

# Introduction to Continuous Control Systems

EEME E3601



Week 10

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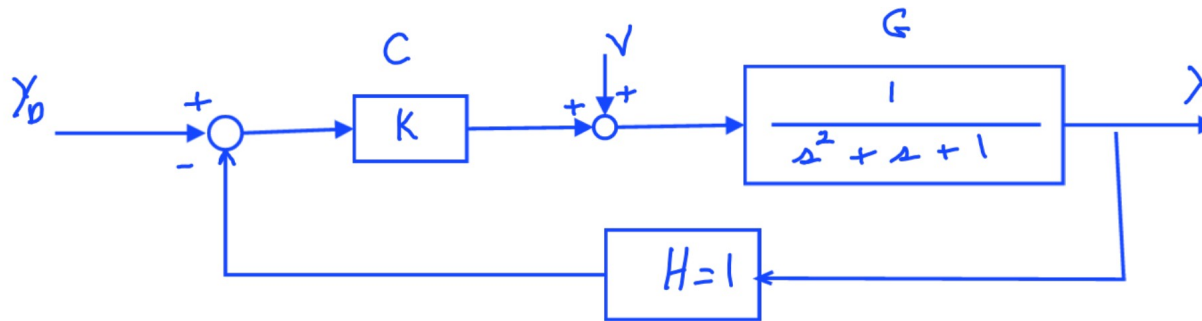
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$$Y = \underbrace{\left[ \frac{C G}{1 + C G H} \right]}_K Y_d + \underbrace{\left[ \frac{G}{1 + C G H} \right]}_1 V$$

$$\frac{K}{s^2 + s + (1+K)} \quad \frac{1}{1 + \frac{K}{s^2 + s + 1}} = \frac{1}{s^2 + s + (1+K)}$$

$$[s^2 + s + (1+K)]Y = K Y_d + V$$

$$\ddot{y} + \dot{y} + (1+K)y = K y_d + r$$

Solution  $y(t) = y_H(t) + y_{PD}(t) + y_{PV}(t)$

①  $y_H = c_1 y_1(t) + c_2 y_2(t)$  function of initial conditions

$$s^2 + s + (1+K) = 0 \quad s_{1,2} = \frac{-1 \pm \sqrt{1 - 4(1+K)}}{2}$$

$$y_1(t) = e^{s_1 t} \quad y_2(t) = e^{s_2 t}$$

want  $y_H \rightarrow 0$  as  $t \rightarrow \infty \forall y(0)$

②  $y_{PD}$ : particular solution for  $y_D$  want  $y_{PD} \rightarrow y_D$  as  $t \rightarrow \infty$

③  $y_{PV}$ : particular solution for  $\checkmark$  want  $y_{PV} \rightarrow 0$  as  $t \rightarrow \infty$

$$\lim_{t \rightarrow \infty} y_H(t) \rightarrow 0 \quad \forall y(0) \implies \text{Asymptotically Stable}$$



- asymptotically stable:  $y_H \rightarrow 0$  as  $t \rightarrow \infty$   $\forall y(0)$  if  $u(t)=0$
- unstable: at least one solution where  $y_{H_i} \rightarrow \infty$  as  $t \rightarrow \infty$   
At least one of the homogeneous solutions
- Lyapunov stable: no solution  $\rightarrow \infty$   
some solution may not go to zero
- Marginal stability: some solution does not decay  
(For all practical purposes not considered to be stable) but it does not blow up either  
the rest do the same or decay
- Bounded Input- Bounded Output (BIBO) stability:  
if  $\vec{x}(0) = \vec{0}$   
 $\|\vec{u}(t)\| \leq K_1 < \infty \quad \forall 0 \leq t < \infty \implies \|\vec{y}(t)\| \leq K_2 < \infty \quad \forall 0 \leq t < \infty$   
BIBO  $\implies$  all poles are strictly in the left half plane

## Asymptotic Stability

$y(t) \rightarrow 0$  as  $t \rightarrow \infty \forall y(0)$  if  $u(t) = 0$

**Theorem** (Asymptotic Stability)

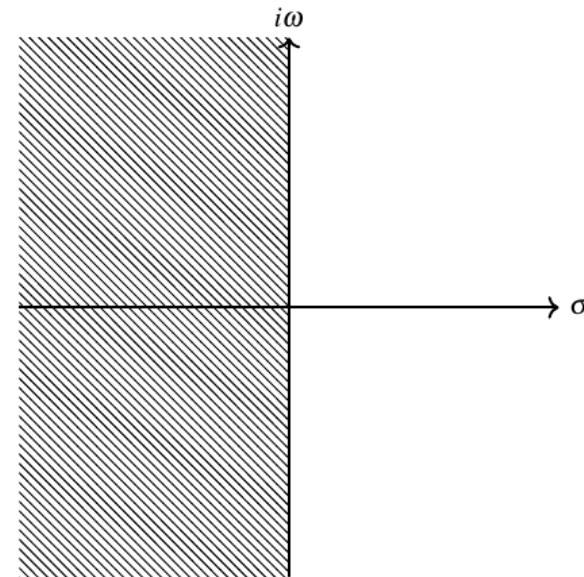
$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad (267)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

The system of Equation 267 is asymptotically stable if and only if all of its poles lie strictly in the left half-plane,

$$\operatorname{Re}\{\lambda_i\} < 0 \quad i \in \{1, 2, \dots, n\}$$

where  $\lambda_i$  are the eigenvalues of  $\mathbf{A}$ .



