The Mathematics of Socionics

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In this paper we develop the mathematics underlying socionics and Model A. My research has uncovered some surprising facts, like the existence of an alternative dichotomy system (2011). [1]

If I am type t and F is a relationship, there is a type $F(t)$ that has that relationship with me. E.g. if t is SLI and F is beneficiary, then $F(t)$ is ESI, SLI's beneficiary. (We distinguish between the beneficiary and benefactor; a type's beneficiary and benefactor are not the same.) Therefore, a relationship can be seen as a specific way of producing one type from another type, in other words a function. In this sense, there are 16 different relationships.

We can also compose these functions: for example, the dual of the activator of a type is always its mirror. We can summarize this by saying dual $\cdot$ activator = mirror. So, the set of relationships is a closed system where the composition of two relationships produces another one in the system. There is also an identity relationship, which associates a type with itself.[2]

From now on I will use the following abbreviations:

e	identity, or identical
d	dual
a	activator
m	mirror
g	superego
c	conflictor
q	quasi-identical
x	extinguishment, contrary, or contrast
S	supervisor
B	benefactor or request transmitter
k	kindred or comparative
h	semidual (h for 'half') or partial dual
s	supervisee or revisee
b	beneficiary or request receiver
ℓ	lookalike or business
i	illusionary or mirage

1 Structure of the relationships

For some relationships, like mirror, if you apply them twice you get the type you started with: the mirror of my mirror is my own type. But for others, like supervisor, you have to apply them four times to get back to the original type: I am the supervisor of the supervisor of the supervisor of my

[1] Some of this content has been researched independently before by Russian researchers like Grigori Reinin, Alexander Bukalov, and Mikhail Gut. I was not aware of any Russian research into the subject when I first began investigating it, and to my knowledge all other research has been fairly rudimentary, limited primarily to the Reinin dichotomies and ignoring the non-abelian group structure. Gut notably constructed a geometric model, as did Bukalov for his Model B. Gut also noted the presence of certain non-Reinin dichotomies in his model but apparently did not construct the full system.

[2] It's a convention in mathematics to call the identity e for eigen, which means "self" in German.

supervisor. So we say that *s has order 4*. Every relationship in socionics has order 1, order 2, or order 4. A relationship is symmetric exactly when its order is 2 or 1. Let's call the entire set of relationships $\mathbb{S}$. This forms a mathematical structure called a *group*, meaning essentially that it is a set of functions satisfying the following requirements:

1) $\mathbb{S}$ contains the identity function (written as e), which gives you the same type back again.

2) If you take two functions f and g in $\mathbb{S}$, then their composition $f \cdot g$ is also in $\mathbb{S}$.

3) Every f in $\mathbb{S}$ has an *inverse* f^{-1} that does the reverse assignment of types, meaning that applying f then f^{-1} gives you the original type, and applying f^{-1} then f also gives you the original type, meaning, $f \cdot f^{-1} = f^{-1} \cdot f = e$.

In $\mathbb{S}$ most elements are their own inverse, e.g. mirror $\cdot$ mirror = identity. This just says that mirror is a symmetric relationship, i.e. it has order 1 or 2.

Abstractly, we define a group as a set X with an operation $\cdot$ and an element e such that $(f \cdot g) \cdot h = f \cdot (g \cdot h)$, $f \cdot e = f = e \cdot f$, and $f \cdot f^{-1} = f^{-1} \cdot f = e$.

A group is a structure that describes symmetries. For example, there is the group of symmetries of a square, which includes rotations and reflections (8 operations in all). This group is called D_4. Integers also form a group under addition and with 0 as the identity element. Another group is *the integers mod n*, denoted $\mathbb{Z}_n$, where we add two integers and then take the remainder mod n. It can alternatively be seen as rotational symmetries of a regular polygon with n sides (excluding reflections).

$\mathbb{S}$ can be described in terms of the above groups: if we take pairs (x, y) where x is in D_4 and y is in the integers mod 2 ($\mathbb{Z}_2$), and multiply them by multiplying the first components and adding the second component, then we get a group that has the same structure as the group of relationships in socionics.[3] This is the first "axiom" of socionics:

Axiom. The group of relationships is isomorphic to $D_4 \times \mathbb{Z}_2$.

No theory that dispenses with this can claim to call itself socionics. Myers-Briggs theory has the same number of types and calls them by similar names, but it has no concept of this group or its importance.[4]

Notice that rotating the square 90° and then reflecting it vertically is not the same as reflecting it vertically and then rotating it. This means that we don't have $fg = gf$ for all operations f and g. This is different from group operations like addition and multiplication of numbers, which are *commutative*. This is in fact a very important property of the relationship group. If we were only working with dichotomies, the group would be commutative.

So essentially we can visualize a type as two squares, which are simultaneously rotated or reflected, and can be switched with each other. Each vertex can be labeled with N, E, S, W depending on its orientation and with 0 or 1 depending on which square it's in, giving exactly the pairs above. This is just a way of describing Model A, with one square being the mental loop and the other the vital loop.

In fact, if we put these squares on top of one another we obtain a cube. And $\mathbb{S}$ can indeed be described as certain symmetries of a cube, namely the ones that take the top and bottom faces to themselves or each other. This model will be described in Section 7.

2 The action

The other axioms have to do with the types. We say that every person has a *type*, and the types have certain relationships with one another. There are also various models, like Model A, which invoke

[3] This pairing is called a *group product* - not the same as the operation in the group. We says $\mathbb{S} \cong D_4 \times \mathbb{Z}_2$.

[4] In dealing with theories describing real content, the idea of an axiom doesn't always make sense: you could say that the Myers-Briggs types have the same relationships as the socionics ones just by the fact that there are 16 of them too.

further theoretical constructs like functions and IM elements. Let's describe the set of types first and how it relates to the group of relationships.[5]

We mentioned above that elements of $\mathbb{S}$ (the relationships) can be seen as functions that take a type t and produce another one $f(t)$ in a specific way. This is what is called a group action of $\mathbb{S}$ on the set of types T. We say then that the ordered pair of types $(t, f(t))$ *have the relation f.*

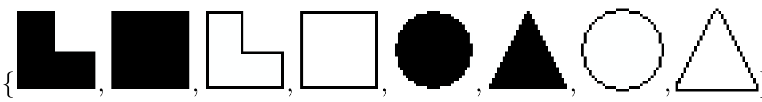
For the theory to be a complete theory of intertype relations, there should be a relationship linking any two types. And moreover, there should not be more than one relationship between two given types. In terms of group actions the first condition says that the action is *transitive* and the second says that it's *free.* The two together mean that there is a **unique** relationship between any two types. Namely, this means that for types s and t, there is exactly one relationship r such that $r(s) = t$. This kind of group action is called regular.[6] So we have our second axiom:

Axiom. The group of relationships $\mathbb{S}$ has a regular action on the set of types T.

So, if you take my type for example, it has one distinct relationship with each of the other types. So taken together, these two requirements guarantee that there are the same number of types as relationships: 16.

3 Model A

In Model A we also have a set F of functions and a set

$I = \{\text{Fe, Te, Fi, Ti, Se, Ne, Si, Ni}\} = \{$  $\}$

of information elements. Traditionally the functions have an order and are identified with the numbers 1 through 8. The order is somewhat arbitrary, but I will use the numbers as a convention. Every relationship rearranges the functions in a specific way, for example the comparative relationship exchanges functions 2 and 4, and also functions 6 and 8. Therefore we write it as (24)(68). The supervisee relationship shifts functions 1-4 and also functions 5-8, so we write it as (1234)(5678).

$\mathbb{S}$ can be defined as the group generated by certain permutations such as $s = (1234)(5678)$, $d = (15)(26)(37)(48)$, and $a = (16)(25)(38)(47)$. That is, $s(1) = 2$, $s(2) = 3$, etc. It can be easily verified that this group is the same as $D_4 \times \mathbb{Z}_2$. Now, Model A also defines the types in terms of the IM elements and functions: a type is a particular one-to-one way of associating the eight IM elements with the eight functions. Not every ordering of the IM elements forms a legal type. That is, the action of $\mathbb{S}$ on arrangements of IM elements is not transitive.[7]

In addition to Model A, there are many other possible models for socionics, i.e. representations of relationships as ways of permuting n functions (making $\mathbb{S}$ a subgroup of the group of permutations, S_n), and types as sequences of those n functions. Notice, when we are looking at models, $\mathbb{S}$ acts on both types *and* functions. It's important not to confuse these two actions.

