ELEN 4810 Midterm Exam

Monday, October 27, 2025, 4:10-6:10 PM. One sheet of handwritten notes is allowed. No electronics of any kind are allowed. Please record your answers in the exam booklet. Raise your hand if you need additional scratch paper.

There are a total of 3 questions. Good luck!

Name:

Uni:

1. Systems in Time and Frequency. Consider the following two LTI systems: a **forward difference**

$$x[n] \longrightarrow \boxed{\text{Forward Diff.}} \longrightarrow y[n] = x[n+1] - x[n]$$

and a **moving average** of two samples

$$x[n] \longrightarrow \boxed{\text{Moving Avg.}} \longrightarrow y[n] = \frac{1}{2}x[n-1] + \frac{1}{2}x[n-2]$$

Notice that for each of these basic systems, the difference equation relating the input $x[n]$ and the output $y[n]$ is given in the diagram.

Consider the composite system

$$x[n] \longrightarrow \boxed{\boxed{\text{Forward Diff.}} \longrightarrow \boxed{\text{Moving Avg.}}} \longrightarrow y[n]$$

Please answer the following questions:

(i) What is the frequency response $H(e^{j\omega})$ of the overall system?

(ii) For what choices of n is the impulse response $h[n]$ nonzero?

(iii) Is the system stable? Why or why not?

(iv) Is the system causal? Why or why not?

(v) Consider an input of the form $x[n] = \exp(j\omega_0 n)$. For what choices of ω_0 is $y[n] = 0$?

(vi) If the two blocks were switched, producing a new system

$$x[n] \longrightarrow \boxed{\boxed{\text{Moving Avg.}} \longrightarrow \boxed{\text{Forward Diff.}}} \longrightarrow y[n]$$

would your answers to Part (v) change? Why or why not?

Answer to Problem 1:

(i) Taking the fourier transform of both sides of their defining difference equations, the forward difference satisfies $Y(e^{j\omega}) = (e^{j\omega} - 1)X(e^{j\omega})$ with frequency response

$$H_{fd}(e^{j\omega}) = e^{j\omega} - 1,$$

while the moving average satisfies $Y(e^{j\omega}) = (\frac{1}{2}e^{-j\omega} + \frac{1}{2}e^{-j2\omega})X(e^{j\omega})$ with frequency response

$$H_{ma}(e^{j\omega}) = \tfrac{1}{2}e^{-j\omega} + \tfrac{1}{2}e^{-j2\omega}. \tag{1}$$

The overall frequency response is

$$H(e^{j\omega}) = H_{fd}(e^{j\omega})H_{ma}(e^{j\omega}) = \tfrac{1}{2}e^{-j2\omega}(e^{j\omega} - 1)(e^{j\omega} + 1) = \tfrac{1}{2}e^{-j2\omega}(e^{j2\omega} - 1) = \tfrac{1}{2} - \tfrac{1}{2}e^{-j2\omega}. \tag{2}$$

(ii) Taking the inverse fourier tranform of the result from (i), the impulse response is nonzero for $n = 0$ and $n = 2$.

(iii) Stable - it is FIR, $\|h\|_1 = 1 < \infty$.

(iv) Causal - $h[n] = 0$ for $n < 0$.

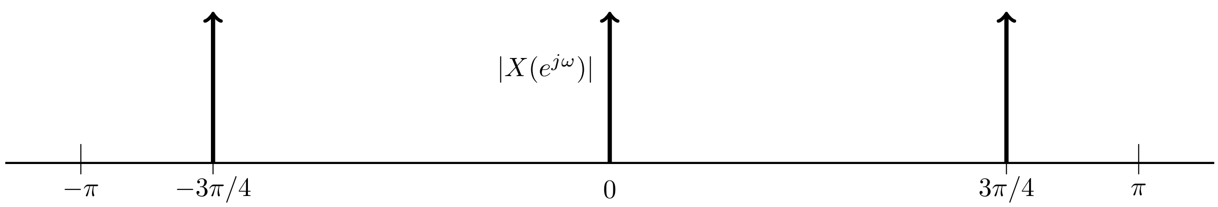
(v) $y[n] = 0$ if and only if $H(e^{j\omega_0}) = 0$ if and only if $1 - e^{j2\omega_0} = 0$ if and only $\omega_0 = k\pi$ for some $k \in \mathbb{Z}$.

