

Lecture 4: Rademacher complexity I

Symmetrization, Bousquet bound, and excess risk bounds

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“*There is Nothing More Practical Than A Good Theory.*”

— Kurt Lewin

1 Introduction

Recall the pre-mentioned aims:

- **A1.** The asymptotics of

$$A_1 = \mathbb{E} \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^n \left(l(\mathbf{Y}_i, f(\mathbf{X}_i)) - \mathbb{E} l(\mathbf{Y}_i, f(\mathbf{X}_i)) \right).$$

- **A2'.** Find a tight upper bound of

$$A_2 = \mathbb{E} \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^n \left(l(\mathbf{Y}_i, f(\mathbf{X}_i)) - \mathbb{E} l(\mathbf{Y}_i, f(\mathbf{X}_i)) \right)^2.$$

The solution to bound those two empirical processes is to introduce **Rademacher complexity** to measure the complexity of the functional space \mathcal{F} . The definition of Rademacher complexity is inspired by the one of the most important properties of the empirical processes, that is, **symmetrization**.

2 Symmetrization

We illustrate with the empirical process in **A1**. Define random variables $\tilde{\mathcal{D}}_n = (\tilde{\mathbf{X}}_i, \tilde{\mathbf{Y}}_i)_{i=1, \dots, n}$ as the independent copy of $\mathcal{D}_n = (\mathbf{X}_i, \mathbf{Y}_i)_{i=1, \dots, n}$, that is, $(\tilde{\mathbf{X}}_i, \tilde{\mathbf{Y}}_i) \stackrel{d}{=} (\mathbf{X}_i, \mathbf{Y}_i)$ and samples in $\{\mathcal{D}_n, \tilde{\mathcal{D}}_n\}$

are all independent.

$$\begin{aligned}
A_1 &= \mathbb{E} \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^n \left(l(\mathbf{Y}_i, f(\mathbf{X}_i)) - \mathbb{E} l(\mathbf{Y}_i, f(\mathbf{X}_i)) \right) = \mathbb{E} \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^n \left(l(\mathbf{Y}_i, f(\mathbf{X}_i)) - \tilde{\mathbb{E}} l(\tilde{\mathbf{Y}}_i, f(\tilde{\mathbf{X}}_i)) \right) \\
&= \mathbb{E} \sup_{f \in \mathcal{F}} \left(\frac{1}{n} \sum_{i=1}^n l(\mathbf{Y}_i, f(\mathbf{X}_i)) - \frac{1}{n} \sum_{i=1}^n \tilde{\mathbb{E}} l(\tilde{\mathbf{Y}}_i, f(\tilde{\mathbf{X}}_i)) \right) \\
&= \mathbb{E} \sup_{f \in \mathcal{F}} \tilde{\mathbb{E}} \frac{1}{n} \sum_{i=1}^n \left(l(\mathbf{Y}_i, f(\mathbf{X}_i)) - l(\tilde{\mathbf{Y}}_i, f(\tilde{\mathbf{X}}_i)) \right) \leq \mathbb{E} \tilde{\mathbb{E}} \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^n \left(l(\mathbf{Y}_i, f(\mathbf{X}_i)) - l(\tilde{\mathbf{Y}}_i, f(\tilde{\mathbf{X}}_i)) \right) \\
&\quad (1)
\end{aligned}$$

$$= \mathbb{E} \tilde{\mathbb{E}} \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^n \rho_i \left(l(\mathbf{Y}_i, f(\mathbf{X}_i)) - l(\tilde{\mathbf{Y}}_i, f(\tilde{\mathbf{X}}_i)) \right) \leq 2 \mathbb{E} \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n \rho_i l(\mathbf{Y}_i, f(\mathbf{X}_i)) \right|, \quad (2)$$

where $(\rho_i)_{i=1,\dots,n}$ are i.i.d. Rademacher random variables independent with \mathcal{D}_n and $\tilde{\mathcal{D}}_n$, with ρ_i taking the values $+1$ and -1 with probability $1/2$ each. The last equality follows from the fact that $(\tilde{\mathbf{X}}_i, \tilde{\mathbf{Y}}_i)$ is the independent copy of $(\mathbf{X}_i, \mathbf{Y}_i)$, thus the joint distribution of $(\mathcal{D}_n, \tilde{\mathcal{D}}_n)$ does not change by switching $(\tilde{\mathbf{X}}_i, \tilde{\mathbf{Y}}_i)$ and $(\mathbf{X}_i, \mathbf{Y}_i)$. Therefore, the equality holds for arbitrary choice of $\rho_i = +1$ or $\rho_i = -1$.

