INTRODUCTION TO CONTINUOUS CONTROL SYSTEMS
COLUMBIA UNIVERSITY MECHANICAL AND ELECTRICAL ENGINEERING DEPARTMENTS: E3601

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Homework 10

Problem 1 (Routh)**.**
Use the Routh criterion on the following polynomials to determine the number of roots in the right-half complex plane.
A.

$$P(s) = s^7 + 3s^6 + 11s^5 + 19s^4 + 36s^3 + 38s^2 + 36s + 24 \tag{1}$$

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Solution

1 A.

$$P(s) = s^7 + 3s^6 + 11s^5 + 19s^4 + 36s^3 + 38s^2 + 36s + 24$$

row					
1	s^7	1	11	36	36
2	s^6	3	19	38	24
3	s^5	$\frac{14}{3}$	$\frac{70}{3}$	28	0
4	s^4	4	20	24	0
~~5~~	~~s^3~~	~~0~~	~~0~~	~~0~~	~~0~~
5	s^3	16	40	0	0
6	s^2	10	24	0	0
7	s^1	1.6	0	0	0
8	s^0	24	0	0	

$24 \times 16 = 144$, 24, 384

$40 - \frac{384}{10}$

$40 - 38.4 = 1.6$

1.6×24

no roots in RHP

$$\frac{d\left[4s^4 + 20s^2 + 24\right]}{ds} = 16s^3 + 40s$$

$16 \times 20 - 4 \times 24$

B.

$$P(s) = s^6 + 2s^5 + 4s^4 + 8s^3 + 6s^2 + 8s + 4 \quad (2)$$

Solution

$$P(s) = s^6 + 2s^5 + 4s^4 + 8s^3 + 6s^2 + 8s + 4$$

1	s^6	1	4	6	4
2	s^5	2	8	8	0
3	s^4	$A = \cancel{0}\ \epsilon$	2	4	0
4	s^3	$B = \frac{8\epsilon - 4}{\epsilon}$	$C = -\frac{8\epsilon - 8}{\epsilon}$	0	0
5	s^2	$D = 2 - \frac{\epsilon(8\epsilon - 8)}{(8\epsilon - 4)}$	4	0	0
6	s^1	E	0	0	0
7	s^0	$4E$	0	$-$	$-$

$$E = \frac{\left(2 - \frac{\epsilon(8\epsilon - 8)}{(8\epsilon - 4)}\right)\left(\frac{-(8\epsilon - 8)}{\epsilon}\right) - 4\left(\frac{8\epsilon - 4}{\epsilon}\right)}{2 - \frac{\epsilon(8\epsilon - 8)}{(8\epsilon - 4)}}$$

$$E = \frac{-16\epsilon + 16}{\epsilon} - \frac{(8\epsilon - 8)^2}{8\epsilon - 4} - \frac{\left(\frac{32\epsilon - 16}{\epsilon}\right)}{\frac{(16\epsilon - 14) - \epsilon(8\epsilon - 8)}{8\epsilon - 4}}$$

$\lim_{\epsilon \to 0^+} A = 0^+$

sign change

$\lim_{\epsilon \to 0^+} B = -\infty$

sign change

$\lim_{\epsilon \to 0^+} D = 2$

$\lim_{\epsilon \to 0^+} E = +\infty$

$\lim_{\epsilon \to 0^+} YE = +\infty$

$\lim_{\epsilon \to 0^-} = 0^-$

sign change

$\lim_{\epsilon \to 0^-} B = +\infty$

sign change

$\lim_{\epsilon \to 0^-} D = 2$

$\lim_{\epsilon \to 0^-} E = +\infty$

$\lim_{\epsilon \to 0^-} YE = +\infty$

$\boxed{\text{2 roots in the RHP}}$

Problem 2 (Routh-Hurwitz).
A system has the following characteristic polynomial,

$$s^5 + 8s^4 + 24s^3 + 32s^2 + as + ab = 0 \qquad (3)$$

where a and b are unspecified parameters. Use the Routh criterion to determind the constraints that must be imposed on the values of a and b to make the above system stable.

1	s^5	1	24	a
2	s^4	8	32	ab
3	s^3	20	$a - \frac{ab}{8}$	0
4	s^2	$32 - \frac{a+ab}{20}$	ab	0
5	s^1	Q	0	0
6	s^0	ab		

$$\left(a - \frac{ab}{8}\right) - \frac{12800ab}{a+ab}$$

$$\frac{8a - ab}{8} - \frac{1200ab}{a+ab}$$

$$(8a - ab)(a + ab) =$$
$$8a^2 + 8a^2b - a^2b - a^2b^2$$
$$= -a^2[b^2 - 7b - 8]$$

$$= \frac{-a^2[b^2 - 7b - 8] - 9600ab}{8a(b+1)}$$

$$Q \triangleq \frac{-a[b^2 - 7b - 8] - 9600b}{8(b+1)}$$

conditions!

$$\boxed{ab > 0}$$

$$\frac{640 - a(b+1)}{20} > 0 \Rightarrow \boxed{a(b+1) < 640}$$

$$Q > 0 \Rightarrow \boxed{\begin{array}{l} b \neq -1 \\ a[b^2 - 7b - 8] + 9600b < 0 \end{array}}$$

Problem 3 (Lyapunov Stability).
A. *Does the choice of* $\mathbf{Q} = \mathbf{I}$ *provide a Lyapunov function for the following system,*

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 1 \\ -4 & -1 \end{bmatrix} \mathbf{x}(t) \tag{4}$$

Solution

3 A. $Q = I$

$M = -(A^T Q + Q A)$

$A = \begin{bmatrix} -1 & 1 \\ -4 & -1 \end{bmatrix}$

$M = -\left[\begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ -4 & -1 \end{bmatrix}\right] = -\begin{bmatrix} -2 & -3 \\ -3 & -2 \end{bmatrix}$

$M = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$

$|M| = 4 - 9$
$= -5 < 0 \Rightarrow$ M is negative definite

M needs to be positive definite for Q to provide a Lyapunov function

$\therefore$ $\boxed{\text{Q = I does not provide a Lyapunov function}}$

B. Determine the stability of the above system using Routh-Hurwitz.

Solution

3 B.

$$\dot{\vec{x}}(t) = \begin{bmatrix} -1 & 1 \\ -4 & -1 \end{bmatrix} \vec{x}(t)$$

$$|(A - \lambda I)| = \left| \begin{bmatrix} -1-\lambda & 1 \\ -4 & -1-\lambda \end{bmatrix} \right|$$

$$= (-1-\lambda)(-1-\lambda) + 4$$

$$= \lambda^2 + 2\lambda + 5 = 0$$

$$\begin{array}{ccccc} 1 & \lambda^2 & 1 & 5 \\ 2 & \lambda^1 & 2 & 0 \\ 3 & \lambda^0 & 5 & 0 \end{array}$$

stable

C. Find a Lyapunov function for the above system.

Solution

3 c.

$$\dot{\vec{x}}(t) = \begin{bmatrix} -1 & 1 \\ -4 & -1 \end{bmatrix} \vec{x}(t)$$

if $M = I$

$$A^T Q + QA = -M$$

$$\begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

① $-Q_{11} - 4Q_{12} - Q_{11} - 4Q_{12} = -1 \Rightarrow 2Q_{11} + 8Q_{12} = 1$

$\boxed{Q_{11} = \frac{1}{2} - 4Q_{12}}$ ⑤

② $-Q_{12} - 4Q_{22} + Q_{11} - Q_{12} = 0 \Rightarrow 4Q_{22} + 2Q_{12} - Q_{11} = 0$

③ $Q_{11} - Q_{12} - Q_{12} - 4Q_{22} = 0 \Rightarrow 4Q_{22} + 2Q_{12} - Q_{11} = 0$

④ $Q_{12} - Q_{22} + Q_{12} - Q_{22} = -1 \Rightarrow \boxed{2Q_{22} = 1 + 2Q_{12}}$ ⑥

plug ⑤ into ② $\Rightarrow 4Q_{22} + 2Q_{12} - \frac{1}{2} + 4Q_{12} = 0$

$4Q_{22} + 6Q_{12} = \frac{1}{2}$

$8Q_{22} + 12Q_{12} = 1$ ⑦

plug ⑥ into ⑦ $\Rightarrow 4 + 8Q_{12} + 12Q_{12} = 1$

$20Q_{12} = -3$

$\boxed{Q_{12} = \frac{-3}{20}}$ ⑧

plug ⑧ into ⑥ $2Q_{22} = 1 - \frac{3}{10}$

$\boxed{Q_{22} = \frac{7}{20}}$

plug (8) into (5) $\Rightarrow$ $$Q_{11} = \frac{1}{2} - 4\left(-\frac{3}{20}\right)$$
$$= \frac{1}{2} + \frac{3}{5} = \frac{11}{10}$$

$$Q = \begin{bmatrix} \frac{11}{10} & -\frac{3}{20} \\ -\frac{3}{20} & \frac{7}{20} \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 22 & -3 \\ -3 & 7 \end{bmatrix}$$

$$|Q| = \frac{1}{20}\left(22 \times 7 - 9\right)$$
$$= \frac{1}{20}\left(154 - 9\right)$$
$$= \frac{145}{20} > 0$$

$\therefore$ $\boxed{\text{system is asymptotically stable}}$

Problem 4 (Root Locus)**.**
Consider the closed-loop system of Figure 1 where the transfer functions for the corresponding blocks are given by the following,

$$C(s) = s+4 \tag{5}$$

$$G(s) = \frac{1}{s(s+2)(s+3)(s+6)} \tag{6}$$

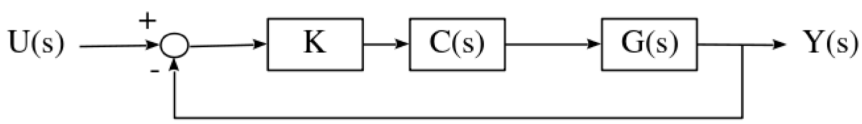
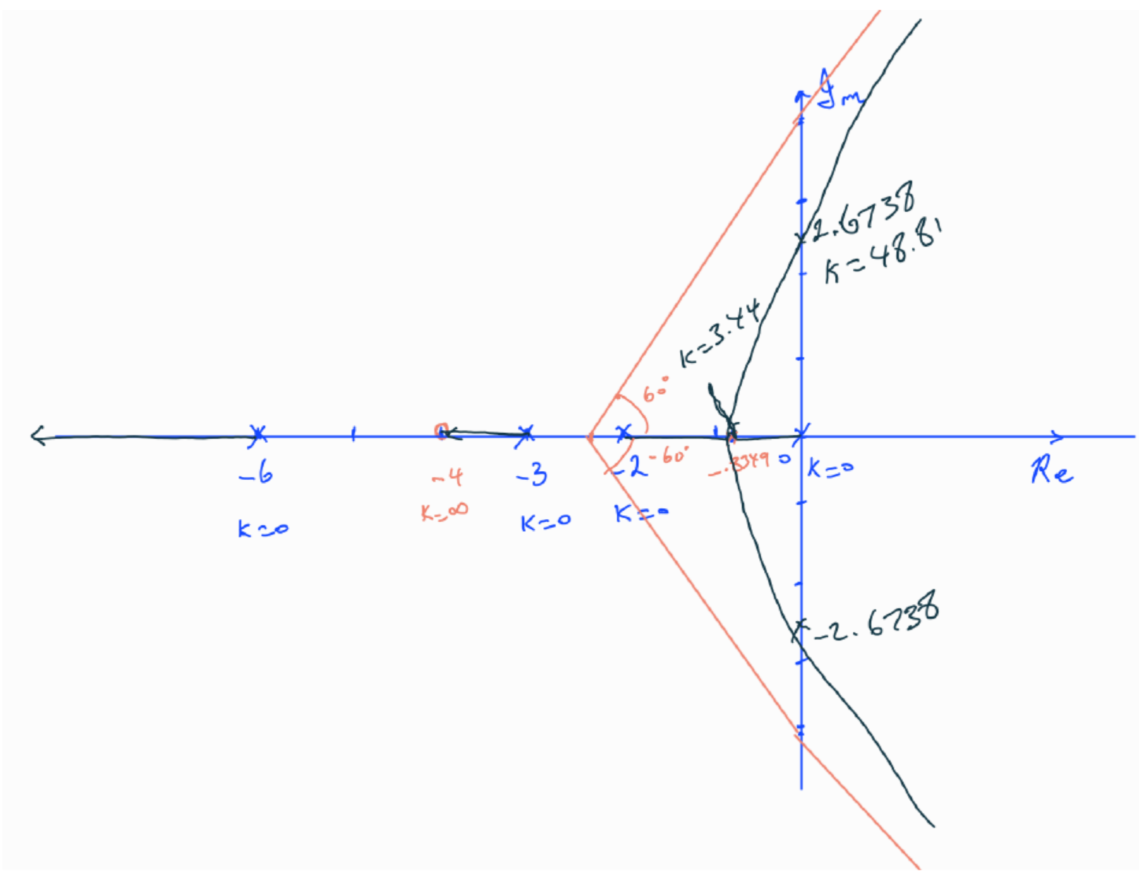


Fig. 1: Closed-Loop Block Diagram

A. *Find all the pertinent information according to rules 1 through 11 and plot the root locus according to the outcome of these rules.*

B. *Using the above application of rules, dettermine at what K the system will go unstable.*

Solution A and B



$$1 + K\frac{(s+4)}{s(s+2)(s+3)(s+6)} = 0$$

$$\left|\frac{(s+4)}{s(s+2)(s+3)(s+6)}\right| = \left|-\frac{1}{K}\right|$$

$$= \left|\frac{1}{K}\right|$$

Rule 5: Angles of the asymptotes

$$\theta_l = \frac{2l+1}{|n-m|} \times 180^\circ$$

$$l = 0,\ 1,\ \dots,\ |n-m|-1$$

$$\theta_0 = \frac{1}{3} \times 180^\circ = 60^\circ$$

$$\theta_1 = \frac{3}{3} \times 180^\circ = 180^\circ$$

$$\theta_2 = \frac{5}{3} \times 180^\circ = 300^\circ = -60^\circ$$

Rule 6: Intersection of the asymptotes with the real axis

$$\sigma = \frac{\sum \mathrm{Re}(\text{poles}) - \sum \mathrm{Re}(\text{zeros})}{n-m}$$

$$= \frac{0-2-3-6+4}{3}$$

$$= -2.3\overline{3}$$

Rule 10: Break away points

$$-K = \frac{s(s+2)(s+3)(s+6)}{s+4}$$

$$-\frac{dK}{ds} = \frac{d\left[\frac{s(s+2)(s+3)(s+6)}{s+4}\right]}{ds}$$

$$= d\left[\frac{s^4+11s^3+36s^2+36s}{s+4}\right]$$

$$= \frac{(4s^3+33s^2+72s+36)(s+4)-(s^4+11s^3+36s^2+36s)}{(s+4)^2}$$

$$= \frac{3s^4+38s^3+168s^2+288s+144}{s^2+8s+16} = 0$$

$$3s^4+38s^3+168s^2+288s+144=0$$

not for K > 0

Real roots: $\cancel{-2.5967}$ & $\boxed{-.8349}$ ✓

Breakaway point

$$|K| = \left|\frac{s(s+2)(s+3)(s+6)}{s+4}\right|_{s=-.8349}$$

$$= 3.4369$$

Imaginary axis:

$$-K = \left.\frac{s(s+2)(s+3)(s+6)}{s+4}\right|_{s=i\omega}$$

$$-K = \left.\frac{s^4+11s^3+36s^2+36s}{s+4}\right|_{s=i\omega}$$

$$-K = \frac{\left[\omega^4 - i11\omega^3 - 36\omega^2 + i36\omega\right]}{(4+i\omega)} \; \frac{(4-i\omega)}{(4-i\omega)}$$

$$= \frac{4\omega^4 - i44\omega^3 - 144\omega^2 + i144\omega - i\omega^5 - 11\omega^4 + 36i\omega^3 + 36\omega^2}{\omega^2+16}$$

$$= \frac{\left[-7\omega^4 - 108\omega^2\right] - i\left[\omega^5 + 8\omega^3 - 44\omega^2 - 144\omega\right]}{(\omega^2+16)}$$

$$= \frac{-\omega^2\left[7\omega^2 + 108\right] - i\omega\underbrace{\left[\omega^4 + 8\omega^2 - 44\omega - 144\right]}_{=0}}{\omega^2+16}$$

$$\omega^4 + 8\omega^2 - 44\omega - 144 = 0$$

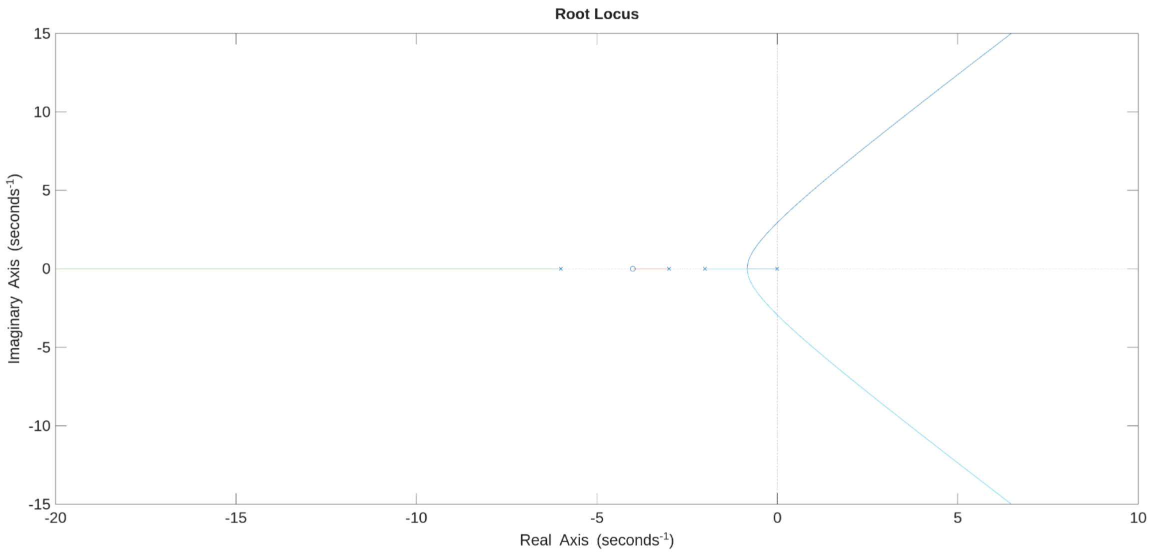
$\omega = 2.6738$ rad/s (only positive real root

$$K = \frac{(2.6738)^2\left[7(2.6738)^2 + 108\right]}{2.6738^2 + 16}$$

$$\boxed{K = 48.81}$$

C. *Plot the root locus using Matlab and compare to your hand-drawn plot.*

Solution



Problem 5 (Amplitude and Phase)**.**
Express the following solutions by combining cosine and sine pairs to be represented in terms of sine only, with an amplitude and a phase.

A.

$$y(t) = \frac{10}{13}e^{-2t} - \frac{17}{10}e^{-t} - \frac{9}{130}\cos(3t) - \frac{7}{130}\sin(3t) \tag{7}$$

Solution

$$= \frac{10}{13}e^{-2t} - \frac{17}{10}e^{-t} - \frac{1}{130}\left[9\cos 3t + 7\sin 3t\right]$$

$C = 9 \quad D = 7 \quad A = \sqrt{9^2+7^2} = 11.4$

$\frac{C}{A} > 0 \quad \frac{D}{A} > 0 \Rightarrow \varphi = \tan^{-1}\left[\frac{9}{7}\right]$

$= 0.91 \text{ rad}$

$= 52.125^\circ$

$$y(t) = \frac{10}{13}e^{-2t} - \frac{17}{10}e^{-t} - 0.0877 \sin\left(3t + 0.91\right)\Big]$$

B.

$$y(t) = \frac{103}{100}e^{-2t} + \frac{29}{10}te^{-2t} - \frac{3}{100}\cos(4t) + \frac{1}{25}\sin(4t) \tag{8}$$

Solution

$$y(t) = \frac{103}{100}e^{-2t} + \frac{29}{10}te^{-2t} - \frac{1}{25}\left[\frac{3}{4}\cos 4t - \sin 4t\right]$$

$C = \frac{3}{4}$ $D = -1$ $A = \sqrt{\left(\frac{3}{4}\right)^2 + 1^2} = 1.25$

$\frac{C}{A} > 0$ $\frac{D}{A} < 0$

$$\begin{aligned}\varphi &= \tan^{-1}\left(\frac{C}{D}\right) + \pi \\ &= \tan^{-1}\left(-\frac{3}{4}\right) + \pi \\ &= 2.5 \text{ rad} \\ &= 143^\circ\end{aligned}$$

$$y(t) = \frac{103}{100}e^{-2t} + \frac{29}{10}te^{-2t} - \frac{5}{100}\sin(4t + 2.5)$$

c.

$$y(t) = 1 - \frac{35}{1261}\cos(6t) + \frac{6}{1261}\sin(6t) - \frac{2}{1261}e^{-\frac{t}{2}}\left(613\cos(\frac{\sqrt{3}}{2}t) + \frac{1262}{\sqrt{3}}\sin(\frac{\sqrt{3}}{2}t))\right) \tag{9}$$

Solution

$$= 1 - \frac{1}{1261}\left[35\cos 6t + 6\sin 6t\right] - \frac{2}{1261}e^{-\frac{t}{2}}\left[613\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1262}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$$

$35\cos 6t + 6\sin 6t$:

$C = 35 \quad D = 6 \quad A = \sqrt{35^2+6^2} = 35.5$

$\frac{C}{A} > 0 \quad \frac{D}{A} > 0 \Rightarrow \varphi = \tan^{-1}\left(\frac{C}{D}\right) = 1.4\ \text{rad.}$
$= 80.3°$

$613\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1262}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)$:

$C = 613 \quad D = \frac{1262}{\sqrt{3}} \quad A = \sqrt{613^2 + \frac{1262^2}{3}} = 952.2$

$\frac{C}{A} > 0 \quad \frac{D}{A} > 0 \quad \varphi = \tan^{-1}\left(\frac{C}{D}\right) = \tan^{-1} 0.84$
$= 0.7\ \text{rad}$
$= 40°$

$$y(t) = 1 - \frac{1}{1261}\left[35.5\sin(6t + 1.4)\right] - 1.51\sin\left(\frac{\sqrt{3}}{2}t + 0.7\right)$$

D.

$$y(t) = 1 + \frac{1}{5}e^{-2t} - \frac{6}{5}\cos(t) + \frac{2}{5}\sin(t) \tag{10}$$

Solution

$$= 1 + \frac{1}{5}e^{-2t} - \frac{3}{5}\left[2\cos t + \sin t\right]$$

$C = 2 \quad D = 1 \quad A = \sqrt{2^2+1} = \sqrt{5}$

$\frac{C}{A} > 0 \quad \frac{D}{A} > 0 \Rightarrow \varphi = \tan^{-1}\left(\frac{C}{D}\right) = \tan^{-1}(2)$

$= 1.1$ rad

$= 63.43^\circ$

$$\boxed{y(t) = 1 + \frac{1}{5}e^{-2t} - 1.34\sin(t + 1.1)}$$