(vi) No: LTI systems in series commute; changing the order does not change the frequency response.

2. Sampling: A continuous-time signal

$$x_c(t) = 1 + 2\cos(\Omega_0 t) \tag{3}$$

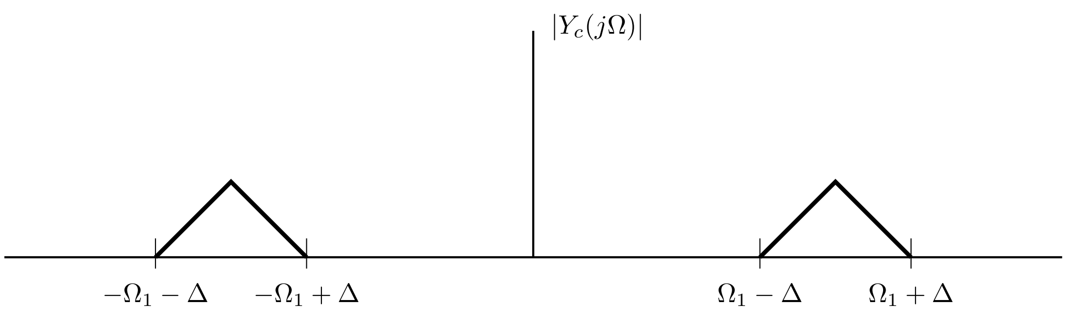
is sampled with sampling period $T = 0.001$ seconds to produce a discrete-time signal $x[n]$. The magnitude spectrum $|X(e^{j\omega})|$ is visualized below:



We do not know Ω_0 ahead of time.

Part (a). Suppose we know that the sampling rate obeys the Nyquist criterion, i.e., $\Omega_s > 2\Omega_0$. Please determine the frequency Ω_0.

Part (b). Now suppose we wish to transmit a continuous-time signal $y_c(t)$ with the following magnitude spectrum:



Here, $\Delta = 10\pi$ rad/s, and we are free to choose Ω_1.

We wish to choose Ω_1 to ensure that $|X_c(j\Omega)| = 0$ whenever $|Y_c(j\Omega)| > 0$, i.e., the nonzero frequencies of X_c and Y_c do not overlap. *What choices of Ω_1 satisfy this property?* For full credit, please indicate all possible choices of Ω_1.

Part (c). Now suppose that we know $|X(e^{j\omega})|$ and the sampling period $T = 0.001$ seconds, but we do not know Ω_0 and do not know whether the sampling rate satisfies the Nyquist criterion.

For what choices of Ω_1 can we guarantee that $|X_c(j\Omega)| = 0$ whenever $|Y_c(j\Omega)| > 0$?

Answer to Problem 2:

(a). In this case, there is no aliasing. Continuous frequency Ω_0 maps to discrete frequency $\omega_0 = \Omega_0 T$. Hence, $3\pi/4 = \Omega_0 \times 0.001$, giving

$$\Omega_0 = 1000 \times 3\pi/4 = 750\pi \, \tfrac{\text{rad}}{\text{s}}. \tag{4}$$

(b). The continuous time signal has nonzero frequency content at $\Omega \in \{-750\pi, 0, 750\pi\}$ rad/s. We need $|\Omega_1 - 750\pi| > 10\pi$, $|\Omega_1| > 10\pi$, $|\Omega_1 + 750\pi| > 10\pi$. Or,

$$\Omega_1 \in (-\infty, -760\pi] \cup [-740\pi, 740\pi] \cup [760\pi, \infty). \tag{5}$$