[5] If you want to get really technical, there is also the set of *people* P, and a function assigning a person their type too.

[6] It is also sometimes called a freely transitive action, or an $\mathbb{S}$-torsor

[7] Formally, $\mathbb{S}$ acts via pre-composition on the set of bijections from F to I and we define the set of types T to be all arrangements of, e.g., ILE = [Ne Ti Se Fi Si Fe Ni Te], which result from applying every permutation in $\mathbb{S}$ (also known as the *orbit* containing ILE). Bijections always act freely by composition (cancellation law), and because of the way we defined T, $\mathbb{S}$ will act transitively, so the action is indeed regular.

4 Subgroups of $\mathbb{S}$

It's helpful to know all the subgroups of $\mathbb{S}$. A subgroup is a subset of the group that is also a group - i.e. contains the identity, inverses for all elements in it, and contains the product of any two elements. A special subgroup of any group G is the set of elements commuting with all other elements, also known as the *center* $Z(G)$. In this case $Z(\mathbb{S}) \cong Z(D_4) \times Z(\mathbb{Z}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, so $Z = \{e, g\}\{e, x\} = egxd$.[8] Cosets of a subgroup are the sets you get when you multiply everything in the subgroup by some group element, and they partition the whole group. Cosets of the center are

$egxd$
$bBsS$
$aqcm$
$ihkl$

Now, $\mathbb{S}$ can be completely described by taking the elements a, b, x (a, b for D_4 and x for $\mathbb{Z}_2$) and requiring the relations

1. $a^2 = b^4 = x^2 = e$
2. $ax = xa, bx = xb$ (x commutes with both a and b), and
3. $(ab)^2 = e$, or $ab^{-1} = ba$.

It turns out a can be replaced with any relation from the last two cosets of Z, b can be replaced with any in the second coset, and x can be replaced only with d. The resulting set will still satisfy the same relations, and thus gives an automorphism of the group. These automorphisms will help us find the subgroups more easily.

So from now on I will call the relations in the second coset **quadralities** and the ones in the last two cosets **odd relations**.[9]

4.1 List of subgroups

Lagrange's theorem says that a subgroup's size must divide the size of the whole group. So in our case they must have $1, 2, 4, 8$, or 16 elements.

First, there are the subgroups $\{e\}$ and the whole group, as in any group. The subgroups of order 2 are those generated by one of the 11 relationships of order 2:

Elements	Normal?	Quotient
eg	yes	$\mathbb{Z}_2^3$
ed	yes	D_4
ex	yes	D_4
ea	no	-
eq	no	-
ec	no	-
em	no	-
ei	no	-
eh	no	-
ek	no	-
el	no	-

[8] Normally sets are written with commas and braces, but I will remove them when no ambiguity results.

[9] To show that $(ab)^2 = e$ for any b of order 4 and a odd, it's enough to show that $|ab| \neq 4$. Well, notice that the set of odd elements is the complement of the group generated by the elements of order 4, but if $|ab| = 4$, $a = (ab)b^{-1}$, a contradiction.

[10]

The subgroups of order 4 are either generated by a relationship of order 4:

Elements	Description	Normal?	Quotient
$ebgB$	benefit ring	yes	$\mathbb{Z}_2^2$
$esgS$	supervision ring	yes	$\mathbb{Z}_2^2$

or only have elements of order 2:[11]

Elements	Description	Normal?	Quotient
$egxd$	$Z =$ Dem/Arist $\cap$ J/P	yes	$\mathbb{Z}_2^2$
$egaq$	I/E $\cap$ Dem/Arist	yes	$\mathbb{Z}_2^2$
$egcm$	Stat/Dyn $\cap$ Dem/Arist	yes	$\mathbb{Z}_2^2$
$egih$	J/P $\cap$ Neg/Pos	yes	$\mathbb{Z}_2^2$
$egk\ell$	temperament=I/E $\cap$ J/P	yes	$\mathbb{Z}_2^2$
$exac$	Dem/Arist $\cap$ Emot/Const	no	-
$exqm$	club	no	-
$exik$	Xi or Xe leading	no	-
$exh\ell$	Xi or Xe creative	no	-
$edam$	quadra	no	-
$edqc$	Dem/Arist $\cap$ Far/Care	no	-
$edi\ell$	2nd value same	no	-
$edhk$	1st value same	no	-

All these intersect the center non-trivially because if r, s have order 2 and aren't in the center, then they are odd, so $rsb = brs$, and $rss = srs$ because odd elements conjugate each other trivially. [12]

Subgroups of order 8 correspond to certain dichotomies. It turns out if you take a particular type and apply all of the 8 relations in the subgroup, you'll get 8 types that share a particular Reinin trait. If you apply the other eight, you get the types with the opposite trait.

Elements	Description	Type	Complements
$egxdbBsS$	Result/Process	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\langle r \rangle$ for r odd
$egxdaqcm$	Democratic/Aristocratic	$\mathbb{Z}_2^3$	k...
$egxdihk\ell$	Rational/Irrational	$\mathbb{Z}_2^3$	a...
$egaqbBk\ell$	Introverted/Extroverted	D_4	m...
$egaqsSih$	Negativist/Positivist	D_4	m...
$egcmbBih$	Questioner/Declarer	D_4	a...
$egcmsSk\ell$	Static/Dynamic	D_4	x...

So for example, if two types are both Process or both Result, they will necessarily have 1 of the 8 relationships $egxdbBsS$; and if one is Process and the other Result they'll have one of the other 8 relationships. [13]

[10] See `https://en.wikipedia.org/wiki/List_of_small_groups`. The quotient of eg has no element of order 4, so it must be $\mathbb{Z}_2^3$. Quotienting by d or x just removes the $\mathbb{Z}_2$ factor.

[11] Hence, isomorphic to $\mathbb{Z}_2^2$.

[12] Technical note: It turns out that a subgroup is normal iff it is contained in the center or it contains g. The $\Longrightarrow$ implication follows as $\mathbb{S}$ is a p-group. As for the rest, if x is conjugate to y and $x \neq y$, $x = gy$, so $xy^{-1} = g$.

[13] Technical note: All of these are normal because they have index 2. The last four are isomorphic to D_4 because they have an element of order 4 and aren't abelian.

Notice that the dichotomies represented here are exactly the ones that Superego partners share, because they all contain the Superego relation. We call these *orbital* dichotomies.

For a group H of order 8 it is sufficient to take the group generated by a single element r as a complement. It is only necessary to check that $\langle r \rangle$ does not intersect the group. Clearly this doesn't occur if $|r| \leq 2$, so any element with order at most 2 in $H's$ complement will do. And H must contain g so none of the elements of order 4 will work.

5 Type dichotomies

A dichotomy is a way of dividing a set into two (equal) parts. A system of dichotomies is a set of dichotomies that uniquely specify each type, when a half of each dichotomy is chosen.

In other words, a dichotomy system is a way of describing each type as a set of binary choices, e.g. a vector like (I, S, T, P). However, the set of types is not really a vector space because a vector space has a way of adding vectors and a special element 0 such that $v + 0 = v$ for all v. [14] We don't necessarily have a way to "add" types together to get other types. The way to avoid creating an addition operation on the types is to instead think about how one can "subtract" one type from another to get a vector of 0s and 1s, where 1 means a flip and 0 means no flip. This means that the set of types is actually acted on by the vector space $\mathbb{Z}_2^4$.[15] This fits in nicely with our description of intertype relationships, and actually about half of the vectors line up with relationships. E.g., the vector (1, 0, 0, 0) represents extinguishment.[16]

To put this more rigorously: a **dichotomy** is $\{T, U\}$, where T and U are complements inside a boolean (i.e., complemented, bounded, and distributive) lattice $(L, \wedge, \vee, \perp, \top, \neg)$, i.e. $T \wedge U = \perp$ and $T \vee U = \top$. This lattice represents subsets of a set.

We also write $T = D_0$ and $U = D_1$ for the two poles or halves of a dichotomy D.

If we take two dichotomies they divide the set into a 2-by-2 grid. If we group the diagonal squares together they give a new dichotomy. We can call this dichotomy addition or **interleaving**. Rigorously, for D, E dichotomies, let

$$D + E = \{(D_0 \wedge D_1) \vee (\neg D_0 \wedge \neg D_1), (D_0 \wedge \neg D_1) \vee (\neg D_0 \wedge D_1)\}$$

It can be more concisely written as $\{D_0 \equiv D_1, D_0 \text{ xor } D_1\}$. $D + E$ is a dichotomy; the proof is straightforward (it requires distributivity). It is obvious that $D + E = E + D$. Moreover, where 0 is the dichotomy $\{\perp, \top\}$, we have $D + 0 = D$ and $D + D = 0$. Therefore, the set of dichotomies in L forms a vector space over $\mathbb{F}_2$, the field with two elements.

