

Introduction to Continuous Control Systems

EEME E3601



Week 11

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Asymptotic Stability (Testing)

Corollary

Second, third, and fourth order polynomials are stable if and only if,

A. *For a second order polynomial*

$$P(s) = s^2 + p_1s + p_0$$

p_1 and p_0 are both positive.

B. *For a third order polynomial,*

$$P(s) = s^3 + p_2s^2 + p_1s + p_0$$

$p_2, p_1, p_0 > 0$ and $p_2p_1 > p_0$.

C. *For a fourth order polynomial,*

$$P(s) = s^4 + p_3s^3 + p_2s^2 + p_1s + p_0$$

$p_3, p_2, p_1, p_0 > 0$ and $p_3p_2p_1 > p_3^2p_0 + p_1^2$.

Proof. The proof is easily done by applying the Liénard-Chipart Theorem directly on these specific polynomials. \square

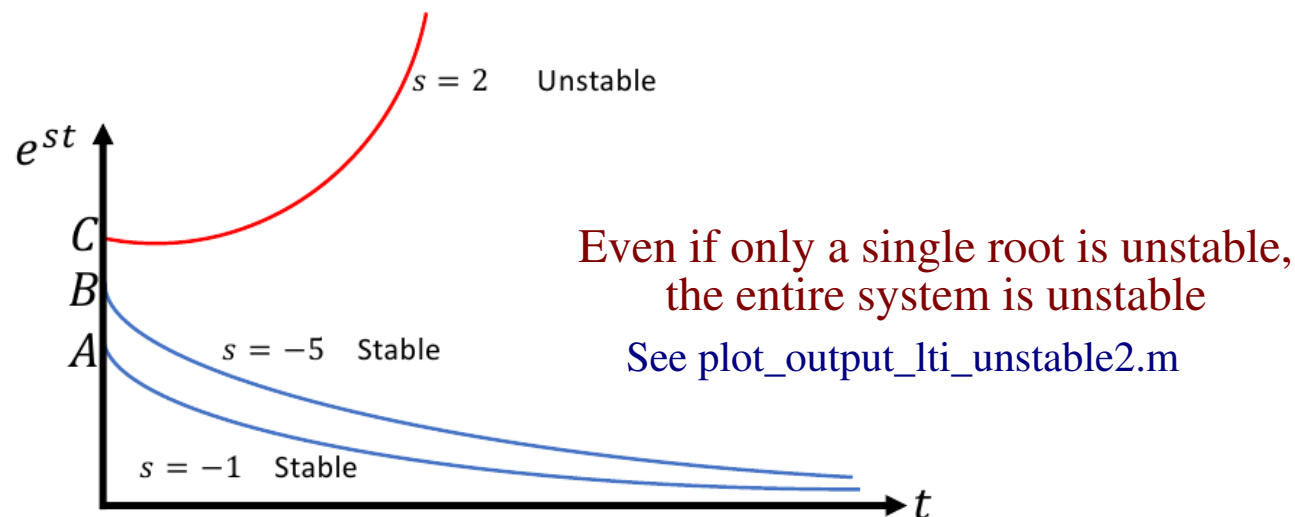


Asymptotic Stability (Testing)

$$H(s) = \frac{1}{s+1} \frac{1}{s+5} \frac{1}{s-2}$$

Partial Fraction Expansion:

$$H(s) = \frac{1}{s+1} \frac{1}{s+5} \frac{1}{s-2} = \underbrace{\frac{A}{s+1}}_{s=-1 \text{ Stable}} + \underbrace{\frac{B}{s+5}}_{s=-5 \text{ Stable}} + \underbrace{\frac{C}{s-2}}_{s=2 \text{ Unstable}}$$





Asymptotic Stability (Testing)

$$H(s) = \frac{1}{s^4 + 9s^3 - 2s^2 + s + 3}$$

Different sign: unstable system

See `plot_output_lti_unstable2.m`

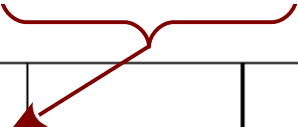
Asymptotic Stability (Routh Criterion)

Theorem (Routh Criterion)

The number of roots of $P(s)$,

$$P(s) = s^n + p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \cdots + p_0 = 0$$

with strictly positive real parts is equal to the number of sign changes in the first column of the Routh array,



Row				
1	1	p_{n-2}	p_{n-4}	\cdots
2	p_{n-1}	p_{n-3}	p_{n-5}	\cdots
3	$b_{3,n-1}$	$b_{3,n-3}$	$b_{3,n-5}$	\cdots
4	$b_{4,n-1}$	$b_{4,n-3}$	$b_{4,n-5}$	\cdots
5	$b_{5,n-1}$	$b_{5,n-3}$	$b_{5,n-5}$	\cdots
\vdots	\vdots	\vdots	\vdots	\ddots
$n+1$	\cdots	\cdots	\cdots	\cdots

Routh Array Table



Asymptotic Stability (Routh Criterion)

Row				
1	1	p_{n-2}	p_{n-4}	\cdots
2	p_{n-1}	p_{n-3}	p_{n-5}	\cdots
3	$b_{3,n-1}$	$b_{3,n-3}$	$b_{3,n-5}$	\cdots
4	$b_{4,n-1}$	$b_{4,n-3}$	$b_{4,n-5}$	\cdots
5	$b_{5,n-1}$	$b_{5,n-3}$	$b_{5,n-5}$	\cdots
\vdots	\vdots	\vdots	\vdots	\ddots
$n+1$	\cdots	\cdots	\cdots	\cdots

Routh Array Table

where $p_{n-1} \triangleq 0 \quad \forall \quad i > n$.

