathematical induction that $4^n + 15n - 1$ is divisible by 9 for all positive integers n .

(5 marks)

4. (a) Let $x = \tan\theta$, show that $\frac{2x}{1+x^2} = \sin 2\theta$.

(b) Using (a), find the greatest value of $\frac{(1+x)^2}{1+x^2}$, where x is real.

(5 marks)

Page Total

5. (a) It is given that $\cos(x+1)+\cos(x-1)=k\cos x$ for any real x. Find the value of k.

(b) Without using a calculator, find the value of $\begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}$.

(6 marks)

6. Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\right)$ from first principles.

(4 marks)

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Page Total

7. Let $f(x) = e^x(\sin x + \cos x)$.

(a) Find $f'(x)$ and $f''(x)$.

(b) Find the value of x such that $f''(x) - f'(x) + f(x) = 0$ for $0 \le x \le \pi$.

(5 marks)

8. (a) Using integration by substitution, find $\int \frac{dx}{\sqrt{4-x^2}}$.

(b) Using integration by parts, find $\int \ln x \, dx$.

(5 marks)

Page Total

9. Find the equations of the two tangents to the curve $x^2 - xy - 2y^2 - 1 = 0$ which are parallel to the straight line $y = 2x + 1$.

(6 marks)

10. (a) Find $\int xe^{-x^2}\,dx$.

(b)

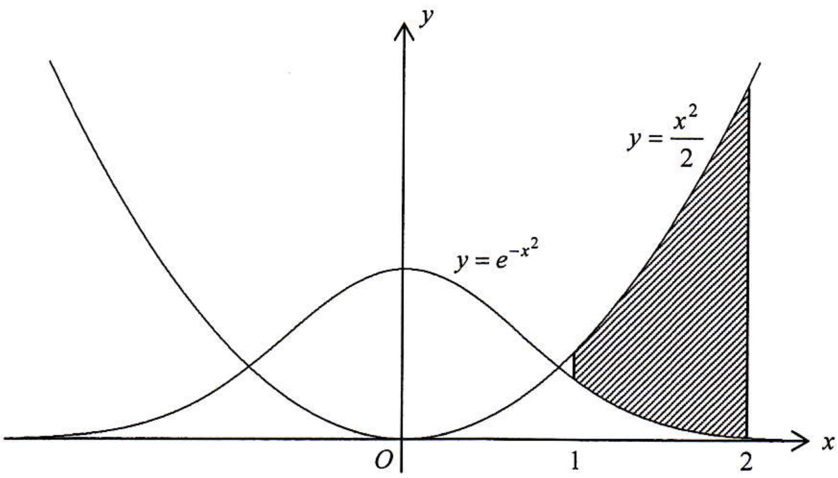

Figure 1

In Figure 1, the shaded region is bounded by the curves $y = \frac{x^2}{2}$ and $y = e^{-x^2}$, where $1 \le x \le 2$. Find the volume of the solid generated by revolving the shaded region about the y-axis.

(6 marks)

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Section B (50 marks)
Answer **ALL** questions in this section and write your answers in the other answer book.

11. Let $A=\begin{pmatrix}\alpha+\beta & -\alpha\beta \\ 1 & 0\end{pmatrix}$ where α and β are distinct real numbers. Let I be the 2×2 identity matrix.

(a) Show that $A^2=(\alpha+\beta)A-\alpha\beta I$.

(2 marks)

(b) Using (a), or otherwise, show that $(A-\alpha I)^2=(\beta-\alpha)(A-\alpha I)$ and $(A-\beta I)^2=(\alpha-\beta)(A-\beta I)$.

(3 marks)

(c) Let $X=s(A-\alpha I)$ and $Y=t(A-\beta I)$ where s and t are real numbers.
Suppose $A=X+Y$.

(i) Find s and t in terms of α and β .

(ii) For any positive integer n , prove that

$$X^n=\frac{\beta^n}{\beta-\alpha}(A-\alpha I) \quad \text{and} \quad Y^n=\frac{\alpha^n}{\alpha-\beta}(A-\beta I) .$$

(iii) For any positive integer n , express A^n in the form of $pA+qI$, where p and q are real numbers.
[Note: It is known that for any 2×2 matrices H and K ,
if $HK=KH=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$, then $(H+K)^n=H^n+K^n$ for any positive integer n .]

(9 marks)

12.

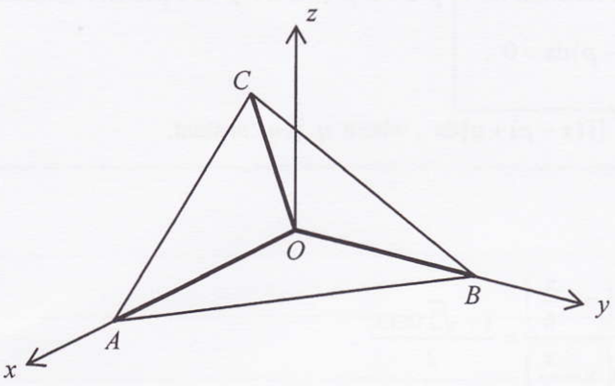

Figure 2

Let $\overrightarrow{OA}=\mathbf{i}$, $\overrightarrow{OB}=\mathbf{j}$ and $\overrightarrow{OC}=\mathbf{i}+\mathbf{j}+\mathbf{k}$ (see Figure 2). Let M and N be points on the straight lines AB and OC respectively such that $AM:MB=a:(1-a)$ and $ON:NC=b:(1-b)$, where $0<a<1$ and $0<b<1$. Suppose that MN is perpendicular to both AB and OC .

(a) (i) Show that $\overrightarrow{MN}=(a+b-1)\mathbf{i}+(b-a)\mathbf{j}+b\mathbf{k}$.

(ii) Find the values of a and b .

(iii) Find the shortest distance between the straight lines AB and OC .

(8 marks)

(b) (i) Find $\overrightarrow{AB}\times\overrightarrow{AC}$.

(ii) Let G be the projection of O on the plane ABC , find the coordinates of the intersecting point of the two straight lines OG and MN .

(5 marks)

13. (a) Let $f(x)$ be an odd function for $-p \le x \le p$, where p is a positive constant.

Prove that $\int_0^{2p} f(x-p)\,dx = 0$.

Hence evaluate $\int_0^{2p} [f(x-p)+q]\,dx$, where q is a constant.

(4 marks)

(b) Prove that $\dfrac{\sqrt{3}+\tan\left(x-\dfrac{\pi}{6}\right)}{\sqrt{3}-\tan\left(x-\dfrac{\pi}{6}\right)} = \dfrac{1+\sqrt{3}\tan x}{2}$.

(2 marks)

(c) Using (a) and (b), or otherwise, evaluate $\int_0^{\frac{\pi}{3}} \ln(1+\sqrt{3}\tan x)\,dx$.

(4 marks)

14. (a)

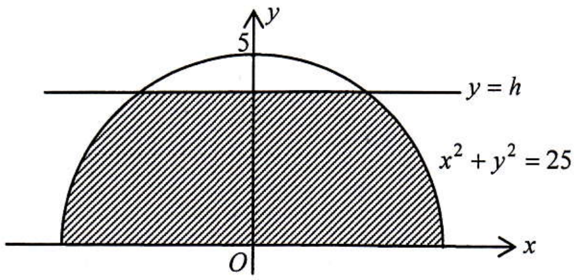

Figure 3

In Figure 3, the shaded region enclosed by the circle $x^2 + y^2 = 25$, the x-axis and the straight line $y = h$ (where $0 \le h \le 5$) is revolved about the y-axis. Show that the volume of the solid of revolution is $\left(25h - \frac{h^3}{3}\right)\pi$.

(2 marks)

(b)

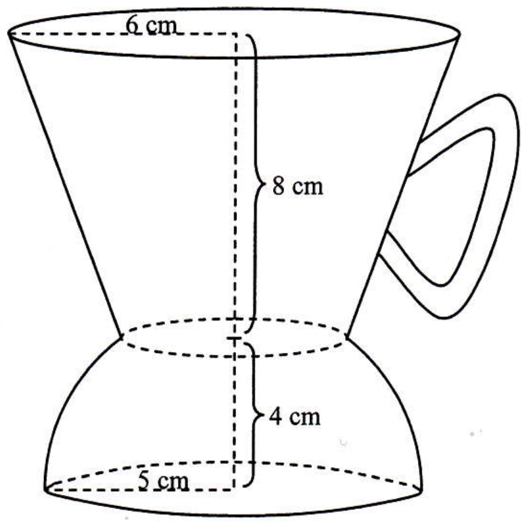

Figure 4

In Figure 4, an empty coffee cup consists of two portions. The lower portion is in the shape of the solid described in (a) with height 4 cm . The upper portion is a frustum of a circular cone. The height of the frustum is 8 cm . The radius of the top of the cup is 6 cm . Hot coffee is poured into the cup to a depth h cm at a rate of 8 cm^3s^{-1} , where $0 \le h \le 12$. Let V cm^3 be the volume of coffee in the cup.

(i) Find the rate of increase of the depth of coffee when the depth is 3 cm .

(ii) Show that $V = \frac{164\pi}{3} + \frac{3\pi}{64}(h+4)^3$ for $4 \le h \le 12$.

(iii) After the cup is fully filled, suddenly it cracks at the bottom. The coffee leaks at a rate of 2 cm^3s^{-1} . Find the rate of decrease of the depth of coffee after 15 seconds of leaking, giving your answer correct to 3 significant figures.

(11 marks)

END OF PAPER