The set of **traits** $\mathfrak{T}(\mathfrak{D})$ generated by a set of dichotomies $\mathfrak{D}$ is defined as the set of all intersections of traits of dichotomies in $\mathfrak{D}$.

A set of dichotomies DS is said to be **independent** iff $\forall D \in DS, \forall T \in \mathfrak{T}(DS \setminus \{D\}), T$ intersects nontrivially with both D_0 and D_1. That is, every dichotomy contributes information everywhere (as opposed to just in certain cases).

Say that a dichotomy D is **relevant** to $X \in L$ iff $X \leq D_0$ or $X \leq D_1$. Also, when a set of dichotomies DS has been fixed, let $R(X)$ be the set of dichotomies in DS that are relevant to X.

For $D \in R(X)$, let the **projection** of $X \neq \perp$ onto D be

[14] In 'introverted socionics' the set of types has a 0, because each type is identified with a relationship.

[15] This action is manifestly free and transitive, and it should be if it is meant to represent a dichotomy system.

[16] This approach may seem to depend on the Jungian basis, because of the way we interpret the vectors, but is actually independent of it. More precisely, the Reinin dichotomies left the same by extinguishment are those generated by N/S, T/F, and J/P.

$$\pi_D(X) = \begin{cases} D_0 & \text{if } X \leq D_0 \\ D_1 & \text{if } X \leq D_1 \end{cases}$$

This is well-defined for $X \neq \bot$ by the assumption that dichotomy poles are disjoint.

Finally, where L is a boolean lattice, $DS \subseteq L$ is a **dichotomy system** for L when 1) DS is independent and 2) every X in L is the meet of its projections onto its relevant dichotomies. This means that the dichotomies give complete information concerning the lattice (so there are no finer-grained partitions of it).

Now, consider a boolean lattice L with a dichotomy system DS. Since DS is a vector space, we can consider the **dual** vector space DS^*. Each element f in DS^* assigns a number, 0 or 1, to each dichotomy in a consistent way. This is exactly what we need to define the "flips" above, with 0 representing a dichotomy that is unchanged, and 1 representing a dichotomy that is flipped. However, we need additional structure to extend this action to types and not just dichotomies. Define

$$f : L \to L$$
$$X \mapsto \bigwedge_{D \in R(X)} \neg^{f(D)} \pi_D(X)$$

Claim. This defines a vector space action of DS^* on L.

Proof. First we prove that $(f + g)(X) = f(g(X))$. Note first that if D is relevant to X, it is also relevant to $f(X)$: if $D \in R(X)$, then $f(X) \leq \neg^{f(D)} \pi_D(X)$ so $D \in R(f(X))$. For the opposite we need independence: e.g., if $D_0 < D_0'$, then D_0' is not contained in D_0 or $\neg D_0$.

But by assumption, the set of dichotomies DS is independent. If D' is relevant to $f(X)$ but not to X, we have that $R(X)$ is *not* independent from D' (because $f(X) \in \mathfrak{T}(R(X))$ and it also intersects D_0' or D_1' nontrivially), so $DS \setminus \{D'\}$ is not independent from D' either.

So

$$f(g(X)) = \bigwedge_{D \in R(X)} \neg^{f(D)} \neg^{g(D)} \pi_D(X) = \bigwedge_{D \in R(X)} \neg^{f(D)+g(D)} \pi_D(X) = (f+g)(X)$$

Moreover, since every element is the meet of its relevant dichotomies,

$$0(X) = \bigwedge_{D \in R(X)} \neg^0 D = X$$

Thus also $\phi(-f)\phi(f) = \phi(0) = id$, and V^* acts on L. □

The orbits of this action are $\mathfrak{O}_{R_i} = \{X : R(X) = R_i\}$. The action preserves the relevance set, and for any two elements with the same relevance set, we can transform one to the other by flipping the appropriate dichotomies (using the spanning property of the dichotomy system). For example, clubs get sent to clubs, temperaments get sent to temperaments, etc.

Moreover, we have an orbit for $R_i = DS$ consisting of minimal (nonbottom or "nonempty") traits (aka **types**), for which all dichotomies are relevant.

Note that an element of DS^* may fix elements that it doesn't flip relevant dichotomies for. But, when restricted to types, the action is free - and thus freely transitive.

It can be recovered as the orbits of a sub-vector space of size 8 of the vector space of dichotomies — for the proof, see the section on dichotomies below.

5.1 The actual dichotomies

The Reinin dichotomies are as follows, expressed in terms of the Jungian basis:

real/unreal	$\emptyset$
extroversion/introversion	E
intuitive/sensing	N
logical/ethical	T
irrational/rational	P
carefree/farsighted	EN
obstinate/compliant	ET
static/dynamic	EP
aristocratic/democratic	NT
tactical/strategic	NP
constructivist/emotivist	TP
positivist/negativist	ENT
reasonable/resolute	ENP
subjectivist/objectivist or merry/serious	ETP
process/result	NTP
questioner/declarer	ENTP

Now I will show how we can derive the dichotomies from the relationships, with the dichotomy system as a vector space action.[17] One can obtain *most* of the system by taking the subgroup D Democratic/Aristocratic (i.e. the relationships that preserve this dichotomy), since these relationships all act by flipping certain Jungian dichotomies. The rest, however, do not, so we only have 8 of the 16 total vectors needed. Therefore we need to add another permutation z that isn't strictly based on relationships. Once we have that, we can multiply it by (add it to) the 8 others to get 16 in all. We can add, for example,

$$z(t) = \begin{cases} B(t) & \text{if t is rational} \\ b(t) & \text{else} \end{cases}$$

This corresponds to flipping T/F and J/P in the Jungian basis, and in combination with everything in D will generate all the rest of the dichotomy flips.

To verify that this makes a 4-dimensional vector space we prove (1) that z has order 2 and (2) that z commutes with everything in D. For (1), if t is rational, $B(t)$ is irrational so $z(z(t)) = z(B(t)) = b(B(t)) = t$, and similarly if t is irrational. Thus $z^2 = e$.

Now for commutativity. r flips rationality iff r is odd. If r is odd, $rB = br$, and if r isn't odd, r commutes with B.

	r odd	r even
t rational	$rz(t) = rB(t) = br(t) = zr(t)$	$rz(t) = rB(t) = Br(t) = zr(t)$
t irrational	$rz(t) = rb(t) = Br(t) = zr(t)$	$rz(t) = rb(t) = br(t) = zr(t)$

The underlying reason that we can use the subgroup D is that it has the same structure as $\mathbb{Z}_2^3$—a $\mathbb{Z}_2$ vector space, which is simply a group in which every element has order 2. This same fact is only true of one other subgroup of order 8: Rational/Irrational (J for short). And in fact, we can use an exactly parallel function to complete that subgroup to a dichotomy system:

$$z'(t) = \begin{cases} B(t) & \text{if t is democratic} \\ b(t) & \text{else} \end{cases}$$

[17] Below we characterize the Reinin dichotomies in terms of what dichotomies they respect, probably this is a more natural view.

Notice how all we did is switch J and D. We can use an identical proof to verify that this creates a complete dichotomy system. Now, what do these new dichotomies represent? We need to work out some of the orbits of dimension-3 subspaces. For example, d, k, z' gives

LII ESE LSI EIE IEI SLE ILI SEE

LIE ESI LSE EII IEE SLI ILE SEI

as our two sets.

Another example: d, ℓ, z' gives

LII ESE EII LSE IEI SLE SEI ILE

LIE ESI EIE LSI IEE SLI SEE ILI

What is the pattern here?

The first dichotomy is base function Beta vs. base function Delta, and the second is creative function Alpha vs. Gamma. We can complete these to a basis with I/E (k, ℓ, z') and J/P. Notice that those two are also based on dichotomous properties of the type's functions, and hence, so are all of the rest. There are 8 information elements, so there are 8 - 1 = 7 dichotomies of information. Whether the first function is I/E, J/P, static/dynamic determines the same for the second. So we get:

4: 1st function alpha/gamma, beta/delta, external/internal, involved/abstract

4: 2nd function alpha/gamma, beta/delta, external/internal, involved/abstract

4: I/E, J/P, static/dynamic, democratic/aristocratic

3: questioner/declarer, negativist/positivist, process/result

= 15 in all, just like the Reinin dichotomies.