Also, the coefficients of the j^{th} row for $j \in \{3, 4, \dots, n+1\}$ are given in terms of the coefficients in the following relations,

Stable only if all these coefficients are strictly positive

$$\begin{array}{l}
 b_{3,n-1} = -\frac{1}{p_{n-1}} \begin{vmatrix} 1 & p_{n-2} \\ p_{n-1} & p_{n-3} \end{vmatrix} \\
 b_{3,n-3} = -\frac{1}{p_{n-1}} \begin{vmatrix} 1 & p_{n-4} \\ p_{n-1} & p_{n-5} \end{vmatrix} \\
 b_{3,n-5} = -\frac{1}{p_{n-1}} \begin{vmatrix} 1 & p_{n-6} \\ p_{n-1} & p_{n-7} \end{vmatrix} \\
 b_{4,n-1} = -\frac{1}{b_{3,n-1}} \begin{vmatrix} p_{n-1} & p_{n-3} \\ b_{3,n-1} & b_{3,n-3} \end{vmatrix} \\
 b_{4,n-3} = -\frac{1}{b_{3,n-1}} \begin{vmatrix} p_{n-1} & p_{n-5} \\ b_{3,n-1} & b_{3,n-5} \end{vmatrix} \\
 b_{4,n-5} = -\frac{1}{b_{3,n-1}} \begin{vmatrix} p_{n-1} & p_{n-7} \\ b_{3,n-1} & b_{3,n-7} \end{vmatrix} \\
 b_{5,n-1} = -\frac{1}{b_{4,n-1}} \begin{vmatrix} b_{3,n-1} & b_{3,n-3} \\ b_{4,n-1} & b_{4,n-3} \end{vmatrix} \\
 b_{5,n-3} = -\frac{1}{b_{4,n-1}} \begin{vmatrix} b_{3,n-1} & b_{3,n-5} \\ b_{4,n-1} & b_{4,n-5} \end{vmatrix} \\
 b_{5,n-5} = -\frac{1}{b_{4,n-1}} \begin{vmatrix} b_{3,n-1} & b_{3,n-7} \\ b_{4,n-1} & b_{4,n-7} \end{vmatrix} \\
 \vdots
 \end{array}$$



Asymptotic Stability (Routh Criterion)

Easy way to remember

sign change = 1 root in the RHP
sign change = 1 root in the RHP
Total of 2 roots in the RHP

1	s^4
2	s^3
-3	s^2
7	s^1
2	s^0

a_5
 a_4

a_3
 a_2

A
 B
 C
 D

a_1
0
0
0

Negative of determinant

 $A = \frac{a_4 a_3 - a_5 a_2}{a_4}$

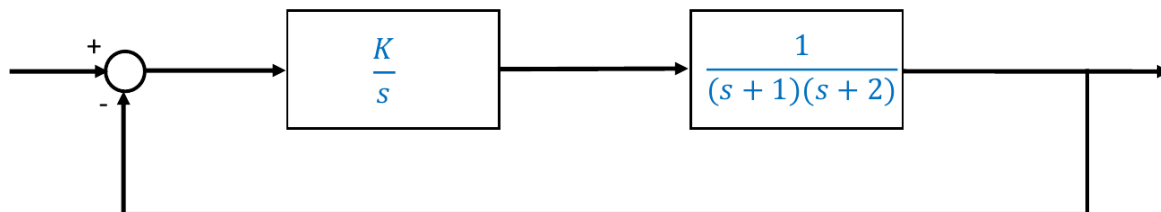
scale

 $B = \frac{a_4 a_1 - a_5 0}{a_4}$ $C = \frac{A a_2 - a_4 B}{A}$ $D = \frac{C B - A 0}{C}$



Asymptotic Stability (Routh Criterion)

Simple Example for Controller Design



$$\frac{Y(s)}{U(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)}$$

$$= \frac{\frac{K}{s(s+1)(s+2)}}{1 + \frac{K}{s(s+1)(s+2)}}$$

$$= \frac{\frac{K}{\cancel{s(s+1)(s+2)}}}{\frac{s(s+1)(s+2) + K}{\cancel{s(s+1)(s+2)}}}$$

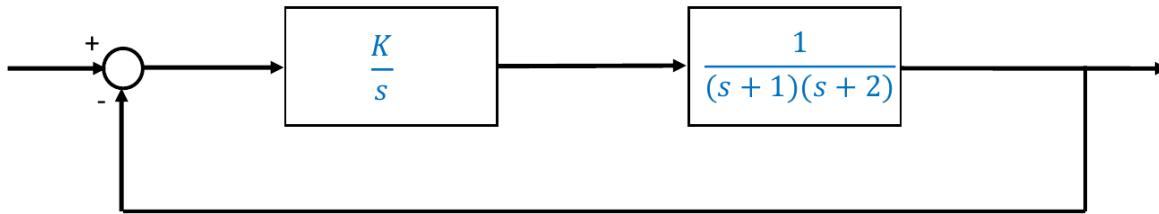
$$= \frac{K}{s(s+1)(s+2) + K}$$

Characteristic Equation

$$s(s+1)(s+2) + K = 0 \longrightarrow s^3 + 3s^2 + 2s + K = 0$$

Asymptotic Stability (Routh Criterion)