(1) and (2) are so-called *symmetrization inequalities*, and (2) indicates that the empirical risk excess process is upper bounded by the Rademacher process. Next, we summarize all the results for a **general empirical process**.

Define a general empirical process on i.i.d. samples $(\mathbf{Z}_i)_{i=1,\dots,n}$ indexed by $h \in \mathcal{H}$ as:

$$\frac{1}{n} \sum_{i=1}^n \left(h(\mathbf{Z}_i) - \mathbb{E} h(\mathbf{Z}_i) \right), \quad h \in \mathcal{H}.$$

Its corresponding Rademacher process is defined as:

$$\mathbf{Rad}_n(h) = \frac{1}{n} \sum_{i=1}^n \rho_i h(\mathbf{Z}_i), \quad h \in \mathcal{H}.$$

Theorem 2.1 (Symmetrization Inequalities). *For any function class \mathcal{H} :*

$$\frac{1}{2} \mathbb{E} \sup_{h \in \mathcal{H}} |\mathbf{Rad}_n(\tilde{h})| \leq \mathbb{E} \sup_{h \in \mathcal{H}} \left| \frac{1}{n} \sum_{i=1}^n \left(h(\mathbf{Z}_i) - \mathbb{E} h(\mathbf{Z}_i) \right) \right| \leq 2 \mathbb{E} \sup_{h \in \mathcal{H}} |\mathbf{Rad}_n(h)|, \quad (3)$$

where $\tilde{h}(\mathbf{Z}) = h(\mathbf{Z}) - \mathbb{E} h(\mathbf{Z})$, $(\tilde{\mathbf{Z}}_1, \dots, \tilde{\mathbf{Z}}_n)$ is independent copy of $(\mathbf{Z}_1, \dots, \mathbf{Z}_n)$, $\mathbb{E} \sup_{h \in \mathcal{H}} \mathbf{Rad}_n(h)$ is the Rademacher complexity of the function class \mathcal{H} , the expectation \mathbb{E} is taken with respect to all randomness.

3 Rademacher complexity

Remark 3.1. Recall the definition of Rademacher process $\mathbf{Rad}_n(h)$, it can be considered as empirical correlation between ρ and $h(\mathbf{Z})$. Suppose h restrains only one constant, say $h(\mathbf{z}) = 1$,

$$\mathbf{Rad}_n(h) = \frac{1}{n} \sum_{i=1}^n \rho_i = O_P\left(\frac{1}{\sqrt{n}}\right);$$

if h is diverse enough, such that $h(\mathbf{z}_i) = \rho_i$:

$$\mathbf{Rad}_n(h) = \frac{1}{n} \sum_{i=1}^n \rho_i^2 = O_P(1).$$

Therefore, the order of $\mathbb{E} \sup_{h \in \mathcal{H}} \mathbf{Rad}_n(h)$ is between $O(n^{-1/2})$ and $O(1)$, measuring the complexity of the function class \mathcal{H} .

In practice, we may want to bound the Rademacher complexity on $\varphi \circ f$. For example, in our case, we tend to investigate the complexity of $l(\mathbf{Y}_i, f(\mathbf{X}_i))$; $f \in \mathcal{F}$. Talagrand's contraction Lemma is proposed to address this target.

Lemma 3.2 (Talagrand's contraction Lemma [Ledoux and Talagrand, 1991]). *Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be a L -Lipschitz function, then*

$$\mathbb{E} \sup_{h \in \mathcal{H}} |\mathbf{Rad}_n(\varphi \circ h)| \leq L \mathbb{E} \sup_{h \in \mathcal{H}} |\mathbf{Rad}_n(h)|. \quad (4)$$

Remark 3.3. Note that φ is a Lipschitz function, it is sensible to believe that the complexity of $\varphi \circ \mathcal{H}$ can be controlled by the complexity of \mathcal{H} .

One important application of Talagrand's contraction Lemma is to upper bound the “second moment” of empirical process (A_2 in our case).

Corollary 3.4. *Suppose that functions in \mathcal{H} are uniformly bounded by a constant U , then*

$$\mathbb{E} \sup_{h \in \mathcal{H}} |\mathbf{Rad}_n(h^2)| \leq 2U \mathbb{E} \sup_{h \in \mathcal{H}} |\mathbf{Rad}_n(h)|.$$

Now, we apply the results to A_1 and A_2 . Denote $h(\mathbf{Z}_i) = l(\mathbf{Y}_i, f(\mathbf{X}_i))$, and suppose the loss function l is uniformly bounded by U , then

$$A_1 = \mathbb{E} \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^n (h(\mathbf{Z}_i) - \mathbb{E}(h(\mathbf{Z}_i))) \leq 2 \mathbb{E} \sup_{f \in \mathcal{F}} |\mathbf{Rad}_n(h)|.$$

Denote $\tilde{h}(\mathbf{Z}_i) = l(\mathbf{Y}_i, f(\mathbf{X}_i)) - \mathbb{E}l(\mathbf{Y}_i, f(\mathbf{X}_i))$, then

$$\begin{aligned}
A_2 &= \mathbb{E} \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^n (\tilde{h}^2(\mathbf{Z}_i) - \mathbb{E}(\tilde{h}^2(\mathbf{Z}_i)) + \mathbb{E}(\tilde{h}^2(\mathbf{Z}_i))) \\
&\leq \mathbb{E} \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^n (\tilde{h}^2(\mathbf{Z}_i) - \mathbb{E}(\tilde{h}^2(\mathbf{Z}_i))) + \sup_{f \in \mathcal{F}} \mathbb{E} \tilde{h}^2(\mathbf{Z}) \\
&\leq 2\mathbb{E} \sup_{f \in \mathcal{F}} |\mathbf{Rad}_n(\tilde{h}^2)| + \sup_{f \in \mathcal{F}} \mathbb{E} \tilde{h}^2(\mathbf{Z}) \leq 4U \mathbb{E} \sup_{f \in \mathcal{F}} |\mathbf{Rad}_n(\tilde{h})| + \sup_{f \in \mathcal{F}} \mathbf{Var}(h(\mathbf{Z})) \\
&\leq 8UA_1 + \sup_{f \in \mathcal{F}} \mathbf{Var}(h(\mathbf{Z})) \leq 16U \mathbb{E} \sup_{f \in \mathcal{F}} |\mathbf{Rad}_n(h)| + \sup_{f \in \mathcal{F}} \mathbf{Var}(h(\mathbf{Z})). \tag{5}
\end{aligned}$$

4 Bousquet bound

Now, we combine all results to have a new updated form of Talagrand's inequality, namely Bousquet bound. For simplicity, we denote:

$$\|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{H}} = \sup_{h \in \mathcal{H}} \frac{1}{n} \left| \sum_{i=1}^n (h(\mathbf{Z}_i) - \mathbb{E}h(\mathbf{Z}_i)) \right|.$$

Theorem 4.1 (Bousquet bound of Talagrand's inequality [Bousquet, 2002]). *Suppose $h(\mathbf{Z})$ is uniformly bounded by U almost surely, then for $t > 0$,*

$$\mathbb{P} \left(\|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{H}} \geq \mathbb{E} \|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{H}} + \sqrt{\frac{2t}{n} (\sigma_{\mathcal{H}}^2 + U \mathbb{E} \|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{H}})} + \frac{Ut}{3n} \right) \leq e^{-t},$$

where $\sigma_{\mathcal{H}}^2$ is defined as

$$\sigma_{\mathcal{H}}^2 = \sup_{h \in \mathcal{H}} \mathbf{Var}(h(\mathbf{Z})).$$

Theorem 4.1 implies the following corollary.

Corollary 4.2. *Suppose $h(\mathbf{Z})$ is uniformly bounded by a constant U almost surely, then for any $\varepsilon_n > 0$,*

$$\mathbb{P} \left(\|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{H}} - \mathbb{E} \|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{H}} \geq \varepsilon_n \right) \leq \exp \left(- \frac{n\varepsilon_n^2}{2(\sigma_{\mathcal{H}}^2 + U \mathbb{E} \|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{H}} + U\varepsilon_n/3)} \right). \tag{6}$$

Furthermore, if

$$\varepsilon_n \geq 4\mathbb{E} \sup_{h \in \mathcal{H}} |\mathbf{Rad}_n(h)|, \quad (\text{thus } \varepsilon_n \geq 2\mathbb{E} \|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{H}})$$

we have,

$$\mathbb{P} \left(\|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{H}} \geq \varepsilon_n \right) \leq \exp \left(- \frac{n\varepsilon_n^2}{8(\sigma_{\mathcal{H}}^2 + 2U\varepsilon_n/3)} \right).$$

Remark 4.3. When $\mathcal{H} = \{h\}$ (only one function), let $W_i = h(\mathbf{Z}_i)$, then $\sigma_{\mathcal{H}}^2 = \mathbf{Var}(W) =: \sigma^2$, (6) yields that

$$\mathbb{P}\left(\frac{1}{n} \sum_{i=1}^n W_i - \mathbb{E}(W) \geq \varepsilon_n\right) \leq \exp\left(-\frac{n\varepsilon_n^2}{2(\sigma^2 + U\varepsilon_n/3)}\right),$$

which is Bernstein inequality. This fact partially indicates that Bousquet bound of Talagrand's inequality is tight.

5 Excess risk bounds

Next, we apply the uniform concentration inequalities to our excess risks. For simplicity, we denote

$$\widehat{R}_n^c(f) = \widehat{R}_n(f) - R(f) = \frac{1}{n} \sum_{i=1}^n \left(l(\mathbf{Y}_i, f(\mathbf{X}_i)) - \mathbb{E}l(\mathbf{Y}_i, f(\mathbf{X}_i)) \right)$$

Corollary 5.1. Suppose the loss function $l(\cdot, \cdot)$ is uniformly bounded by a constant U , then for $t > 0$,

$$\mathbb{P}\left(\sup_{f \in \mathcal{F}} |\widehat{R}_n^c(f)| \geq \mathbb{E} \sup_{f \in \mathcal{F}} |\widehat{R}_n^c(f)| + \sqrt{\frac{2t}{n} (\sigma_{\mathcal{F}}^2 + U \mathbb{E} \sup_{f \in \mathcal{F}} |\widehat{R}_n^c(f)|)} + \frac{Ut}{3n}\right) \leq e^{-t},$$

where $\sigma_{\mathcal{F}}^2$ is defined as

$$\sigma_{\mathcal{F}}^2 = \sup_{f \in \mathcal{F}} \mathbf{Var}(l(\mathbf{Y}, f(\mathbf{X}))).$$

Alternatively, for any $\varepsilon_n > 0$,

$$\mathbb{P}\left(\sup_{f \in \mathcal{F}} |\widehat{R}_n^c(f)| - \mathbb{E} \sup_{f \in \mathcal{F}} |\widehat{R}_n^c(f)| \geq \varepsilon_n\right) \leq \exp\left(-\frac{n\varepsilon_n^2}{2(\sigma_{\mathcal{F}}^2 + U \mathbb{E} \sup_{f \in \mathcal{F}} |\widehat{R}_n^c(f)| + U\varepsilon_n/3)}\right).$$

Furthermore, if

$$\varepsilon_n \geq 4\mathbb{E} \sup_{f \in \mathcal{F}} |\mathbf{Rad}_n(l \bullet f)|, \quad (l \bullet f)(\mathbf{Z}) = l(\mathbf{Y}, f(\mathbf{X}))$$

we have,

$$\mathbb{P}\left(\sup_{f \in \mathcal{F}} |\widehat{R}_n(f) - R(f)| \geq \varepsilon_n\right) \leq \exp\left(-\frac{n\varepsilon_n^2}{8(\sigma_{\mathcal{F}}^2 + 2U\varepsilon_n/3)}\right).$$

From Corollary 5.1, to derive a probabilistic bound for an excess risk, it suffices to compute and upper bound the Rademacher complexity of $(l \bullet f)(\mathbf{Z}) = l(\mathbf{Y}, f(\mathbf{X})); f \in \mathcal{F}$.

References

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- [Ledoux and Talagrand, 1991] Ledoux, M. and Talagrand, M. (1991). *Probability in Banach Spaces: isoperimetry and processes*, volume 23. Springer Science & Business Media.