5.2 Function and element dichotomies

Now, in Model A F and I have size 2^3 so we can construct dichotomy systems on them, call them F^* and I^*. $\mathbb{S}$ already acts on F, so we will simply define $F^* = D$. D is a vector space and it acts transitively on F (think, e.g., Ni can occur in any position in an Aristocratic type). If a certain function occurs in the same place for two Democratic (or Aristocratic) types, they must be the same. Hence, the action is faithful.[18] Dichotomies are

Subgroup	Dichotomy	1st set	2nd set
quadra	valued/subdued	1256	3478
egxd	accepting/producing	1357	2468
egaq	bold/cautious	1368	2457
egcm	conscious/unconscious	1234	5678
exac	contact/inert	1467	2358
edqc	evaluatory/situational	1458	2367
club	strong/weak	1278	3456

Notice, however, that J does not act transitively, which makes it unsuitable for use as a dichotomy system. (Its "dichotomies" are accepting/producing, as well as a few four-element partitions).

[18]Technical note: or, you can just check $|D| = |F|$.

I^* is more tricky. $Z(\mathbb{S})$ acts on I according to its permutation of the accepting functions and producing functions respectively.[19] That gives us "two of the three dichotomies", so to speak. And, remarkably enough, the last is given by z' above! We obtain

Subgroup	Dichotomy	1st set	2nd set
e, g, x, d	irrational/rational	$NeSeNiSi$	$TiFiTeFe$
e, g, z', gz'	extroverted/introverted	$NeSeTeFe$	$TiFiNiSi$
e, x, z', xz'	abstract/involved	$NeNiTeTi$	$FiFeSiSe$
e, d, z', dz'	Delta/Beta	$NeTeSiFi$	$NiTiSeFe$
e, d, xz', gz'	Alpha/Gamma	$NeSiFeTi$	$TeFiSeNi$
e, x, gz', dz'	internal/external	$NeNiFeFi$	$TiTeSiSe$
e, g, xz', dz'	static/dynamic	$NeSeTiFi$	$TeFeNiSi$

5.3 Classification of dichotomies

There are 32768 possible ways of dividing the socion into two groups, so we might ask what makes the Reinin and alternative dichotomies special or "good". Clearly, they need to be compatible with the relationships in some way. In fact, all of them are either preserved or reversed by each relationship in the center of $\mathbb{S}$. This is not true for other relationships: semidual, for example, sometimes reverses the sensing/intuition dichotomy and sometimes preserves it.

So, let's determine which dichotomies have this property. Call such a dichotomy a *geometric* dichotomy.

For each dichotomy, we assign 0 to a relationship if it preserves the dichotomy and 1 if it reverses it. So we can use the numbers assigned to x and d to classify the dichotomies, since they determine whether it preserves g. That is, for $m, n \in \{0, 1\}$, let an mn-dichotomy be one that is preserved by x (resp d) iff $m = 0$ (resp $n = 0$).

Claim: This actually defines a homomorphism $h : GeoDich \to \mathbb{Z}_2 \times \mathbb{Z}_2$.

Lemma: where $\equiv$ denotes logical equivalence and $-$ denotes complementation,

$$-(D \equiv E) = -D \equiv E = D \equiv -E$$

(easy)

Now, for $r \in \mathbb{S}$, let $M = (-1)^m$ (where again negation represents complementation, so M is complementation or identity), and $N = (-1)^n$, and let D, $E \subseteq Type$. Now suppose that $r(D) = MD$ and $r(E) = NE$, so that r is compatible with the dichotomies $(D, -D)$ and $(E, -E)$. Then,

$$r(D \equiv E) = r((D \cap E) \cup (-D \cap -E)) = (MD \cap NE) \cup (-MD \cap -NE) = MD \equiv NE$$

(Note: the function r commutes with $\cap$ because it is a bijection) And that is $M(D \equiv NE) = MN(D \equiv E)$ by the lemma.

This shows that if r is compatible with dichotomies D and E, it's compatible with D * E. And it also shows that h is a homomorphism.

So for example, if we interleave a 01-dichotomy and a 10-dichotomy we get a 11-dichotomy.

The orbital dichotomies are all 11- or 00-dichotomies because they are all preserved by superego.

Now let's classify the geometric dichotomies. The poles of 00-dichotomies must contain the following four groups:

LII ESE LIE ESI — ILE SEI ILI SEE — EIE LSI LSE EII — IEI SLE SLI IEE

[19] Explicitly, $r(Xa) = r(t)(t^{-1}(Xa))$ for any t

There are $\frac{2^4}{2} = 8$ ways of joining these up into a dichotomy. However, only three of these split the set of types evenly, namely Democratic/Aristocratic, Rational/Irrational, and Result/Process. One other is the tautological trait (true/false), and the other four are 4/12 splits.

In every other class of dichotomies, two types out of every orbit of Z must have the opposite trait as the other two.

The 10 class:

LII ESE — LIE ESI

ILE SEI — ILI SEE

LSI EIE — LSE EII

SLE IEI — SLI IEE

When we exchange, respectively, none of the pairs, the first 2, 1 and 3, 1 and 4, only the 1st, 2nd, 3rd, 4th, we obtain:

Merry/Serious, Reasonable/Resolute, yielding/obstinate, carefree/farsighted, 2nd function Beta/Delta, 1st function Beta/Delta, 2nd function Alpha/Gamma, 1st function Alpha/Gamma

The 01 class:

LII LIE — ESE ESI

ILE ILI — SEI SEE

LSI LSE — EIE EII

SLE SLI — IEI IEE

When we exchange, respectively, none of the pairs, the first 2, 1 and 3, 1 and 4, only the 1st, 2nd, 3rd, 4th, we obtain:

T/F, N/S, emotivist/constructivist, tactical/strategic, 2nd function internal/external, 1st function internal/external, 2nd function abstract/involved, 1st function abstract/involved

The 11 class:

LII ESI — ESE LIE

ILE SEE — SEI ILI

LSI EII — EIE LSE

SLE IEE — IEI SLI

When we exchange, respectively, none of the pairs, the first 2, 1 and 3, 1 and 4, only the 1st, 2nd, 3rd, 4th, we obtain:

static/dynamic, questioner/declarer, I/E, negativist/positivist,

E Democratic Positivist or Static Aristocratic Declarer, I Democratic Negativist or Static Aristocratic Declarer, Static Democratic Questioner or E Aristocratic Negativist, I Result Questioner or Static Process Positivist

These last four are **not** Reinin dichotomies, **nor** are they from the alternative system. Call these the odd dichotomies out (ODOs) and label them D1, D2, D3, and D4 respectively.

So altogether we have $8 + \frac{2^4}{2} \cdot 3 = 32 = 2^5$ geometric dichotomies. So they overspecify the set of types by one dichotomy, and if we add any geometric dichotomy to either the Reinin or alternative system, we will get all of the rest.

As for the ODOs:

$(D1 \equiv D2)$ = static Aristocratic or (E Positivist or I Negativist)) = static Aristocratic or Aristocratic = Aristocratic

and (Aristocratic $\equiv$ D3) = negativist Aristocratic or dynamic Democratic = $-D4$

(technically we are using the traits listed above to represent the dichotomies, and computing with them).

So also $D3 * D4 = DA$ and the ODOs have dimension ≤ 3. But $D1 * D2 \neq D3$ by the homomorphism, so the ODOs in fact generate a dimension 3 subspace, which contains all of the even 00 dichotomies. They don't distinguish superegos because they are in the 11 class.

The Jungian dichotomies actually are respected by the DA (Democratic/Aristocratic) subgroup, therefore so are the Reinin dichotomies. And the alternative dichotomies are respected by the JP (rational/irrational) group.

If both DA and JP respect a dichotomy, then so must all of $\mathbb{S}$, since $\mathbb{S}$ is generated by them. But then the dichotomy actually gives a homomorphism $\mathbb{S} \to \mathbb{Z}_2$. So such dichotomies are exactly the orbital ones, and the homomorphism is just the quotient by the dichotomy's subgroup, sending each relationship to 0 if the dichotomy contains it and 1 if it doesn't.

