ELEN 4810 Homework 2

Analytical Questions

2.36 **(a)** We use the basic DTFT relationship

$$\alpha^n u[n] \to \frac{1}{1-\alpha e^{-j\omega}} \tag{1}$$

together with the shifting relationship of the DTFT. We have

$$0.8^n u[n] \to \frac{1}{1-0.8e^{-j\omega}} \tag{2}$$

and

$$0.8^{n-2} u[n-2] \to \frac{e^{-j2\omega}}{1-0.8e^{-j\omega}} \tag{3}$$

and so

$$h[n] = 0.8^n u[n] + 0.8^{n-2} u[n-2] \tag{4}$$

(b) Since $H(e^{j\omega}) = Y(e^{j\omega})/X(e^{j\omega})$, we have that

$$Y(e^{j\omega})\Big(1-0.8e^{-j\omega}\Big) = X(e^{j\omega})\Big(1+e^{-j2\omega}\Big). \tag{5}$$

So, using the time-shifting property of the DTFT and taking the inverse DTFT of both sides, we have

$$y[n] - 0.8y[n-1] = x[n] + x[n-2]. \tag{6}$$

(c) We use the fact that when a complex exponential $x[n] = e^{j\omega_0 n}$ is input to an LTI system with frequency response $H(e^{j\omega})$, the output is $y[n] = H(e^{j\omega_0})e^{j\omega_0 n}$. Hence, the system output for $x[n] = 4 + 2\cos(\omega_0 n)$, is

$$y[n] = 4H(e^{j0}) + H(e^{j\omega_0})e^{j\omega_0 n} + H(e^{-j\omega_0})e^{-j\omega_0 n}. \tag{7}$$

Thus, $y[n] = A$ is constant if and only if

$$H(e^{j\omega_0}) = 0 \quad \text{and} \quad H(e^{-j\omega_0}) = 0. \tag{8}$$

We have $H(e^{j\omega_0}) = 0$ iff $1 + e^{-j2\omega_0} = 0$ iff $e^{-j2\omega_0} = -1$, which is satisfied iff

$$\omega_0 = \tfrac{\pi}{2} + k\pi \quad \text{for} \quad k \in \mathbb{Z}. \tag{9}$$

The constant A is given by $A = 4H(e^{j0}) = 4 \times 2/(1-0.8) = 8/0.2 = 40$.

2.62 Note that[1]

$$\begin{aligned} x[n] &= \cos\left(\frac{15\pi n}{4} - \frac{\pi}{3}\right) && (10)\\ &= \cos\left(-\frac{\pi n}{4} - \frac{\pi}{3}\right) && (11)\\ &= \cos\left(\frac{\pi n}{4} + \frac{\pi}{3}\right). && (12) \end{aligned}$$

The DTFT of x is

$$X(e^{j\omega}) = \pi e^{j\pi/3}\delta(\omega - \pi/4) + \pi e^{-j\pi/3}\delta(\omega + \pi/4) \qquad -\pi < \omega \leq \pi. \tag{13}$$

The DTFT of y is

$$\begin{aligned} Y(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega}) && (14)\\ &= e^{-j[\omega/2+\pi/4]}\left[\pi e^{j\pi/3}\delta(\omega - \pi/4) + \pi e^{-j\pi/3}\delta(\omega + \pi/4)\right] \qquad -\pi < \omega \leq \pi. && (15)\\ & && (16) \end{aligned}$$

Using the sifting property of the Dirac delta[2] to plug in $\omega = \pm\frac{\pi}{4}$ in the term $e^{-j[\omega/2+\pi/4]}$, this becomes

$$\begin{aligned} & e^{-j\pi/4}\left[\pi e^{j\pi/3}e^{-j\pi/8}\delta(\omega - \pi/4) + \pi e^{-j\pi/3}e^{j\pi/8}\delta(\omega + \pi/4)\right] && (17)\\ =\; & e^{-j\pi/4}\mathrm{DTFT}\left[\cos\left(\frac{\pi}{4}n - \pi/3 + \pi/8\right)\right]. && (18) \end{aligned}$$

So,

$$y[n] = e^{-j\pi/4}\cos\left(\frac{\pi}{4}n + \frac{\pi}{3} - \frac{\pi}{8}\right). \tag{19}$$

This should seem like a reasonable answer – $H(e^{j\omega}) = e^{-j\omega/2-j\pi/4}$ consists of a half-sample delay and a multiplication by a complex exponential! Any expression for y which is equivalent to this one will receive full credit. One particularly simple approach is to note that the response of an LTI system with frequency response $H(e^{j\omega})$ to an input of the form $e^{j\omega_0 n}$ is itself an exponential:

$$H(e^{j\omega_0})e^{j\omega_0 n}. \tag{20}$$

So

$$y[n] = \tfrac{1}{2}e^{-j\pi/3}H(e^{j15\pi/4})e^{j15\pi n/4} + \tfrac{1}{2}e^{-j\pi/3}H(e^{-j15\pi/4})e^{j15\pi n/4}, \tag{21}$$

simplifying, we obtain the same answer quoted above.

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[1] (10) is true because $15\pi n/4 = 16\pi n/4 - \pi n/4$, and for any integer n, $16\pi n/4$ is an integer multiple of 2π. This manipulation is helpful, because it makes the frequency fall between $-\pi$ and π, which allows us to immediately write down the DTFT of x.

[2] For continuous f, $f(t)\delta(t-\tau) = f(\tau)\delta(t-\tau)$.

2.71 We can calculate the impulse response directly from the inverse DTFT:

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega/2} e^{j\omega n} d\omega \tag{22}$$

$$= \frac{1}{2\pi j(n-1/2)} e^{j\omega(n-1/2)}\Big|_{-\pi}^{\pi} \tag{23}$$

$$= \frac{1}{\pi(n-1/2)} \sin\left(\pi(n-1/2)\right). \tag{24}$$

The first term, $\frac{1}{\pi(n-1/2)}$ is not zero for any integer n; the second term $\sin\left(\pi(n-1/2)\right)$ takes on value $\sin(\pi/2) = 1$ when n is odd, and $\sin(-\pi/2) = -1$ when n is even. Since $h[n] \neq 0$ for $n < 0$, the system is not causal.

Note, you can contrast this to what happens if $H(e^{j\omega}) = e^{-j\omega k}$ for some positive integer k. This is the frequency response of the ideal delay, and *is* causal. When $H(e^{j\omega}) = e^{-j\omega/2}$ the system actually corresponds to a *half-integer* delay, which can be thought of as follows: we reconstruct a bandlimited continuous time signal from $x[n]$, delay it by half the sampling period, and then re-sample. This system ends up not being causal, because the reconstruction formula depends on the value of $x[n]$ at every n (both past and future).