Simple Example for Controller Design

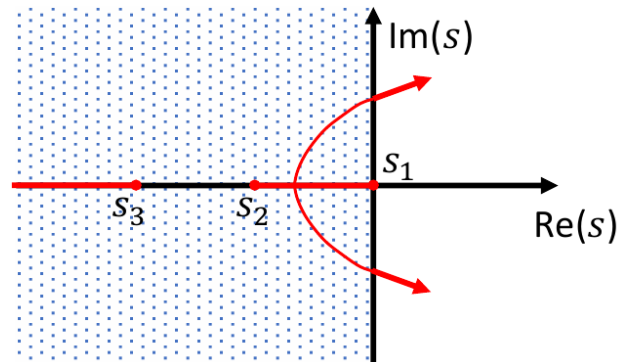


$$s(s+1)(s+2) + K = 0 \longrightarrow s^3 + 3s^2 + 2s + K = 0$$

s^3	1	2	0
s^2	3	K	0
s^1	$\frac{6-K}{3}$	0	0
s^0	K		

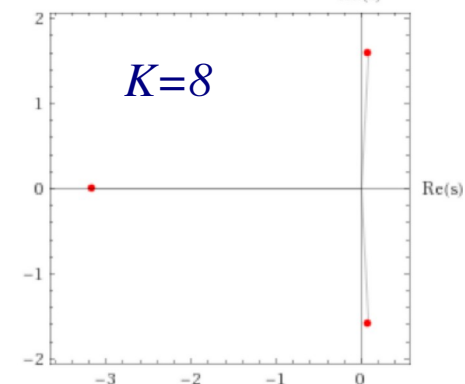
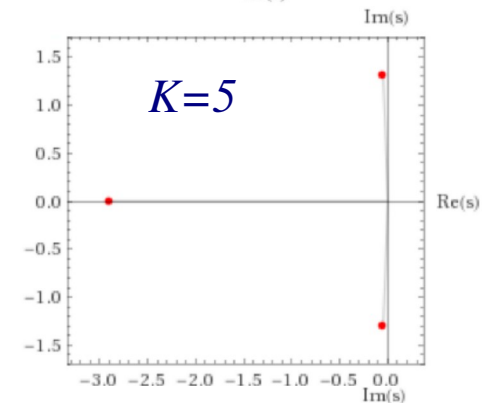
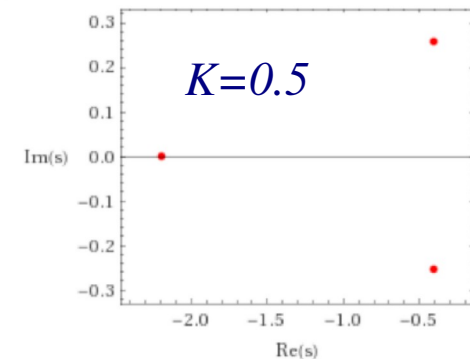
ranging $0 < K < 6$

Range for stability



Root Locus as $0 < K < 6$

Roots in the complex plane:





Asymptotic Stability (Routh Criterion)

Row				
1	1	p_{n-2}	p_{n-4}	\cdots
2	p_{n-1}	p_{n-3}	p_{n-5}	\cdots
3	$b_{3,n-1}$	$b_{3,n-3}$	$b_{3,n-5}$	\cdots
4	$b_{4,n-1}$	$b_{4,n-3}$	$b_{4,n-5}$	\cdots
5	$b_{5,n-1}$	$b_{5,n-3}$	$b_{5,n-5}$	\cdots
\vdots	\vdots	\vdots	\vdots	\ddots
$n+1$	\cdots	\cdots	\cdots	\cdots

two special cases where some intervention is necessary to come to a conclusion

Routh Array Table

1. **Case:** *There is a 0 in the first column at some row, but there are some nonzero values in the later columns of the same row.*
2. **Case:** *There is a row of all zeros.*

Asymptotic Stability (Routh Criterion)

1. **Case:** *There is a 0 in the first column at some row, but there are some nonzero values in the later columns of the same row.*

Procedure:

- *Assumption: there are no pure imaginary roots that are moved to the right-half plane.*
- *Replace the 0 with a very small number, ϵ*
- *Continue with the computation of the rest of the coefficients.*
- *Take the limit as $\epsilon \rightarrow 0$.*
- *Count the number of sign changes like the nominal case and the number of sign changes gives the number of roots with real parts in the right half plane.*
- *In case there is a breakdown in the procedure, factor out all the factors of the form $(s^2 + \omega^2)$ and start over.*

Asymptotic Stability (Routh Criterion)

2. **Case:** *There is a row of all zeros.*

Procedure:

- *If $P(s)$ has no roots at $s = 0$, then $n - j$ must be even.*
- *Use the coefficients in row $j - 1$ to build the auxiliary polynomial of order $n - j + 2$. (ie, This will be a factor of the $P(s)$).*
- *Differentiate the created auxiliary polynomial and enter the coefficients in place of the zeros and continue.*
- *May need to repeat the procedure more than once.*



Asymptotic Stability (Routh Criterion) Example

Explore the stability of,

$$P(s) = s^4 + 2s^3 + 3s^2 + 4s + 5$$

Compute the Routh array,

Row	Degree					
1	s^4	1	3	5	0	...
2	s^3	2	4	0	0	...
3	s^2	1	5	0	0	...
4	s^1	-6	0	0	0	...
5	s^0	5	0	0	0	...

Two sign changes

Two roots in the right half plane



Asymptotic Stability (Routh Criterion) Example

Explore the stability of,

$$P(s) = s^3 + 7s^2 + 14s + 8$$

Compute the Routh array,

Row	Degree				
1	s^3	1	14	0	...
2	s^2	7	8	0	...
3	s^1	$\frac{90}{7}$	0	0	...
4	s^0	8	0	0	...