In fact, for any group G and G-set X there is actually a contravariant functor CD associating each subgroup of G with the set of dichotomies that it respects, so that $H \leq K$ implies $CD(K) \subseteq CD(H)$, and each dichotomy in $CD(H)$ can be associated with a (not necessarily unique) homomorphism $H \to \mathbb{Z}_2$. Note: the dichotomies need not be even.

Above we calculated $CD(Z)$. $CD(\mathbb{S})$ is the seven orbital dichotomies. $CD(DA)$ is the Reinin dichotomies. $CD(JP)$ is the alternative dichotomy system. It turns out that CD applied to any other subgroup of order eight is again the orbital dichotomies, owing to the group structure of the other subgroups (they have less subgroups of index two, and only two generators instead of three).

In general, when a group G acts transitively on a set X, the G-invariant dichotomies correspond directly to index-2 subgroups of G.

Proof: let H be an index-2 subgroup of G. Then H has two orbits and these partition X. Why is it an invariant partition? H has index 2 so it's normal, so $gH = Hg$ for all $g \in G$. Therefore $gHx = Hgx$ for all $x \in X$.

Now suppose we have a G-invariant dichotomy D (partition with two elements whose equivalence relation is G-invariant). Then we get a homomorphism $h : G \to \mathbb{Z}_2$. So let $K = \ker(h)$. Since G acts transitively, we have, for all x and y in X, $g \in G$ such that $gx = y$. If x and y are in the same block then $h(g) = 0$ so $g \in K$. Both orbits are therefore Kx for some x. $\square$

Notice that homomorphisms $G \to \mathbb{Z}_2$ have kernel of index *at most two*. That is, G itself corresponds to the zero homomorphism. Call the corresponding G-invariant equivalence relations *almost-dichotomies*. Since the sum of homomorphisms $G \to \mathbb{Z}_2$ is also a homomorphism $G \to \mathbb{Z}_2$, we conclude that the set of index-2 subgroups along with G itself forms an $\mathbb{F}_2$ vector space, so G must have $2^k - 1$ index-2 subgroups for some k, and they are closed under the operation of interleaving (which corresponds to addition of the dual homomorphisms). In particular, if G is a vector space acting freely transitively on X then its almost-dichotomies give a dichotomy system on X. That is, if x and y are equivalent under all almost-dichotomies, then they are equal.

Proof: if we choose a basis (e_i) and take the homomorphism f_i defined by dot product with each of the basis vectors, then $K_i = \ker(f_i)$ is just the vectors with a zero in the ith coordinate. We may identify X with the vector space V itself, and the action with left addition, so the orbit of x under K_i is the set of vectors that agree with x in the ith coordinate. Therefore, if x and y are in the same orbit for all K_i, all their coordinates are equal, so they are equal. $\square$

The above has the corollary that G-invariant dichotomies for G acting transitively must all be even, i.e. they must split the set evenly. This is perhaps not obvious and in fact fails when G is not transitive, as we saw above while classifying the Z-invariant dichotomies.

It's a result of p-group theory that in fact the dimension of the space of maximal subgroups equals the minimal size of a generating set of G, which in this case is 3.

Any geometric dichotomy D can be symmetrized to a dichotomy $\bar{D}$ which is in fact orbital: just take it and add it to its rotation QD (i.e. any quadrality applied to it, since they all give the same result). Then $Q(D + QD) = QD + D$ so $Q(\bar{D}) = \bar{D}$.

Now, since the quadralities act as rotations on traits, and wall dichotomies are reversed by g, we have that

$$Q(\bar{D}^+) = Q(D^+ \& QD^+ or D^- \& QD^-) = QD^+ \& gD^+ or QD^- \& gD^- = QD^+ \& D^- or QD^- \& D^+ = \bar{D}^-$$

so Q reverses $\bar{D}$. This means that the symmetrization of a wall dichotomy must be Democratic/Aristocratic, Rational/Irrational, or the tautological dichotomy.

The orbital dichotomies produce the tautological dichotomy.

The Reinin dichotomies produce

6 Set Models

A *model* of a group is a representation as a concrete group of transformations, for example, as a set of bijections on a set X. Model A is such a model for $\mathbb{S}$, where the set X is the set of functions. Let us call these *set models* to distinguish them from other models, like geometric ones.

Group actions on sets can be split into their *transitive* parts: simply take the orbits of the group G and each one gives a sub-action. Further, transitive group actions can be classified according to the stabilizers of a point: that is, if $x \in X$ and $H = Stab_x = \{g \in G : g \cdot x = x\}$, then the action of G on X is isomorphic to the action of G on the set of left cosets of H, $\{gH : g \in G\}$. Moreover, the stabilizer of any other element in X is always conjugate to H. Therefore, transitive set models correspond to conjugacy classes of subgroups of G.[20]

Note that transitive actions of $\mathbb{S}$ model two distinct phenomena in socionics: the set of sociotypes as well as the set of functions.[21] Then, we further model types as labeled sequences of information elements. We consider the set of types as "fixed", but with different potential theoretical constructs representing either the 16 types or a coarser-grained set of types. Here we interpret the positions of X (considered as a labeled set) as generalized "functions", which contain generalized "information elements" that are permuted by $\mathbb{S}$. Therefore, if we label the set and declare that a certain labeling of it represents a type, say ILE, each permutation by $\mathbb{S}$ of this sequence represents the other types, possibly in a coarse-grained way where there are less than 16 types. However, the action of $\mathbb{S}$ on labelings of X may not be transitive—and hence not every labeling will correspond to a type. This fact is familiar in Model A: not every sequence of the eight elements gives a valid type. So, let's classify these models. Notice also that most of the transitive models will not be faithful; here we will characterize which ones are both faithful and transitive.

[20] Much of this material can be found at http://math.ucr.edu/home/baez/groupoidification/. Upon closer inspection Socionics is not really about the group $\mathbb{S}$, it's about a groupoid where the objects are people and the morphisms are the relationships between them. This groupoid (call it P) is equipped with a faithful functor F to $\mathbb{S}$ which classifies the relationships. This functor is the same as a group action of $\mathbb{S}$ on P but it more closely reflects the "physical" intuition, which is that each two people have a distinct relationship, which is nonetheless classified as one of the sixteen socionic relationships.

[21] Technically, $\mathbb{S}$ does not have a unique action on the set of IM elements in Model A! It can be arbitrarily given one by choosing to identify the elements with the set of Process types or the set of Result types, and orienting the benefit and supervision rings accordingly. But the choice itself is arbitrary. This is one problem that Model A2 solves.

Recall that if $r, r' \in \mathbb{S}$ are conjugate, we have either that $r = r'$ or $r = gr$. Thus, conjugacy classes of subgroups must have either one or two elements.

There is first of all the trivial model corresponding to $\mathbb{S}$ itself, where there is just one "element" and it is permuted in the trivial way.

All index-2 subgroups of $\mathbb{S}$ are normal, so each one forms a conjugacy class. So for each one we obtain a model with two "elements", where their relative order in the type's model says whether that type has one pole of the given dichotomy or the other.

The order-4 subgroups give $\frac{16}{4} = 4$ elements. Note that given two subgroups G and H, the cosets of $K = G \cap H$ correspond to pairs of cosets of G and H, because $u(G \cap H) = uG \cap uH$ (this is trivial to check). This also shows that if G and H are normal, so is $G \cap H$. So for each order-4 group that is an intersection of two order-8 subgroups, the model is given by pairs of dichotomies.

The remaining conjugacy classes of order-4 groups are $C_1 = \{exac, exqm\}$, $C_2 = \{exik, exhl\}$, $C_3 = \{edam, edqc\}$, $C_4 = \{edil, edhk\}$. There is not much to say about a model until we give its "elements" an interpretation. How can we do this in a rigorous way? C_1's can be interpreted as NTSF.

NTSF

The smallest models of $\mathbb{S}$ will not be faithful (i.e. there will be less than 16 types), but let's classify them anyways. We will only consider models that have no fixed functions (which would be redundant).

A group action splits up the set into partitions called *orbits*. Functions x and y are said to be in the same orbit if there is a relationship r such that $r(x) = y$. This is an equivalence relation; if $r(x) = y$ and $s(y) = z$, then $sr(x) = z$. It turns out that we can consider each orbit of a model separately. And whenever we have two models, we can set them side-by-side to create a composite model. Therefore we need only classify the models with one orbit—i.e., the *transitive actions*.

Recall that a model of $\mathbb{S}$ is transitive when for all functions i and j in the model, there is a relationship r such that $r(i) = j$. Model A is a transitive model of order 8. Transitivity puts all the functions on the same footing, so to speak. The models of order 6, for example, are not transitive: they have two unbreachable classes of functions (e.g., NTSF and IE).