## Asymptotic Stability

$$y(t) \rightarrow 0 \text{ as } t \rightarrow \infty \quad \forall \quad y(0) \text{ if } u(t) = 0$$

### Theorem (Asymptotic Stability)

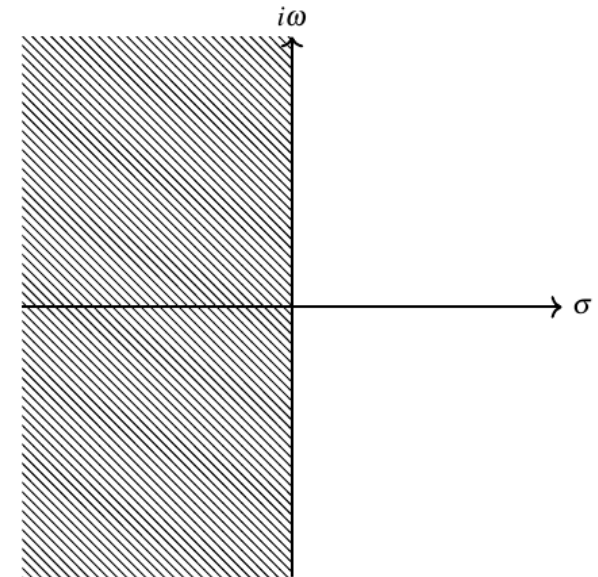
Equivalently, the system described by Equation 270,

$$\begin{aligned} \frac{d^{(n)}y(t)}{dt^{(n)}} + p_{n-1} \frac{d^{(n-1)}y(t)}{dt^{(n-1)}} + p_{n-2} \frac{d^{(n-2)}y(t)}{dt^{(n-2)}} + \cdots + p_0 y(t) \\ = q_{n-1} \frac{d^{(n-1)}u(t)}{dt^{(n-1)}} + q_{n-2} \frac{d^{(n-2)}u(t)}{dt^{(n-2)}} + \cdots + q_0 u(t) \end{aligned} \quad (270)$$

is asymptotically stable if and only if all of its poles, roots of the characteristic polynomial,

$$P(s) = s^n + p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \cdots + p_0 \quad (271)$$

are strictly in the left half plane.



(270)

(271)



## Bounded-Input Bounded-Output (BIBO) Stability

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

$$\|\mathbf{u}(t)\| \leq K_1 < \infty \quad \forall \quad 0 \leq t < \infty \implies \|\mathbf{y}(t)\| \leq K_2 < \infty \quad \forall \quad 0 \leq t < \infty$$

$$\begin{aligned} \frac{d^{(n)}y(t)}{dt^{(n)}} &+ p_{n-1} \frac{d^{(n-1)}y(t)}{dt^{(n-1)}} + p_{n-2} \frac{d^{(n-2)}y(t)}{dt^{(n-2)}} + \cdots + p_0 y(t) \\ &= q_{n-1} \frac{d^{(n-1)}u(t)}{dt^{(n-1)}} + q_{n-2} \frac{d^{(n-2)}u(t)}{dt^{(n-2)}} + \cdots + q_0 u(t) \end{aligned}$$

If all the initial conditions are zero,  $y(0) = \dot{y}(0) = \ddot{y}(0) = \cdots = y^{(n-1)}(0) = 0$   
and,

$$|u(t)| \leq K_1 < \infty \quad \forall \quad 0 \leq t < \infty \implies |y(t)| \leq K_2 < \infty \quad \forall \quad 0 \leq t < \infty$$



## Asymptotic Stability (Testing)

**Lemma** (Positive Coefficients for Stability). *Necessary – But not Sufficient*

$$P(s) = s^n + p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \cdots + p_0 = 0 \quad (275)$$

*If  $P(s)$  is stable, then all of its coefficients,  $p_i$ , including the coefficient for  $s^n$  (usually 1), must have the same sign,*

$$p_i > 0 \quad \forall \quad i \in \{0, 1, \dots, n-1, n\}$$

*assuming that  $P(s)$  may be multiplied by  $-1$  to make  $p_{n-1} > 0$ , if needed.*





## Asymptotic Stability (Testing)

**Lemma** (Positive Coefficients for Stability). *Necessary – But not Sufficient*

$$P(s) = s^n + p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \cdots + p_0 = 0 \quad (275)$$

If  $P(s)$  is stable, then all of its coefficients,  $p_i$ , including the coefficient for  $s^n$  (usually 1), must have the same sign,

$$p_i > 0 \quad \forall i \in \{0, 1, \dots, n-1, n\}$$

assuming that  $P(s)$  may be multiplied by  $-1$  to make  $p_{n-1} > 0$ , if needed.