No sign changes

Stable

Asymptotic Stability (Routh Criterion)

Example Case 1

Explore the stability of,

$$P(s) = s^4 + s^3 + 2s^2 + 2s + 1$$

Compute the Routh array,

Row	Degree					
1	s^4	1	2	1	0	...
2	s^3	1	2	0	0	...
3	s^2	0	1	0	0	...
3	s^2	ϵ	1	0	0	...
4	s^1	$2 - \frac{1}{\epsilon}$	0	0	0	...
5	s^0	1	0	0	1	...

Two sign changes for small ϵ , $|\epsilon| < \frac{1}{2}$

Two roots in the right-half plane



Asymptotic Stability (Routh Criterion)
Example Case 2

Explore the stability of, $P(s) = s^5 + 2s^4 + 3s^3 + 6s^2 - 4s - 8$

Since there is a sign change in the coefficients, we know that the polynomial is unstable. However, we would like to know the number of roots in the right-half plane.

Row	Degree					
1	s^5	1	3	-4	0	...
2	s^4	2	6	-8	0	...
3	s^3	0	0	0	0	...
3	s^3	8	12	0	0	...
4	s^2	3	-8	0	0	...
5	s^1	33.3	0	0	0	...
6	s^0	-8	0	0	0	...

$\leftarrow P_2 = 2s^4 + 6s^2 - 8$

$\leftarrow P_3 = \frac{dP_2}{ds} = 8s^3 + 12s$

One sign change

One roots in the right-half plane



Asymptotic Stability (Routh Criterion) Stability Margin

Have all the poles to the left of -1

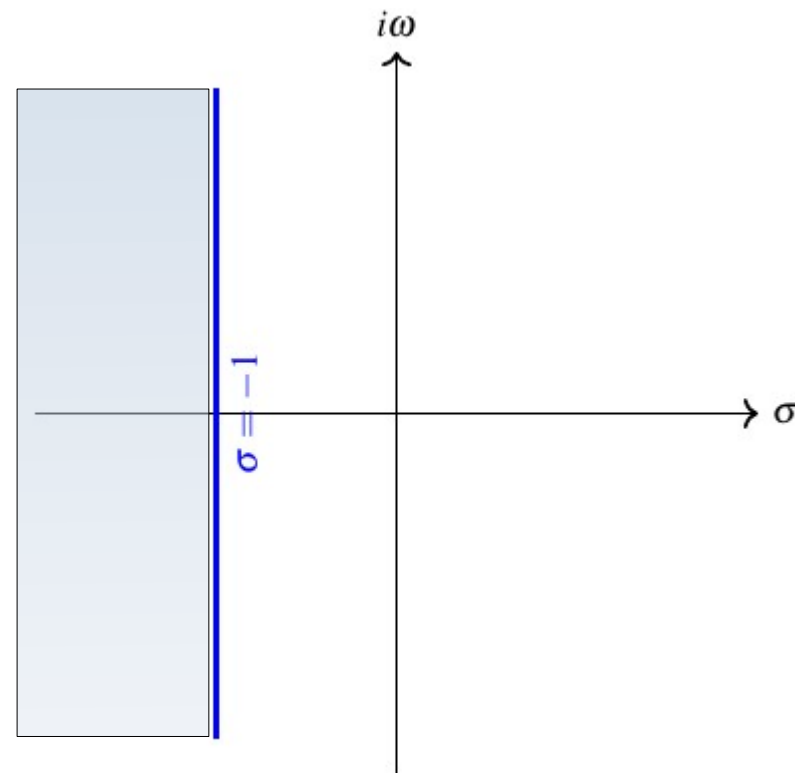
$$\hat{P}(s) = P(s - \sigma_1)$$

If the roots of $P(s)$ are $s_i, i \in \{1, 2, \dots, n\}$,
then, the roots of $\hat{P}(s)$ would be

$$\hat{s}_i = s_i + \sigma_1, i \in \{1, 2, \dots, n\}$$

$$\mathcal{Re}\{\hat{s}_i\} < 0 \implies \mathcal{Re}\{s_i\} < -\sigma_1$$

Use the Routh-Hurwitz test on the new polynomial





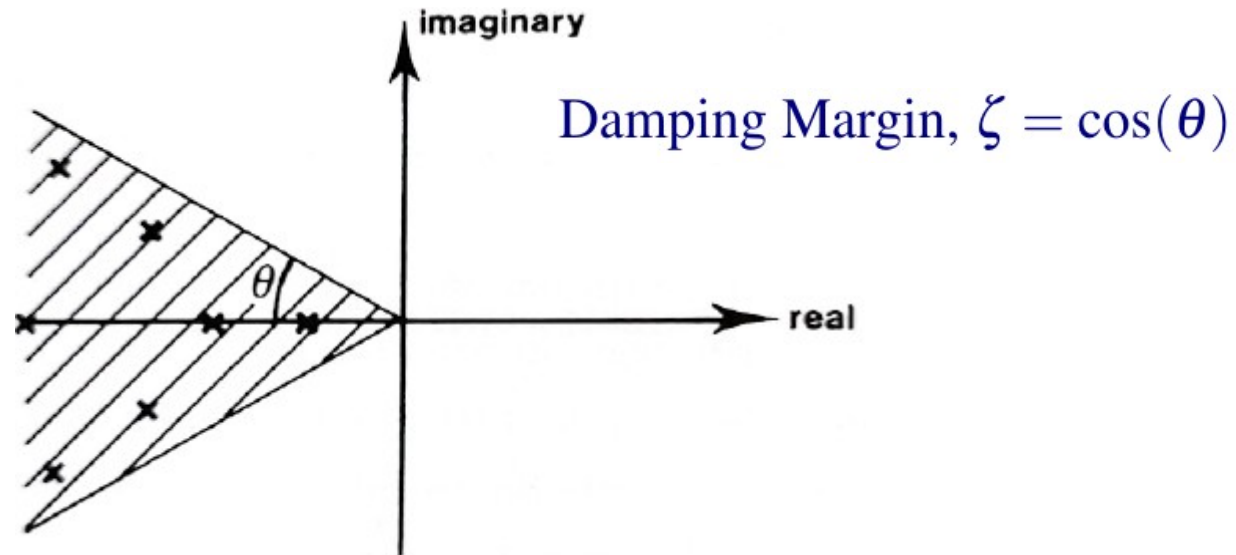
Asymptotic Stability (Routh Criterion) Stability Margin