Orbits of $\mathbb{S}$ must have size dividing the size of $\mathbb{S}$, 16. So if the action is transitive, the number of functions n is even and ≤ 16.

There is a unique, trivial model of order 1.

6.1 Order 2

If $\mathbb{S}$ acts transitively on a set of size 2, the kernel of the homomorphism from $\mathbb{S}$ to S_2 satisfies $|\ker(\alpha)| = 8$. Hence there is a model for each subgroup of order 8 (since all are normal). Here the "types" are very coarse: they are just single dichotomies, and the "functions" are the two dichotomy poles, arranged in order of preference.

6.2 Order 4

For S_4 (size 24), $|\ker(\alpha)| \geq 2$, because $\gcd(16, 24) = 8$.

If $|\ker(\alpha)| = 2$, the only possibilities for the kernel are eg, ex, and ed, whose quotients are $\mathbb{Z}_2^3$[22], D_4[23], and D_4[24]. It is clear that $\mathbb{Z}_2^3$ doesn't embed into S_4 (using the below lemmas), but D_4 does (thinking of the four points as corners of a square). The rotation in D_4 can be made to go to (1234), and the

[22] proof: no element of order 4

[23] The only group of order 8 which is not abelian; the quotient is not abelian because $[ab]$ and $[ba]$ are different.

[24] Same as the previous by automorphism of $\mathbb{S}$.

possibilities for the flip are given by the proof below for S_6; all are in the standard representation of D_4 where $a = (12)(34)$, for example. Thus, there are two unique transitive models, viz.:

$NTSF$ for ILE and ILI

and

$\alpha\beta\gamma\delta$ for ILE and SEI

where it is understood that if the quadras go in reverse order, the type is a Result type.[25] Let's call these models *loops*.

So, what this tells us is that Extinguishers and Duals both have something very much in common in their relationships. We can surmise that the former has to do with strengths and the latter with quadra values.

For $|\ker(\alpha)| = 4$, there is at least one model for every normal subgroup of order 4, namely the 5 containing g.[26] For all, the group of coarse relationships (the quotient and image in S_4) has no element of order 4, as $[b^2] = [g] = [e]$ and same with s. Thus it is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, and is generated by two commuting elements of order 2, which one can pick to be of the form $(12), (34)$ (see below lemmas for why). Thus the type model is just a pair of dichotomies, as listed above. Thus, it is the union of two models on S_2.

It may be worthwhile describing interactions between these coarse types, an example of which is temperaments.

6.3 Order 6

For order 6, there can be no transitive model, since 6 is not a power of 2. Therefore any model of this size is a collection of smaller models. It can be either:

• 3 dichotomies. Notice the only dichotomies allowed are ones that superego partners share; therefore, the only non-redundant choices make Superegos the same type.

• A dichotomy and a loop. If you pick a dichotomy to distinguish duals or extinguishers (depending on the loop), you get a faithful model with 16 types, such as $\alpha\beta\gamma\delta EI$, $NTSF+-$, etc.

6.4 Order 8

Here again we can place smaller models side by side to get the non-transitive models. The possibilities are:

• Four dichotomies,

• One loop and two dichotomies (redundant addition to the models of order 6),

[25]This is the basic cause for the name Left/Right. If b and s both correspond to a simple shift, then the "orientation" of the functions must correspond to Process/Result, the group generated by b and s. We can also construct these models from Model A by identifying "opposite-version' functions or dual functions, respectively, in which case the interpretation of the latter model changes somewhat, e.g.: Reasonable Merry Resolute Serious. But notice that those values apply to α, β, γ, and δ, in that order.

[26]The subgroup must be normal, which means that it is impossible to construct relationships between clubs, for example, because no club can consistently be called the "supervisor" of another club. One NT's supervisor is an ST, while another's is an NF.

• Two loops: $NTSF\alpha\beta\gamma\delta$. This is another faithful model which is not transitive.

So far we haven't seen any models that are both faithful *and* transitive. But of course, Model A is such a model.

If the model is faithful and transitive, $n = 8$ or 16. If $n = 16$, the action must be free as well, by the pigeonhole principle: if $\mathbb{S}x = S$, then application to x is surjective, so it is injective because $\mathbb{S}$ is finite. Thus any transitive model of order 16 is isomorphic to $\mathbb{S}$'s action on T.[27]

Proof. First we make an observation: If $gh = hg$, then $gh^n(x) = h^n g(x)$, and in particular if $h^n(x) = x$, then $g(x) = gh^n(x) = h^n g(x)$.

The latter condition also holds if $gh = h^{-1}g$: if $h^n(x) = x$, then $g(x) = gh^n(x) = h^{-n}g(x)$ so $h^n g(x) = g(x)$. In either of these cases, g preserves elements' "order" with respect to h: it can't move elements between orbits of different sizes, and in particular must preserve fixpoints (orbits of size 1).

Let's say that in either of these situations h **forces** g. Thus any element of order 4 in $\mathbb{S}$ forces every other element, and so does any element in the center.

For a model of $\mathbb{S}$ to be transitive, any element which forces every other element must have orbits that all have the same size, because each element of the set is linked by some transformation. And their size is equal to the order of the element, because the gcd of any divisors of 2^n is the same as their maximum. So WLOG, $S = (1234)(5678)$. Now we consider the possibilities for an element $x \in Z(\mathbb{S})$, which must have order 2 and also commute with S (so they force one another). Because x is in the center, it in fact forces everything, and its orbits must all have size 2. So first consider the case where e.g., $x(1)$ is in the same orbit of S as 1 is. If $x(1) = S(1)$, then $S^2(1) = S(x(1)) = x(S(1)) = x(x(1)) = 1$, which is false. The same argument with S^{-1} shows that $x(1) \neq S^{-1}(1) = S^3(1)$. Therefore $x(1) = S^2(1)$, and $x(S(1)) = x(S(1)) = S^3(1)$, and the other orbit follows a similar pattern. But then we obtain $[x] = [S^2]$, so the action is not faithful.

What about when x moves 1 (everything, actually) to the opposite orbit? We can then calculate that x is one of

(15)(26)(37)(48) (d_{MA})

(16)(27)(38)(45) (x_0)

(17)(28)(35)(46) (x_{MA})

(18)(25)(36)(47) (d_0)

where MA denotes the standard elements of Model A. Either of these two plus S generates the other, and the same with the alternative pair. However, these are all the same up to conjugation (i.e. permutation of the set of functions) by (5678), which fixes S.

So WLOG $x = x_{MA}$. Now we must find the possible values for a, a generic odd element.

Note that a and x commute, so $a(x(k)) = x(a(k))$, meaning that a induces a pairing of the orbits of x. We also need $aS^n(i) = S^{-n}a(i)$ so $a(3) = S^{-2}(a(1))$ and $a(2) = S^{-1}(a(1))$, and $a(6) = S^{-1}(a(5))$, $a(7) = S^2(a(5))$. Therefore, it suffices to choose $a(1)$ and $a(5)$. But $ax = xa$ so $a(7) = a(x(1)) = x(a(1))$ so it actually suffices to just choose $a(1)$. We then obtain eight elements, one for each choice of $a(1)$, which are exactly the odd elements of Model A. □

Are there any other transitive models of order 8? By the Orbit-Stabilizer theorem, the stabilizer of a type must have size $\frac{16}{8} = 2$. Therefore, the kernel has at most two elements, since it's the intersection of all the stabilizers. If it has one element, then the action is free, and it must be Model A. What about when it has two? Then the kernel is among the two-element groups generated by g, d, and x

[27] Gulenko's Model G involves 16 elements called +Fi, -Fi, etc. But one could also say an ILE's leading function is not +Ne, but ILE!

respectively, and the image is $\mathbb{Z}_2^3$, D_4, or D_4 respectively. Thus, it suffices to consider whether these two groups have a transitive action on eight elements.

Well, these groups both have eight elements, so if they act transitively on eight elements, then the action must be isomorphic to the action of each on itself by multiplication. It is best to think of these as quotients of a 16 function model. For example, for $\langle g \rangle$, we can write Ji_D for rational democratic introversion, etc. For $\langle d \rangle$, the functions would be J_α, etc.

7 Geometric models

7.1 3D models

The functions of Model A can be placed at the vertices of a cube. When you arrange the IM elements at the corners of the cube, it represents a type, and every intertype relationship is a transformation of the cube that produces another type.

This is called a *linear representation* of the group $\mathbb{S}$. It is characterized as the symmetries (isometries) of the cube that preserve a chosen axis, here the vertical axis. Or equivalently, as the symmetries of a "fat cube", a rectangular prism in which the height is not equal to the width. Let us call this representation Geometric Model A (GMA). In Coxeter notation, this group is called $[4, 2]$ or $[2, 4]$; in Conway's orbifold notation it is $*332$.[28]

In this model, the rotation about the z-axis is an order-four relationship, so the bottom and top of the cube are either benefit rings or supervision rings. Vertical reflection is then either duality or extinguishment. Finally, we choose an odd element to represent a horizontal reflection, in this case mirror (so mirror is literally a mirror reflection). There are always two horizontal reflections so they generate essentially the same model. We can also specify a diagonal horizontal reflection instead of the "parallel" ones, here kindred.

The significance of the kindred relation is that it actually fixes the leading function (and in fact all the accepting functions), preventing us from identifying the vertices with types. We can say that the vertices are pairs of types, but the action does not respect this interpretation: the supervisor of LII is Se-leading while the supervisor of LSI is Ne leading.

So the types are actually identified with *oriented edges* of the cube, e.g. we can identify a type with its leading and creative function together, or its leading and vulnerable function (when we use supervision rings), or its leading and mobilizing function (with benefit rings), etc. In fact the two vertices need not be adjacent—we can pick any pair as long as they aren't above/below one another, antipodal on the circle, or antipodal on the sphere. So they are either horizontal edges or vertical diagonals.

A geometric representation is a homomorphism $\mathbb{S} \to O(n)$.[29] When we precompose with an automorphism of $\mathbb{S}$ we obtain a new representation. Notice, most of these representations will have different "IM elements" than Model A. For example we might have quasi-identical pairs with the same "leading function" or activator pairs (meaning, those relationships fix half the vertices). But in all cases they are the symmetries of a cube fixing an axis going through the middle of the cube.

Claim. This is the only 3D representation of $\mathbb{S}$.

Every finite subgroup of $SO(3)$ (that is, orientation-preserving symmetries) is one of the following (Artin, p100):

[28] See https://en.wikipedia.org/wiki/Octahedral_symmetry

[29] In this paper we will only be concerned with representations consisting of isometries—maps that preserve distance. Any representation of a finite group by isometries must fix some point (see Artin), so it can in fact be viewed as a linear representation.

- $\mathbb{Z}_n$
- D_n
- T (tetrahedral group: order 12)
- O (octahedral group: order 24)
- I (icosahedral group: order 60)

Let α be the representation (in $O(3)$, not $SO(3)$). Then we have $\alpha(\mathbb{S}) \cap SO(3) = \ker(sgn|_{\mathbb{S}})$.

If the sign map is trivial ($\text{im}(sgn\alpha) = 1$) then $\text{im}\alpha$ must be isomorphic to one of the groups in the list. But then it must be $\mathbb{Z}_{2^i}$ for $i \leq 4$, or D_k for $k \leq 8$ just by checking the order. In either case, the representation isn't essentially 3-dimensional and is not faithful. So $\mathbb{S}$ has no faithful representation as 3D rotations.

If the sign map is surjective then $|\text{im}| = 2$ so $|\ker sgn\alpha| = 8$ where α is the representation, by the isomorphism theorem. Thus, $\ker sgn$ (in $\text{im}(\alpha)$) must be $\mathbb{Z}_8$ or D_4. But $\mathbb{S}$ has no order-8 element so the kernel must be D_4.[30] This also implies that the representation is faithful, since we also have at least one orientation-reversing element.

Therefore, the orientation-preserving elements in the representation are IE, PN, QD, or SD (see the list of subgroups above). Moreover, none of them contain x or d.

To complete the proof, let C be either x or d. C is an order-2 element so it acts as negation (reflection across the origin) on some nontrivial subspace and as the identity on its orthogonal complement.

I claim that C must either be negation on all of space, or the reflection across the plane of rotation. To prove this, take some vector v for which $Cv = -v$ (this must exist). Then $CQv = QCv = Q(-v) = -Qv$ where Q is the rotation quadrality. Either 1) all such v are in the axis preserved by Q, so we're done. or 2) v isn't in the axis preserved by Q, so that $\{v, Cv, C^2v, C^3v\}$ are all distinct and form a square. This square may go through the origin, but not for all v, because then C = g. If the square doesn't go through the origin, then in fact $\{v, Cv, C^2v\}$ are all linearly independent so they span all of space and C is total negation. Since C can be either x or d, both of these possibilities are realized—we just have to choose which is which. □

All of these transformations manifestly preserve the axis of rotation.

Now we present an alternative proof of uniqueness using representation theory. While the above reasoning is more concrete it becomes unwieldy when we go to the 4-dimensional case.

Proof. Let V be a 3D real representation of G, and let V' be its complexification, which has complex dimension 3. V' is a direct sum of irreducible representations (irreps). Moreover, since $\mathbb{S}$ is the direct product $\mathbb{Z}_2 \times D_4$, its irreps are of the form $X \boxtimes Y$ where X is an irrep for $\mathbb{Z}_2$ and Y is an irrep for D_4. So we get $V' = \bigoplus X_i \boxtimes Y_i$. We take D_4 to be generated by $\{a, b\}$ and $\mathbb{Z}_2$ to be generated by x.

$\mathbb{Z}_2$ has two irreps: the trivial one and its faithful representation as negation on $\mathbb{C}^1$. [31] Call the negation rep N.

D_4 has four irreps: the trivial one, two "sign reps" with image equal to $\mathbb{Z}_2$, and the canonical faithful rep S (corresponding to symmetries of a square).[32] The sign reps are 1D and the rotation rep is 2D.

Notice: all of these representations are given by signed permutation matrices (matrices with entries equal to 0, 1, or -1) so they are real and orthogonal. This property is inherited by the decomposition above.

[30]Notice: the "reflection" in the D_4 subgroup is orientation-preserving in 3-space because it can be implemented by a rotation through the third dimension. It's orientation-reversing in 2D.

[31]https://groupprops.subwiki.org/wiki/Linear_representation_theory_of_cyclic_group:Z2

[32]https://groupprops.subwiki.org/wiki/Linear_representation_theory_of_dihedral_group:D8

Also note that $g = b^2$ is in the kernel of all of D_4's irreps *except* the canonical one. We aim to classify only faithful representations, so the canonical representation must be included.[33] Thus we have exactly two summands of dimension 1 and 2 respectively: $V' = W \boxtimes X \oplus Y \boxtimes S$. If Y is N but W isn't, then $x = g$ so the rep isn't faithful. Therefore $W = N$. X must be one of D_4's four 1D reps (the trivial irrep is 1D), and Y is either one of $\mathbb{Z}_2$'s irreps. So we get eight.

However, it is possible that some of these are isomorphic to others. In fact, they are all the same up to automorphism of $\mathbb{S}$. First, note that $\mathbb{S}$ has 64 automorphisms, D_4 has 8, and $\mathbb{Z}_2$ has just one. Thus, there are exactly 8 $(8 \cdot 1)$ automorphisms of $\mathbb{S}$ that are given by automorphisms of its factors. These were already taken into account when we said that D_4 has one injective representation, for example. So quotienting out by these, we obtain 8 essentially different automorphisms of $\mathbb{S}$. When we precompose a faithful representation by each one of these it will always give a different representation: each automorphism f exchanges some pair of elements r, p, but if exchanging them does not change the action then the action is not faithful. So every faithful 3D representation of $\mathbb{S}$ is the same up to $\mathbb{S}$-automorphism. □

7.2 4D models

We classify the possible models using representation theory as before.

We have already seen that any representation of $\mathbb{S}$ will be equivalent to a hypercube representation. This means that all of them are expressible in terms of dichotomies. In 3D, we have a faithful representation that is however not "free" in that there is no orbit where it acts regularly: there is always a centralizing element.

In a hypercube model of any group, the group elements permute the basis vectors (from here on we on we only consider the standard basis).

In the 4D reps of $\mathbb{S}$ the action on basis vectors is also *transitive*: We can take any signed basis vector to any signed basis vector. This is true simply because there are $2^4 = 16$ of them.

Finally, since $\mathbb{Z}_2$ always acts by negation or trivially, we have that the central elements of $\mathbb{S}$ all preserve each basis vector or negate it. That is, the symmetries of the (hyper)cube permute its faces, and therefore permute the midpoints of faces, which are algebraically represented as *signed basis vectors*. So the axes in the representation are in fact the even **geometric dichotomies** as classified above. That is, each axis must either have a 1 or -1, corresponding to the two poles of the dichotomy.[34] If we consider other orbits or non-isometric models (say, on "hyperboxes" where the sides need not all be the same length) then we may generalize this to T_i is true of type t (represented as a nonzero vector) iff $sgn(t \cdot e_i) > 0$.

In fact every even geometric dichotomy is realized (or, can be taken as a basis vector) in some model. First of all, in every **geometric basis** (i.e. a basis of dichotomies that occurs in a 4D geometric model), we must have a pair of dichotomies that are "rotated" into one another, i.e. when you apply a quadrality to $D+$ you obtain $E+$ and when you apply it to $E+$ you get $D-$. If this is true for one quadrality it is true for all of them, and we call the pair of dichotomies a **rotation pair**.

Since an orbital dichotomy is always sent to itself, rotation pairs must consist of non-orbital geometric dichotomies, which we call **wall dichotomies** as they occur as the walls of the 3D cube. The pairs are as follows:

[33] This shows that there can be no faithful 1D representation. And there can't be any faithful 2D representation either, because both g and x must act as negation.

[34] Faces of the cube of various dimensions correspond to intersections of dichotomous traits.

Dichotomies	Preserved by	description
Intuitive/Sensing	x m q	strong/weak == N/S [Club]
Tactical/Strategic	x c a	contact/inert == N/S
1st A/Γ	d k h	
2nd A/Γ	d l i	
Logical/Ethical	x m q	strong/weak == T/F [Club]
Constructivist/Emotivist	x c a	contact/inert == T/F
Merry/Serious	d m a	valued/unvalued = TiFe/TeFi [Quadra]
Yielding/Obstinate	d q c	evaluatory/situational == TiFe/TeFi
Reasonable/Resolute	d m a	valued/unvalued = SiNe/SeNi [Quadra]
Carefree/Farsighted	d q c	evaluatory/situational == SiNe/SeNi
1st ext/int	x k i	??? == external/internal
2nd ext/int	x l h	
1st abs/inv	x k i	
2nd abs/inv	x l h	
1st B/Δ	d k h	
2nd B/Δ	d l i	

A rotation pair of Reinin wall dichotomies consists of D and D + Rational/Irrational. A rotation pair of post-Reinin wall dichotomies consists of D and D + Aristocratic/Democratic. This says that Rational/Irrational and Democratic/Aristocratic are actually *diagonal slices* of the hypercube (second-order traits).

We can either have two rotation pairs or one.

If we have two, then they cannot be both Reinin or both post-Reinin, because then their sum is equal and they will be linearly dependent. However, if one is Reinin and the other is post-Reinin, they will generate dichotomies that are neither Reinin nor post-Reinin. Proof: if D is a Reinin wall dichotomy and E is a post-Reinin wall dichotomy and $D+E$ is Reinin, then $E = (D+E)+D$ so E is also Reinin, a contradiction. Similarly, $D+E$ cannot be post-Reinin. □

Therefore, if it is even (and it must be even in order to occur in a geometric basis), their sum must actually be one of the ODOs found above. And since we have two Reinin wall dichotomies and two post-Reinin wall dichotomies, all of the four ODOs are realized in a two-loop model (as second-order dichotomies).

Moreover, this says that each Reinin rotation pair corresponds to exactly one post-Reinin rotation pair with which it forms a geometric basis—and vice versa: simply add all of the ODOs to the space generated by the pair. This also tells us that the ODOs can be completed to a geometric basis.

Thus, there are exactly four two-loop models (up to basis permutation)—although only one up to $\mathbb{S}$-automorphism and basis permutation (*structural similarity*).

The rest of the 4D models have one rotation pair and two orbital dichotomies—because, by the representation-theoretic classification, the other two dichotomies are always sent to themselves (i.e. preserved or negated) by any relation. As long as we have four such dichotomies that form a basis, they will give a geometric model. Since every Reinin or post-Reinin dichotomy is either orbital or a wall dichotomy, this shows that they can all occur in a geometric basis—hence, are truly geometric.

Perhaps the simplest 4D model is the (1-loop) 2-cubes model, where we take the standard action of PR on two copies of the cube, but redefine the action of an odd element as the exchange of the two cubes. So a centrality C is the vertical reflection as in the 3D model, and an odd relation O is the exchange of the cubes:

- the dichotomy between the two cubes is always Process/Result since each cube is preserved by the chosen quadrality Q and the centrality C.

- the vertical dichotomy is one of the D_4 subgroups—which one depends on which relations are used. For example, it's Static/Dynamic in the *sdk* hypercube.
- the other two are wall dichotomies that depend on O and C.

So the 2-cubes model can contain any dichotomy as a basis vector except Rational/Irrational and Democratic/Aristocratic.

Algebraically, the hypercube symmetries are *signed permutation matrices*, that is, matrices with only one nonzero entry in each row and each column, and no entries other than 0, 1, -1. So the action actually permutes the spaces generated by basis vectors.

We write cycles as, for example (-1 -3 2). This means that 1 gets sent to -3, 3 gets sent to 2, and 2 gets sent to -1. So we use the *signed output* convention, where you ignore the sign on the input basis vector. We could also use a notation like (1-3=2-) which is more convenient for taking inverses (just reverse the sequence of symbols).

The transformation can be written in many different bases, but the ones that align with the hypercube are all related by permutation or changing signs. By the classification of representations, for $\mathbb{S}$ all cycles must have length 1 or 2. A 2-cycle where the signs are the same ((a=b=) or (a-b-)) is a diagonal reflection. A 2-cycle where the signs are different is a 90° rotation.

7.3 Interpolation

If we represent order-4 elements of Model A as rotations, it is natural to consider the group obtained by including the rotations "in between" these, induced by the actual physical movement of rotating the cube.[35] Interpolating, we obtain a 1-dimensional Lie group over $\mathbb{R}$, isomorphic as a manifold to $\bigsqcup_4 S^1$, whose four connected components represent each of the four rings. This kind of interpolation is only possible in special kinds of groups: for a compact Lie group H with r in the connected component of the identity, a unique left-invariant vector flow "points" from the identity to r.[36]

So for $G \leq H$ Lie groups, we define $Interp(G, r)$ to be the Lie group generated by applying this vector flow to all elements of G. However, there is also a unique right-invariant flow. For the interpolation given by these two to be equal, Rigorously: the vector flow pointing from e to r gives maps $\phi_t : G \to H, t \in \mathbb{R}$ such that 1) $\phi_\bullet$ is a group homomorphism, 2) $\phi_t(hg) = h\phi_t(g)$ for all $g \in G, h \in H$, and 3) $\phi_1(e) = r$. This implies that $\phi_t(g) = \phi_t(ge) = g\phi_t(e)$.

Then, $\phi_s(g)\phi_t(g') = g\phi_s(e)g'\phi_t(e)$. $\phi_t(\phi_s(g)g')$. If the flow is also right-invariant, we can conclude that this equals $\phi_t(\phi_s(gg')) = \phi_{t+s}(gg')$, proving that the set $\{\phi_t(g)\}$ is a subgroup and in fact the product of S^1 and G. What is needed to conclude that the flow is right invariant as well? Well, notice that $r = \phi_1(e)$ so where $[g] = \{\phi_t(g) : t \in \mathbb{R}\}$, $r[e] = [e]r$. Moreover, the condition of right-invariance implies that $r\phi_t(g) = \phi_1(e)\phi_t(g) = \phi_{1+t}(g) = \phi_{t+1}(ge) = \phi_t(g)\phi_1(e) = \phi_t(g)r$. But then r is actually in the center of G. In fact a weaker condition is sufficient: assume that for every $g \in G$, there is n such that $gr = r^n g$. Then $\phi_t(g) = \phi_t(ge) = g\phi_t(e) =$

$[e][g] = [g][e]$ for all $g \in G$. Therefore, since $r \in [e]$, we must have that $\langle r\rangle g = g\langle r\rangle$, i.e. that r generates a normal subgroup of H and in particular of G also.

[35] Never mind that it is impossible to *reflect* a physical cube! This is because GMA actually represents change in *description* rather than change in physical location.

[36] This does not exist for any H and r; it is sufficient for H to be compact, however: https://math.stackexchange.com/questions/163123/flows-of-left-invariant-vector-fields-on-a-lie-group