*Proof.*

Equation 275 may be rewritten in terms of a set of complex conjugate and real roots,

$$P(s) = \left[ (s - \sigma_1)^2 + \omega_1^2 \right] \cdots \left[ (s - \sigma_{m-1})^2 + \omega_{m-1}^2 \right] (s - \lambda_{m+1}) \cdots (s - \lambda_n) \quad (277)$$

with  $\sigma_i \pm i\omega_i$ ,  $i \in \{1, 3, \dots, m-1\}$ , and  $\lambda_j$ ,  $j \in \{m+1, m+2, \dots, n\}$  as its roots.

$$\sigma_i < 0 \quad i \in \{1, 3, \dots, m-1\} \quad \lambda_j < 0 \quad j \in \{m+1, m+2, \dots, n\}$$

all  $\{-\sigma_i, \omega_i^2, -\lambda_j\}$  must be positive

multiplying the terms in 277 would cause all the coefficients of 275 to be positive.  $\square$

## Asymptotic Stability (Testing)

**Theorem** (Hurwitz Criterion) *Necessary and Sufficient*

*$P(s)$  is stable if and only if the Hurwitz determinants are all positive.*

$$D_1 = |p_{n-1}| = p_{n-1}$$

$$D_2 = \begin{vmatrix} p_{n-1} & p_{n-3} \\ 1 & p_{n-2} \end{vmatrix}$$

$$D_3 = \begin{vmatrix} p_{n-1} & p_{n-3} & p_{n-5} \\ 1 & p_{n-2} & p_{n-4} \\ 0 & p_{n-1} & p_{n-3} \end{vmatrix}$$

...

$$D_n = \begin{vmatrix} p_{n-1} & p_{n-3} & p_{n-5} & \cdots & 0 & 0 \\ 1 & p_{n-2} & p_{n-4} & \cdots & 0 & 0 \\ 0 & p_{n-1} & p_{n-3} & \cdots & 0 & 0 \\ 0 & 1 & p_{n-2} & \cdots & 0 & 0 \\ 0 & 0 & p_{n-1} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & p_0 & 0 \\ 0 & 0 & 0 & \cdots & p_1 & 0 \\ 0 & 0 & 0 & \cdots & p_2 & p_0 \end{vmatrix}$$

$$D_i > 0 \quad i \in \{1, 2, \dots, n\}$$



## Asymptotic Stability (Testing)

**Theorem** (Liénard-Chipart). *Necessary and Sufficient*

*Assuming that the coefficients of  $P(s)$  are all positive,*

$$p_i > 0 \quad i \in \{0, 1, 2, \dots, n-1\}$$

*then,  $P(s)$  is stable if and only if,*

$$D_i > 0 \quad i \in \{2, 4, \dots, n-1\} \quad (\text{when } n \text{ is odd})$$

*or*

$$D_i > 0 \quad i \in \{3, 5, \dots, n-1\} \quad (\text{when } n \text{ is even})$$

## Asymptotic Stability (Testing)

### Example

Consider the following system,

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & -4 & -3 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & -6 & 4 & 0 \end{bmatrix} \mathbf{x}(t)$$

with the characteristic polynomial,  $P(s) = s^4 + 2s^3 + 3s^2 + 4s + 5$

Since  $n = 4$  is even, we require,  $D_i > 0 \quad i \in \{3, 5, \dots, n-1\}$

Therefore, we need to evaluate only one determinant,

$$D_3 = \begin{vmatrix} 2 & 4 & 0 \\ 1 & 3 & 5 \\ 0 & 2 & 4 \end{vmatrix} = -12 \quad \text{See plot\_output\_lti\_statespace.m}$$

Since  $D_3$  is negative, according to the combination of Hurwitz Theorem and Liénard-Chipart Theorem, this system is unstable.

## Asymptotic Stability (Testing)

### Example

Consider the following system,

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & -4 & -3 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & -6 & 4 & 0 \end{bmatrix} \mathbf{x}(t)$$

Note that using the Hurwitz theorem only, we would have also had to compute,

$$D_1 = |2| = 2 \quad D_2 = \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 2$$

$$D_4 = \begin{vmatrix} 2 & 4 & 0 & 0 \\ 1 & 3 & 5 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 3 & 5 \end{vmatrix} = -60$$

## Asymptotic Stability (Testing)

### Example

*Consider the following system,*

$$\ddot{y}(t) + 7\dot{y}(t) + 14y(t) = \dot{u}(t) + 3u(t)$$

*with the following characteristic polynomial,*

$$P(s) = s^3 + 7s^2 + 14s + 8 \quad \text{See plot_output_lti.m}$$

*All the coefficients are positive, so we need to check,*

$$D_2 = \begin{bmatrix} 7 & 8 \\ 1 & 14 \end{bmatrix} = 90$$

*Since  $D_2$  is positive, the system is stable.*