(c). In this situation, there could be aliasing. We have $\omega_0 = \Omega_0 T + 2\pi\, k$ for some integer k. Solving for Ω_0, we have

$$\begin{aligned} \Omega_0 &= 1000\,\omega_0 + 2000k\pi \qquad rad/s \qquad (6) \\ &= 750\pi + 2000k\pi \qquad \text{rad/s} \qquad (7) \end{aligned}$$

for some integer k. We need to choose Ω_1 such that $|\Omega_1| > 10\pi$ and

$$\left|\Omega_1 - \Big(750\pi + 2000k\pi\Big)\right| > 10\pi \quad \text{for all integers } k. \tag{8}$$

3. Discrete Fourier Transform and System Identification. Consider a linear, time invariant system with impulse response $h[n]$:

$$x[n] \longrightarrow \boxed{h} \longrightarrow y[n]$$

Suppose we know that the system is FIR and h has length N, so

$$h[n] = 0 \quad \text{for} \quad n < 0 \text{ and } n \geq N, \tag{9}$$

but *we do not know* h, and our goal is to determine h by observing the outputs $y_i = h * x_i$ of the system for particular input signals $x_i[n]$:

$$\begin{array}{ccc} x_1[n] & \mapsto & y_1[n] \\ x_2[n] & \mapsto & y_2[n] \\ & \vdots & \\ x_M[n] & \mapsto & y_M[n]. \end{array}$$

Part (a). Complex Exponential Inputs. Suppose that the inputs are complex exponentials

$$x_i[n] = \exp\Big(\mathrm{j}\, \omega_i n\Big) \tag{10}$$

and we are allowed to choose the (i) the number of input signals M, and (ii) frequencies $\omega_1, \ldots, \omega_M$. *Please answer the following:*

- What is the minimum number of inputs M required to determine h?
- How should we choose the frequencies $\omega_1, \ldots, \omega_M$?
- How can we determine h based on the output signals $y_1, \ldots, y_M$?

Part (b). Now suppose that N is an even integer and that $h[n]$ satisfies

$$h[n] = 0 \quad \text{for} \quad n = 1, 3, 5, 7, \ldots, N-1, \tag{11}$$

i.e., $h[n]$ is only nonzero for $n = 0, 2, 4, \ldots N-2$. How should you choose M and $\omega_1, \ldots, \omega_M$ to minimize the number of input-output pairs required?

Part (c). Other Types of Input? Returning to the setting of Part (a), where $h[n]$ could be nonzero from $n = 0, \ldots, N-1$. Suppose we are allowed to choose inputs $x_i[n]$ that are not complex exponentials. What is the minimum number of inputs $x_i[n]$ needed to determine h ?

Answer to Problem 3:

(a) We can choose $M = N$, set $\omega_i = \frac{2\pi(i-1)}{N}$. We have

$$y_i[n] = H(e^{j\omega_i})x_i[n] = H[i]x_i[n], \tag{12}$$

so the observation y_i gives us the value $H[i-1]$ of the length-N DFT of h at $\omega = \omega_i$. We can obtain $h[n]$ by taking the inverse DFT of $H[0], H[1], \ldots, H[N-1]$.

(b) Inspired by the decimation-in-time FFT, can express the length-N DFT of h in terms of the DFTs of the even and odd parts of h:

$$H[k] = H_e[k \mod N/2] + e^{-j\omega k}H_o[k \mod N/2]. \tag{13}$$

Since $h_o = 0$, we have

$$H_e[k \mod N/2] = H[k] \qquad k = 0, 1, \ldots, N/2 - 1. \tag{14}$$

We can therefore set $\omega_i = 2\pi(i-1)/N$ (as previously), for $i = 1 \ldots N/2$, and reconstruct the even part $h_e[n]$ of $h[n]$ by the inverse DFT.

(c) We can determine $h[n]$ from a single input, by setting $x[n] = \delta[n]$. Then the output $y[n]$ is equal to $h[n]$.

Scratch paper: