INTRODUCTION TO CONTINUOUS CONTROL SYSTEMS
COLUMBIA UNIVERSITY MECHANICAL AND ELECTRICAL ENGINEERING DEPARTMENTS: E3601

Homayoon Beigi†

1340 Mudd Building
Columbia University, New York City, NY 10027
hb87@columbia.edu

Homework 11

Problem 1 (Bode Plot of a First Order System)**.** *Consider the first-order low-pass filter with the following transfer function:*

$$H(s) = \frac{10}{s+5}$$

1. *Determine the corner frequency (also known as the cutoff frequency) of the system.*
2. *Sketch the Bode plot for the given transfer function, including both the magnitude and phase plots.*
3. *Calculate and describe the behavior of the magnitude and phase as the frequency approaches zero and infinity.*

Solution

Step 1: Corner Frequency

The standard form for a first-order low-pass filter is:

$$H(s) = \frac{K}{s+\omega_c}$$

where ω_c is the corner frequency.
From the given transfer function $H(s) = \frac{10}{s+5}$, we identify:

$$\omega_c = 5\frac{rad}{s}$$

†Homayoon Beigi is Professor of Professional Practice in the department of mechanical engineering and in the department of electrical engineering at Columbia University

Step 2: Bode Plot

Magnitude Plot

- *At low frequencies* $(\omega \ll \omega_c)$*, the magnitude is approximately the constant gain,* $20\log_{10}(K)$.
- $|H(i\omega)| = \left|\frac{10}{i\omega+5}\right| = \frac{10}{\sqrt{\omega^2+25}}$
- *At* $\omega = 0$*,* $|H(i0)| = \frac{10}{5} = 2$
- *In dB:* $20\log_{10}(2) \approx 6.02$ *decibel*

- *At the corner frequency* $(\omega = \omega_c = 5\frac{rad}{s})$*, the magnitude drops by 3 dB.*
- $|H(i5)| = \frac{10}{\sqrt{25+25}} = \frac{10}{\sqrt{50}} \approx 1.414$
- *In dB:* $20\log_{10}(1.414) = 3.01$ *decibel*

- *For high frequencies* $(\omega \gg \omega_c)$*, the magnitude decreases at a rate of -20 dB/decade.*

Phase Plot:

- *At low frequencies* $(\omega \ll \omega_c)$*, the phase is approximately* 0°.
- *At the corner frequency, the phase shift is* -45°.
- *For high frequencies* $(\omega \gg \omega_c)$*, the phase approaches* -90°.

Step 3: Behavior

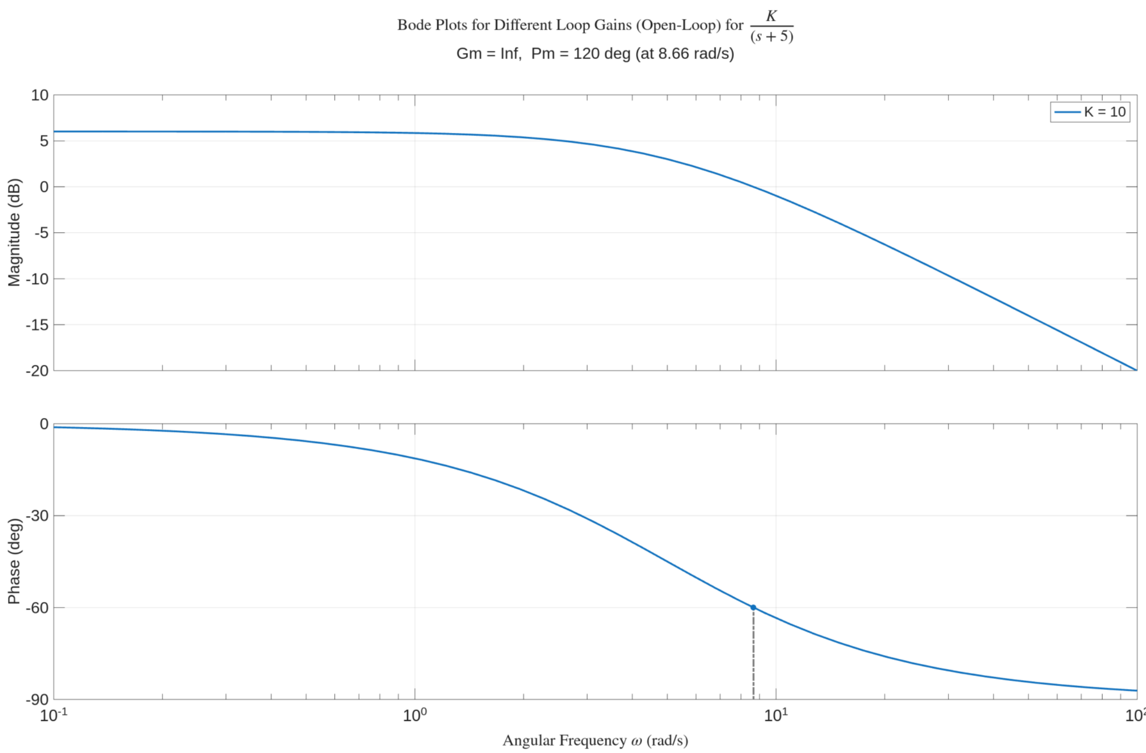
- ***As*** $\omega \to 0$***:***

- *Magnitude approaches* 6.02 *decibel*
- *Phase approaches* 0°

- ***As*** $\omega \to \infty$***:***

- *Magnitude approaches* $-\infty$ *dB (decreases continuously)*
- *Phase approaches* -90°

Bode Plot



Problem 2 (Bode Plot of a Second Order System)**.** *Consider the second order transfer function:*

$$H(s) = \frac{4}{s^2 + 0.4s + 4}$$

This system has:

$$\omega_n = 2, \quad \zeta = 0.1$$

Sketch the Bode Plot by following these steps

A. Break Down the Transfer Function Parts
B. Approximate the parts of the Magnitude plot by computing values for different regions along the frequency.
C. Approximate the parts of the Phase plot by computing values for different regions along the frequency.

Calculations

- The system is already in its standard form.

- At low frequencies ($\omega << \omega_n$)*,* $|H(i\omega)| \approx 1$.
- At high frequencies ($\omega >> \omega_n$)*, the magnitude decreases at a rate of -40 dB/decade.*

3. - At frequencies much less than ω_n*, the phase is approximately* 0°.
- Near $\omega = \omega_n$*, the phase decreases and crosses* -90°.
- At frequencies much greater than ω_n*, the phase approaches* -180°.

Magnitude Plot
The magnitude in dB is

$$|H(i\omega)|_{dB} = 20\log_{10}(|H(i\omega)|)$$

$$|H(i\omega)| = \left|\frac{4}{-\omega^2 + i(0.4\omega) + 4}\right|$$

Simplify and calculate the magnitudes at:
- Low frequencies ($\omega \approx 0$)*:*

$$|H(i0)| = 1, \quad 20\log_{10}(1) = 0\ dB$$

- Resonant frequency ($\omega = \omega_n = 2$)*:*

$$|H(i2)| = \left|\frac{4}{4 - i0.8 + 4}\right|$$

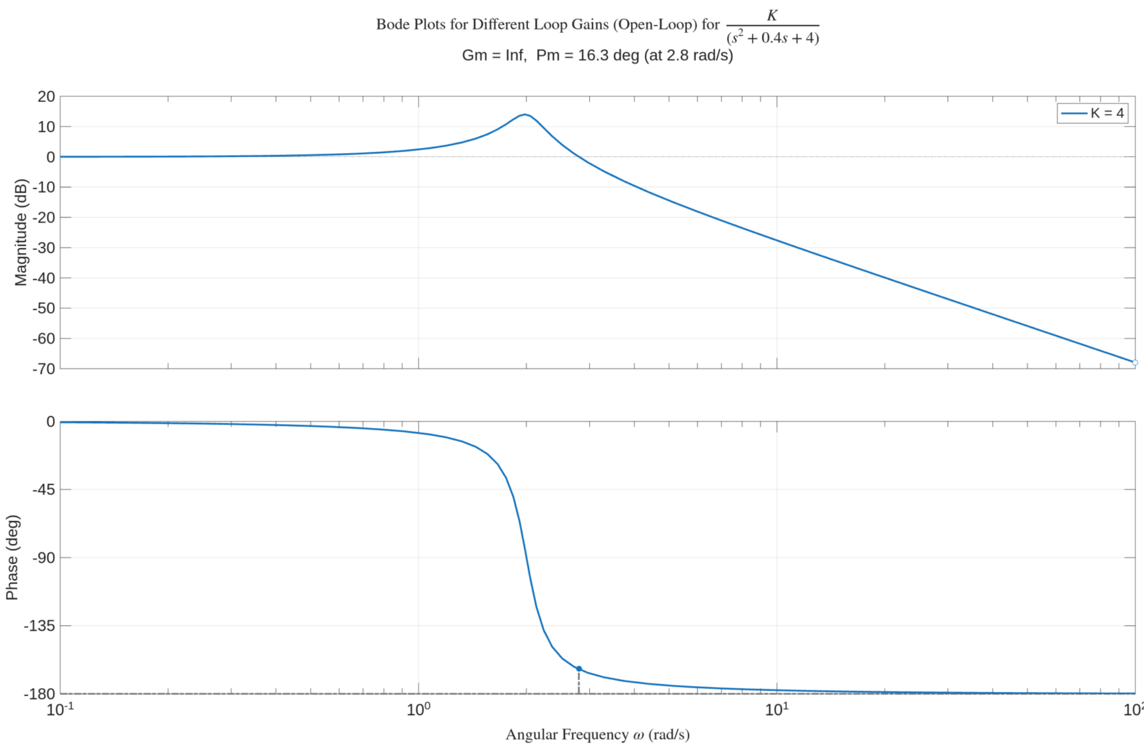
- High frequencies ($\omega \to \infty$)*:*

$$|H(i\omega)| \approx \frac{4}{\omega^2}$$

Phase Plot
The phase is given by

$$Phase = \tan^{-1}\left(\frac{-0.4\omega}{4 - \omega^2}\right)$$

Evaluate at:
- $\omega = 0$*: Phase* $= 0^\circ$
- $\omega = \omega_n = 2$*: Phase* $\approx -90^\circ$ *-* $\omega \to \infty$*:*
Phase $\to -180^\circ$



Problem 3 (Bode Plot of a Third Order System)**.**

Consider the third-order transfer function:

$$H(s) = \frac{10}{(s+1)(s^2+2s+4)}$$

1. *Identify the corner and/or resonance frequencies of the system.*
2. *Sketch the Bode plot for the given transfer function, including both the magnitude and phase plots.*
3. *Discuss the behavior of the magnitude and phase as the frequency approaches zero and infinity.*

Solution

Step 1: Frequencies

The system consists of:
- *A first-order pole at* -1 *with a corner frequency* $\omega_c = 1\,\frac{rad}{s}$
- *A second-order section:* s^2+2s+4
- *Natural frequency* $\omega_n = \sqrt{4} = 2$
- *Damping ratio* $\zeta = 0.5$

Step 2: Bode Plot

Magnitude Plot

- At low frequencies $(\omega \ll \omega_c)$*, the magnitude is* $\approx 20\log_{10}(2.5) = 7.96$ *decibel.*
- At $\omega = 1$ *rad/s, the slope changes to -20 dB/decade.*
- At $\omega = 2$ *rad/s, the slope becomes -40 dB/decade due to the second-order effect.*

Phase Plot

- Initial phase near 0°*.*
- Drops by 45° *at* $\omega_c = 1$ *rad/s.*
- Further decreases by additional 90° *by* $\omega_n = 2$ *rad/s to total* -135°*.*

Step 3: Behavior

- ***As*** $\omega \to 0$***:***

- *Magnitude approaches* 7.96 *decibel*
- *Phase approaches* 0°

- ***As*** $\omega \to \infty$***:***

- *Magnitude decreases continuously at* -60 *dB/decade*
- *Phase approaches* -180°

Bode Plot

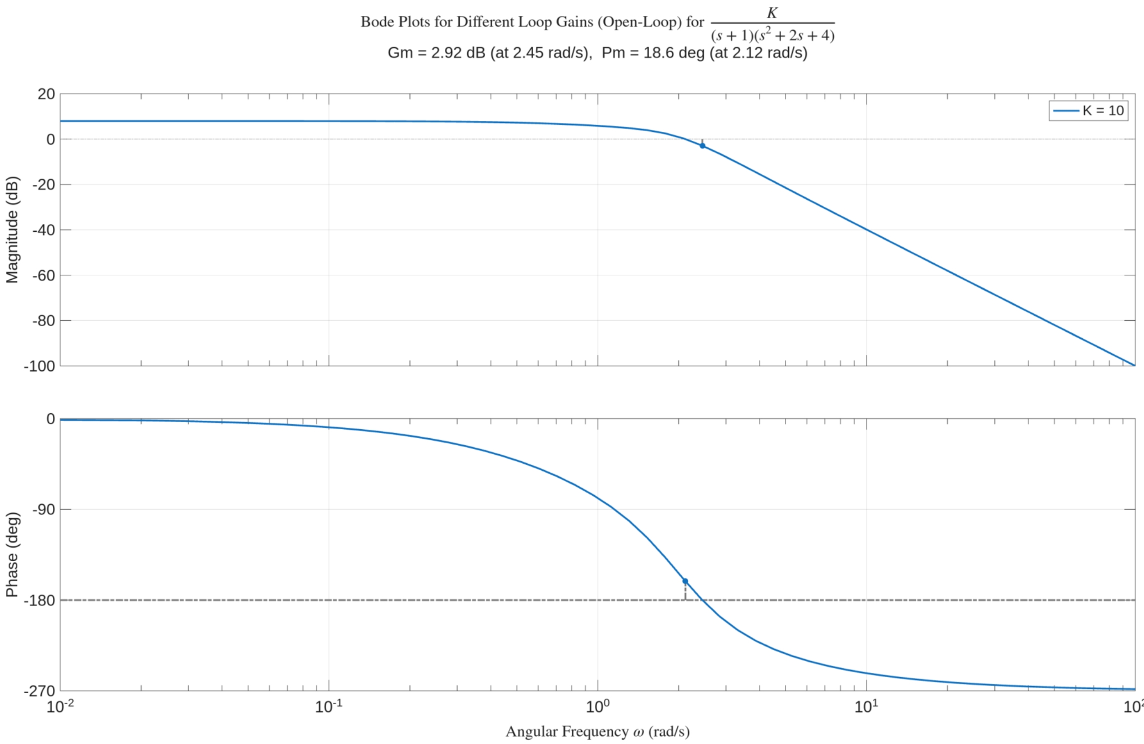


Fig. 1: Bode Plot of the Transfer Function $H(s) = \frac{10}{(s+1)(s^2+2s+4)}$