Notes on Fewster–Verch

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1 Introduction

Let $\mathbf{M}$ and $\mathbf{N}$ be two spacetimes of equal dimension. In the covariant functorial quantum field theory Brunett–Fredenhagen–Verch [1], there exists a morphism $\varphi : \mathbf{M} \longrightarrow \mathbf{N}$ in the category **Loc**. For every compact open $O \subset \mathbf{M}$, there is an associated algebra $\mathscr{A}(O)$ via the functor

$$\mathbf{Loc} \longrightarrow \mathbf{Phys}$$

and the morphism φ gives rise to a functor

$$\varphi_{\mathscr{A}}(O) : \mathscr{A}(O) \longrightarrow \mathscr{A}(O') \tag{1.1}$$

where the closure of O' lies in $\mathbf{N}$.

If O and O' are contractible, then we can contract them both to points to get the following:

Proposition 1.1. *Let $\iota_{X;\Sigma}$ and $\iota_{X;\Sigma'}$ be two canonical inclusions, as defined in ([2], p.8). If $\Sigma' \subset \Sigma$, then $\iota_{X;\Sigma} \circ \iota_{X;\Sigma'}$ is well-defined, and there exists $x_0 \in \Sigma'$ and $x_1 \in \Sigma$ such that*

$$\varphi(x_0) = x_1$$

is Cauchy.

Proof. Recall that a Cauchy morphism is a morphism which takes Cauchy regions to Cauchy regions. Assume that Σ' is Cauchy. Then, there exists a net $\mathcal{N}(x)$ such that $\mathcal{N}|_0 = x_0$ and $\mathcal{N}|_1 = 1$ by Urysohn's lemma (see Appendix §A).

Since the intersection $\Sigma \cap \Sigma'$ is non-trivial, then Σ is not causally disjoint from Σ'. Thus, there exists some $y \in (0, 1)$ and a causal diamond $\mathrm{D}_{\mathbf{M}}(x_y) \subset \varphi(\mathrm{D}_{\mathbf{M}}(x_y))$ centered at a point

$$\mathcal{N}|_y = x_y$$

which is the *accumulation point* of the net $\mathcal{N}$. Thus, for every inextensible timelike curve γ, the intersection $\gamma \cap \mathcal{N}|_y$ is nonempty. $\square$

By lemma (A.2), we can construct a fiber bundle with contractible fiber such that the combination

$$(\varphi \circ \psi)(\mathcal{U}) = x_1$$

and by functoriality the composition is Cauchy ([2], p.9), and we have

$$\begin{aligned}\mathscr{A}(\varphi \circ \psi) &= \mathscr{A}(\varphi) \circ \mathscr{A}(\psi) && (1.2)\\ &= \varphi_{\mathscr{A}} \psi_{\mathscr{A}}(\mathcal{U}) && (1.3)\\ &= \mathscr{A}(\Sigma) \circ \mathscr{A}(\Sigma') \circ \mathscr{A}(\mathcal{U}). && (1.4)\end{aligned}$$

If we choose the y from Prop. (1.1) to be the (infinitesimal) difference $1-\varepsilon$, then x_1 becomes an arbitrarily close approximation to our accumulation point x_y. The numerator of our automorphism $f(y)$ from the proof of Lemma (A.1) then vanishes, the resulting algebra $\mathscr{A}(x_y)$ is canonically isomorphic to the algebra $\varphi^*_{\mathscr{A}}(x_0)$. This isomorphism occurs before we even take the limit $\varepsilon \to 0$, meaning that the value ε is renormalizable, and can hold for any quantity.[1]

Of course, in practice we may not necessarily be free to "choose" the value y. Nonetheless, the fact that such a value exists may demonstrate that theories with contractible fiber[2] may not obey the *same physics in all spacetimes* (SPASs) principle of [2]. We could restore the SPASs principle by requiring that

$$\mathscr{A}(x_0) = \mathscr{A}(x_y) \quad \forall y$$

but this seems to severe a restriction, as isotony ([3], p.3) should mean that the first algebra is a subset of the second rather than an equivalence.

A Basic Lemmas

Lemma A.1 (Urysohn)**.** *Let x_0 and x_1 be distinct points in a (normal) topological space X. Then, there exists a function:*

$$f(x) : X \to \mathbb{I}$$

which is zero on x_0 and one on x_1.

Proof. We can always find a function

$$f(x) = \begin{cases} 1 - \frac{\mathrm{d}(x,x_1)+\varepsilon}{\mathrm{d}(x_0,x_1)+\varepsilon} & \text{if } ||x_1|| - ||x|| < ||x_1|| - ||x_0|| \\ 1 - \lim\limits_{\varepsilon\to 0} \frac{\mathrm{d}(x,x_1)+\varepsilon}{\mathrm{d}(x_0,x_1)+\varepsilon} & \text{if } ||x_1|| - ||x|| = ||x_1|| - ||x_0|| \\ 1 - \left(\frac{\mathrm{d}(x,x_1)+\varepsilon}{\mathrm{d}(x_0,x_1)+\varepsilon}\right)^{-1} & \text{if } ||x_1|| - ||x|| > ||x_1|| - ||x_0|| \end{cases}$$

where d denotes the normed distance, and as $\varepsilon \to 0$, we get 0 on x_0 and 1 on x_1. □

Remark A.1. *The above function, as constructed, is a homotopy when we consider $f(x)$ to be the family of functions $f_{(\mathrm{d}(x_0,x_1))}$ and take $f_0(x) = f(x_1)$ and $f_1(x) = f(x_0)$. If X is the unit interval, then $f(x)$ is an endomorphism and we can drop the final conditional from the equation. There are several other possible functions we could have chosen, but this is one such example that holds for any normal topological X.*

[1] Of particular note, we can let ε be a closed form $\varepsilon = dA$, where A is a U(1)-valued 1-form compactified on a circle.

[2] The devil in the details is the assumption expressed in A.3 that the map ψ is a bijection. If we ease this to a condition of monotically increasing cardinality, then we have that $\psi_{\mathscr{A}}$ and $\varphi_{\mathscr{A}}$ (and by functoriality, their composition as well) are both algebra inclusions, but this does not come naturally from the net itself.

Lemma A.2. *Let $P : \mathbf{L} \longrightarrow X$ be a fiber bundle with contractible fiber. Then, for every neighborhood $\mathcal{U} \subset \mathbf{L}$, there exists a deformation retract:*

$$\kappa : \mathcal{U} \longrightarrow x_0.$$

Proof. Since we have trivial homotopy groups $\pi_\bullet(\mathbf{L})$ globally, then by Myers–Vietoris, we also have trivial homotopy groups $\pi_\bullet(\mathcal{U})$. □

Lemma A.3. *Let $\psi = P \circ \kappa$. If $\psi(\mathcal{U})$ is a bijection, then its image is a point.*

Proof. Since the contraction mapping yields a point, then by A.4 we get a monoid $\mathcal{C}_x$ and the projection must also yield a point x. □

Lemma A.4. *Each point x forms a 1-object monoidal category $\mathcal{C}_x$.*

Proof. We have $\mathrm{Ob}(\mathcal{C}_x) = x$, $\mathrm{Mor}(\mathcal{C}_x) = 1_x$, and our binary operation acts trivially as

$$\cdot : \mathcal{C}_x \times \mathcal{C}_x \to \mathcal{C}_x \coloneqq 1_x(x).$$

□

References

[1] R. Brunetti, K. Fredenhagen, R. Verch. *The generally covariant locality principle: A new paradigm for local quantum physics.* Commun. Math. Phys. **237**, 31–68 (2003).

[2] C. J. Fewster and R. Verch. *Dynamical locality and covariance: What makes a physical theory the same in all spacetimes?* arXiv:1106.4785v1 (2011)

[3] C.–L. Sion *The Haag-Kastler aximatic framework.* (2012) Available at: `https://www.physik.uni-hamburg.de/th2/ag-fredenhagen/dokumente/thehaag-kastleraxiomaticframework.pdf`