Do all the poles lie to the left of $\sigma = -1$?

$$P(s) = s^3 + 7s^2 + 14s + 8$$

$$\begin{aligned}\hat{P}(s) &= P(s-1) \\ &= (s-1)^3 + 7(s-1)^2 + 14(s-1) + 8 \\ &= s^3 + 4s^2 + 3s + 0\end{aligned}$$

Asymptotic Stability (Routh Criterion) Damping Margin



- Testing for a specific damping margin may be done by a coordinate rotation and the generation of a new polynomial in the new coordinate system and testing for stability.
- Another method would be through the use of the Nyquist stability criterion.



Asymptotic Stability (Lyapunov Stability)

Definition (Positive Definite Matrix).

$\mathbf{Q} : \mathcal{R}^n \mapsto \mathcal{R}^n$ is a positive definite matrix if,

$$f(\mathbf{x}) = \mathbf{x}^T \mathbf{Q} \mathbf{x} > 0 \quad \forall \quad \mathbf{x} \neq \mathbf{0} \quad (322)$$

or, alternatively, if \mathbf{Q} has all strictly positive Eigenvalues.

N.B., Equation 322 is the Euclidean norm of $\mathbf{Q}^{\frac{1}{2}} \mathbf{x}$



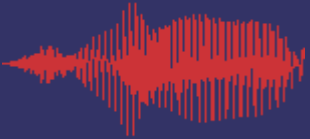
Asymptotic Stability (Lyapunov Stability)

Aleksandr Mikhailovich Lyapunov (1857-1918)

Uses a scalar energy of the system

Also works for time-variant and nonlinear systems

We only consider the linear time-invariant case here



Asymptotic Stability (Lyapunov Stability)

Consider the following homogeneous linear time-invariant system,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

Note that a measure of the energy of the above system may be written as

$$\begin{aligned}\mathcal{E}(\mathbf{x}) &\triangleq \|\mathbf{x}(t)\|^2 \\ &= \mathbf{x}^T \mathbf{x}\end{aligned}$$

If $\mathcal{E}(\mathbf{x}(t)) \rightarrow 0$ as $t \rightarrow \infty$, then the system is stable.



Asymptotic Stability (Lyapunov Stability)

In fact, this would work for any quadratic function,

$$v(\mathbf{x}) \triangleq \mathbf{x}^T \mathbf{Q} \mathbf{x}$$

where \mathbf{Q} is a symmetric positive definite matrix, in which case,

$$\lambda_{\min} \|\mathbf{x}\|^2 \leq \mathbf{x}^T \mathbf{Q} \mathbf{x} \leq \lambda_{\max} \|\mathbf{x}\|^2$$

Let's impose a slightly stronger condition than requiring $v(\mathbf{x}) \rightarrow 0$

Asymptotic Stability (Lyapunov Stability)

Definition (Quadratic Lyapunov Function).

The following function,

$$v(\mathbf{x}) \triangleq \mathbf{x}^T \mathbf{Q} \mathbf{x}$$

is a quadratic function for the following system,

$$\dot{\mathbf{x}}(t) = \mathbf{A} \mathbf{x}(t)$$

if $\mathbf{Q} > \mathbf{0}$ and there exists a constant $k > 0$ such that,

$$\dot{v}(\mathbf{x}) \leq -k \|\mathbf{x}\|^2 \quad \forall t$$

In other words the generalized energy measure, $v(\mathbf{x})$, decreases monotonically as time increases.



Asymptotic Stability (Lyapunov Stability)

Proof.

Since,

$$\lambda_{\min} \|\mathbf{x}\|^2 \leq \mathbf{x}^T \mathbf{Q} \mathbf{x} \leq \lambda_{\max} \|\mathbf{x}\|^2$$

$\dot{v}(\mathbf{x})$ may be bounded with respect to the largest Eigenvalue of A as follows,

$$\dot{v}(\mathbf{x}) \leq -k \|\mathbf{x}\|^2 \leq -\frac{k}{\lambda_{\max}} v(\mathbf{x})$$

Asymptotic Stability (Lyapunov Stability)

Proof.

Since,

$$\lambda_{\min} \|\mathbf{x}\|^2 \leq \mathbf{x}^T \mathbf{Q} \mathbf{x} \leq \lambda_{\max} \|\mathbf{x}\|^2$$

$\dot{v}(\mathbf{x})$ may be bounded with respect to the largest Eigenvalue of A as follows,

$$\dot{v}(\mathbf{x}) \leq -k \|\mathbf{x}\|^2 \leq -\frac{k}{\lambda_{\max}} v(\mathbf{x})$$

If we write the above differential equation for the equality case,

$$\dot{v}(\mathbf{x}) = -\frac{k}{\lambda_{\max}} v(\mathbf{x})$$

then,

$$v(\mathbf{x}) = e^{-\left(\frac{k}{\lambda_{\max}}\right)t} v(\mathbf{x}_0)$$

Asymptotic Stability (Lyapunov Stability)

Proof. (continued)

$$v(\mathbf{x}) = e^{-\left(\frac{k}{\hat{\mathcal{A}}_{\max}}\right)t} v(\mathbf{x}_0)$$

Since $v(\mathbf{x}) \geq 0$, then v may decay even faster than the above solution,

$$v(\mathbf{x}) \leq e^{-\left(\frac{k}{\hat{\mathcal{A}}_{\max}}\right)t} v(\mathbf{x}_0)$$

Using the following inequality again,

$$\dot{v}(\mathbf{x}) \leq -k\|\mathbf{x}\|^2 \leq -\frac{k}{\hat{\mathcal{A}}_{\max}} v(\mathbf{x})$$

then,

$$\begin{aligned} \|\mathbf{x}\|^2 &\leq \frac{v(\mathbf{x}(t))}{\hat{\mathcal{A}}_{\min}} \\ &\leq \frac{e^{-\left(\frac{k}{\hat{\mathcal{A}}_{\max}}\right)t} v(\mathbf{x}_0)}{\hat{\mathcal{A}}_{\min}} \xrightarrow[t \rightarrow \infty]{} 0 \end{aligned}$$

Therefore the system is asymptotically stable.



Asymptotic Stability (Lyapunov Stability)

Since

$$v(\mathbf{x}) = \mathbf{x}^T \mathbf{Q} \mathbf{x}$$

using the homogeneous differential equation,

$$\dot{\mathbf{x}} = \mathbf{A} \mathbf{x}$$

then

$$\begin{aligned} \dot{v}(\mathbf{x}) &= \frac{d(\mathbf{x}^T \mathbf{Q} \mathbf{x})}{dt} \\ &= \dot{\mathbf{x}}^T \mathbf{Q} \mathbf{x} + \mathbf{x}^T \mathbf{Q} \dot{\mathbf{x}} \\ &= \mathbf{x}^T \mathbf{A}^T \mathbf{Q} \mathbf{x} + \mathbf{x}^T \mathbf{Q} \mathbf{A} \mathbf{x} \\ &= \mathbf{x}^T (\mathbf{A}^T \mathbf{Q} + \mathbf{Q} \mathbf{A}) \mathbf{x} \end{aligned}$$

Asymptotic Stability (Lyapunov Stability)

Let us define \mathbf{M} such that,

$$\mathbf{M} \triangleq -(\mathbf{A}^T \mathbf{Q} + \mathbf{Q} \mathbf{A})$$

then,

$$\begin{aligned} \dot{v}(\mathbf{x}) &= \mathbf{x}^T (\mathbf{A}^T \mathbf{Q} + \mathbf{Q} \mathbf{A}) \mathbf{x} \\ &= -\mathbf{x}^T \mathbf{M} \mathbf{x} \\ &\leq -k \mathbf{x}^T \mathbf{x} \quad \forall t \end{aligned}$$

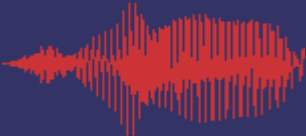
The above must hold for $t = 0$ and any initial state \mathbf{x}_0 , which means that $\mathbf{M} > \mathbf{0}$.

$$\mathbf{M} \geq k\mathbf{I} > \mathbf{0}$$

Asymptotic Stability (Lyapunov Stability)

To determine whether v is a Lyapunov function,

- Choose any $\mathbf{Q} > \mathbf{0}$
- Compute \mathbf{M} from its definition, above
- If $\mathbf{M} > \mathbf{0}$, then $v(\mathbf{x}) = \mathbf{x}^T \mathbf{Q} \mathbf{x}$ is a Lyapunov function for the homogeneous system of \mathbf{A} .
- If $\mathbf{M} \not> \mathbf{0}$, $v(\mathbf{x})$ is not a Lyapunov function, but there is no conclusion about the stability.



Asymptotic Stability (Lyapunov Stability) Better Alternative

To determine whether v is a Lyapunov function,

- Choose a positive definite matrix, $\mathbf{M} > \mathbf{0}$
- Solve for \mathbf{Q} , using this \mathbf{M} in $\mathbf{A}^T \mathbf{Q} + \mathbf{Q} \mathbf{A} = -\mathbf{M}$
- Test for positive definiteness of \mathbf{Q}
- This procedure is definitive, so that if $\mathbf{Q} \not> \mathbf{0}$, then the system cannot be stable.

Asymptotic Stability (Lyapunov Stability) Better Alternative

Theorem (Lyapunov Function Determination).

Choose $\mathbf{M} > \mathbf{0}$, then \mathbf{A} is stable if and only if

$$\mathbf{A}^T \mathbf{Q} + \mathbf{Q} \mathbf{A} = -\mathbf{M}$$

has a unique solution, $\mathbf{Q} > \mathbf{0}$.

Proof.

For any \mathbf{M} ,

$$\mathbf{Q} = \int_0^\infty e^{\mathbf{A}^T t} \mathbf{M} e^{\mathbf{A} t} dt \quad (352)$$

is a solution to the above linear equation. The integral of 352 must converge for \mathbf{A} to be stable. Plugging in for \mathbf{Q} ,

$$\begin{aligned} -\mathbf{M} &= \mathbf{A}^T \int_0^\infty e^{\mathbf{A}^T t} \mathbf{M} e^{\mathbf{A} t} dt + \int_0^\infty e^{\mathbf{A}^T t} \mathbf{M} e^{\mathbf{A} t} dt \mathbf{A} \\ &= \int_0^\infty \left(\mathbf{A}^T e^{\mathbf{A}^T t} \mathbf{M} e^{\mathbf{A} t} + e^{\mathbf{A}^T t} \mathbf{M} e^{\mathbf{A} t} \mathbf{A} \right) dt \\ &= \left[e^{\mathbf{A}^T t} \mathbf{M} e^{\mathbf{A} t} \right]_0^\infty \end{aligned}$$

Asymptotic Stability (Lyapunov Stability) Better Alternative

Theorem (Lyapunov Function Determination).

Proof. (Continued)

The linear map,

$$\mathbf{L}(\mathbf{Q}) \triangleq \mathbf{A}^T \mathbf{Q} + \mathbf{Q} \mathbf{A}$$

is such that $\mathbf{L}(\mathbf{Q}) : \mathcal{R}^{n^2} \mapsto \mathcal{R}^{n^2}$, where, $\mathbf{L}(\mathbf{Q}) = -\mathbf{M}$

has a solution for all \mathbf{M} , therefore, $\mathbf{L}(\mathbf{Q})$ has full rank and hence it is invertible. To verify that $\mathbf{Q} > \mathbf{0}$,

$$\begin{aligned} \mathbf{x}_0^T \mathbf{Q} \mathbf{x}_0 &= \int_0^\infty \mathbf{x}_0^T e^{\mathbf{A}^T t} \mathbf{M} e^{\mathbf{A} t} \mathbf{x}_0 dt \\ &= \int_0^\infty \mathbf{x}^T(t) \mathbf{M} \mathbf{x}(t) dt \end{aligned}$$

where $\mathbf{x}(t)$ can only be $\mathbf{0}$ if $\mathbf{x}_0 = \mathbf{0}$ since $e^{\mathbf{A} t}$ is nonsingular.

$$\mathbf{A} > \mathbf{0} \implies \mathbf{x}_0^T \mathbf{Q} \mathbf{x}_0 > 0 \text{ unless } \mathbf{x}_0 = \mathbf{0}$$



Asymptotic Stability (Lyapunov Stability)

Example

Find a Lyapunov function for

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix} \mathbf{x}(t)$$

Choose $\mathbf{M} = \mathbf{I}$ and solve the following linear equation,

$$\begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{aligned} -2Q_{11} - 2Q_{12} &= -1 \\ -3Q_{11} - Q_{12} + Q_{22} &= 0 \\ -6Q_{12} - 4Q_{22} &= -1 \end{aligned}$$

which leads to the following slution for \mathbf{Q} ,

$$\mathbf{Q} = \begin{bmatrix} 3 & -3.5 \\ -3.5 & 5.5 \end{bmatrix} \quad \text{Therefore the system is stable since } \mathbf{Q} > \mathbf{0}.$$



Root Locus Plots

$$F(s) = P(s) + KQ(s) = 0$$

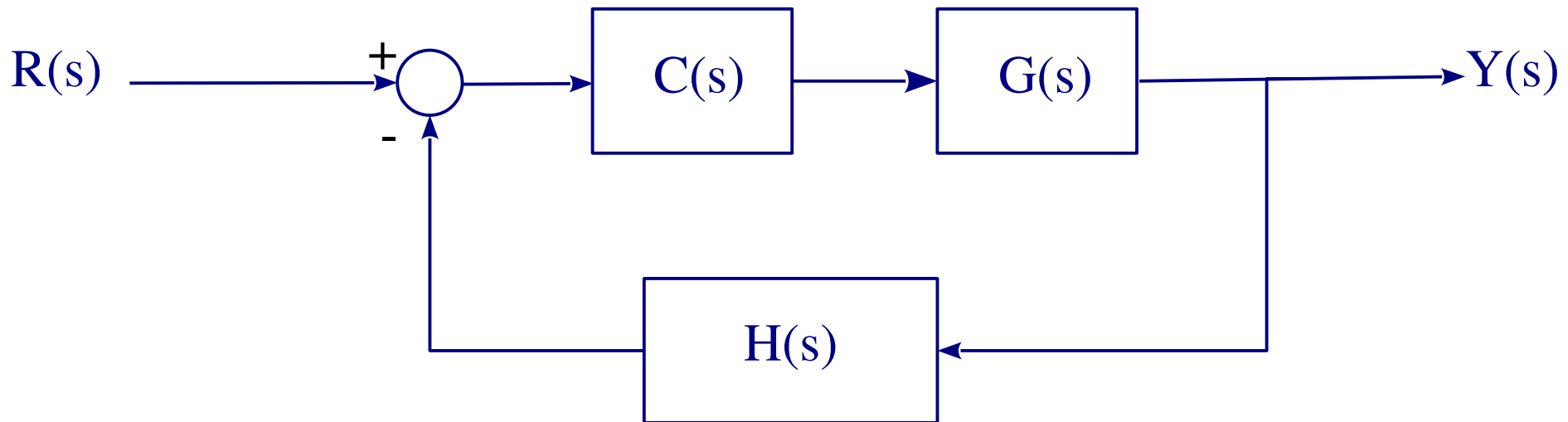
$$Q(s) = s^m + q_{m-1}s^{m-1} + q_{m-2}s^{m-2} + \cdots + q_0$$

$$P(s) = s^n + p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \cdots + p_0$$

$$1 + K \frac{Q(s)}{P(s)} = 0 \quad -\infty < K < \infty$$



Root Locus Plots



$$\frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)H(s)}$$



Root Locus Plots

Closed-loop transfer function

$$\frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)H(s)}$$

Closed-loop characteristic equation

$$1 + C(s)G(s)H(s) = 0$$

Assume $C(s)G(s)H(s)$ has a free parameter, K $C(s)G(s)H(s) = \frac{KQ(s)}{P(s)}$

$$\begin{aligned} 1 + C(s)G(s)H(s) &= 1 + \frac{KQ(s)}{P(s)} \\ &= \frac{P(s) + KQ(s)}{P(s)} \quad \leftarrow F(s) = P(s) + KQ(s) = 0 \\ &= 0 \end{aligned}$$



Root Locus Plots

What if K is not a multiplicative factor in $C(s)G(s)H(s)$

$$\begin{aligned}1 + C(s)G(s)H(s) &= s(s+1)(s+2) + s^2 + (3+2K)s + 5 \\ &= 0\end{aligned}$$

Rearrange the equation to have K as a factor

$$s(s+1)(s+2) + s^2 + 3s + 5 = -2Ks$$

$$1 = \frac{-2Ks}{s(s+1)(s+2) + s^2 + 3s + 5}$$

$$\begin{aligned}1 + \frac{2Ks}{s(s+1)(s+2) + s^2 + 3s + 5} &= 1 + \frac{KQ(s)}{P(s)} \\ &= 0\end{aligned}$$



Root Locus Plots

$$\begin{aligned} 1 + \frac{2Ks}{s(s+1)(s+2) + s^2 + 3s + 5} &= 1 + \frac{KQ(s)}{P(s)} \\ &= 0 \end{aligned}$$

$$Q(s) = 2s$$

$$\begin{aligned} P(s) &= s(s+1)(s+2) + s^2 + 3s + 5 \\ &= (s^2 + s)(s+2) + s^2 + 3s + 5 \\ &= s^3 + 2s^2 + s^2 + 2s + s^2 + 3s + 5 \\ &= s^3 + 4s^2 + 5s + 5 \end{aligned}$$

$$\frac{Q(s)}{P(s)} = \frac{2s}{s^3 + 4s^2 + 5s + 5}$$



Root Locus Plots

$$\frac{Q(s)}{P(s)} = \frac{2s}{s^3 + 4s^2 + 5s + 5}$$

We can define new transfer functions that do not contain K ,

$$C(s)G(s)H(s) = KC_1(s)G_1(s)H_1(s)$$

Such that,

$$C_1(s)G_1(s)H_1(s) = -\frac{1}{K}$$

Therefore we can say the following about the magnitude of the above equation,

$$|C_1(s)G_1(s)H_1(s)| = \frac{1}{|K|} \quad \text{where } -\infty < K < \infty$$



Root Locus Plots

$$C_1(s)G_1(s)H_1(s) = -\frac{1}{K}$$

Also, the following phase angle conditions also hold

$$\begin{aligned}\angle C_1(s)G_1(s)H_1(s) &= (2l+1)\pi \quad \text{where } K \geq 0 \\ &= \text{Odd multiples of } \pi\end{aligned}$$

$$\begin{aligned}\angle C_1(s)G_1(s)H_1(s) &= 2l\pi \quad \text{where } K \leq 0 \\ &= \text{Even multiples of } \pi\end{aligned}$$

where $l \in \{0, \pm 1, \pm 2, \dots\}$



Root Locus Plots

$$\begin{aligned} C(s)G(s)H(s) &= KC_1(s)G_1(s)H_1(s) \\ &= \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \end{aligned}$$

$$\begin{aligned} |C_1(s)G_1(s)H_1(s)| &= \frac{\prod_{j=1}^m |s+z_j|}{\prod_{k=1}^n |s+p_k|} \\ &= \frac{1}{|K|} \end{aligned}$$



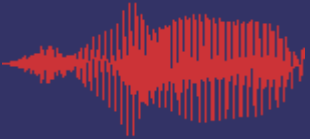
Root Locus Plots

$$\begin{aligned} C(s)G(s)H(s) &= KC_1(s)G_1(s)H_1(s) \\ &= \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \end{aligned}$$

Use to draw
Root Loci

$$\left\{ \begin{aligned} \angle C_1(s)G_1(s)H_1(s) &= \sum_{j=1}^m \angle(s+z_j) - \sum_{k=1}^n \angle(s+p_k) \\ &= (2l+1)\pi \quad \text{where } K \geq 0 \\ \\ \angle C_1(s)G_1(s)H_1(s) &= \sum_{j=1}^m \angle(s+z_j) - \sum_{k=1}^n \angle(s+p_k) \\ &= 2l\pi \quad \text{where } K \leq 0 \end{aligned} \right.$$

where $l \in \{0, \pm 1, \pm 2, \dots\}$



Root Locus Plots

$$\begin{aligned} |C_1(s)G_1(s)H_1(s)| &= \frac{\prod_{j=1}^m |s + z_j|}{\prod_{k=1}^n |s + p_k|} \\ &= \frac{1}{|K|} \end{aligned}$$

After the Root Locus is created, use the following to compute the value of K

$$|K| = \frac{\prod_{k=1}^n |s + p_k|}{\prod_{j=1}^m |s + z_j|}$$