INTRODUCTION TO CONTINUOUS CONTROL SYSTEMS

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Homework 9

Problem 1 (Hurwitz Criterion)**.**

Using the Hurwitz Criterion, determine whether the following two characteristic equations describe asymptotically stable systems.
A.

$$P(s) = s^3 + 2s^2 + s + 3 \tag{1}$$

Solution

$$P_{n-1} = 2 \tag{2}$$
$$P_{n-2} = 1 \tag{3}$$
$$P_{n-3} = 3 \tag{4}$$
$$\tag{5}$$

$$\begin{vmatrix} 2 \end{vmatrix} > 0 \tag{6}$$

$$\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -3 < 0 \tag{7}$$

Therefore the system is Unstable.

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B.

$$P(s) = s^4 + 2s^3 + 2s^2 + 4s + 2 \tag{8}$$

Solution

$$P_{n-1} = 2 \tag{9}$$
$$P_{n-2} = 2 \tag{10}$$
$$P_{n-3} = 4 \tag{11}$$
$$P_{n-4} = 2 \tag{12}$$
$$\tag{13}$$

$$\begin{vmatrix} 2 \end{vmatrix} > 0 \tag{14}$$

$$\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = 0 \tag{15}$$

Therefore the system is not asymptotically syable. It is Marginally Stable.

Problem 2 (Eigensystem)**.**

Solve the following equation by using the Eigenvalue-Eigenvector approach:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 3 & -4 \\ 1 & -2 \end{bmatrix} \mathbf{x}(t) \tag{16}$$
$$y(t) = [1 \ \ 0] \, \mathbf{x}(t) \tag{17}$$

where the initial conditions are given by,

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{18}$$

Solution

$$\mathbf{A} = \begin{bmatrix} 3 & -4 \\ 1 & -2 \end{bmatrix} \tag{19}$$
$$\mathbf{A} - \lambda \mathbf{I} = \begin{bmatrix} 3-\lambda & -4 \\ 1 & -2-\lambda \end{bmatrix} \tag{20}$$

$$
\begin{aligned}
|\mathbf{A}-\lambda\mathbf{I}| &= (3-\lambda)(-2-\lambda)+4 && (21)\\
&= -6-3\lambda+2\lambda+\lambda^2+4 && (22)\\
&= \lambda^2-\lambda-2 && (23)\\
& && (24)
\end{aligned}
$$

$$
\begin{aligned}
\lambda_{1,2} &= \frac{1\pm\sqrt{1+8}}{2} && (25)\\
&= 2,-1 && (26)
\end{aligned}
$$

$$
(\mathbf{A}-2\mathbf{I})\vec{v}_1 = \begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix}\vec{v}_1 \tag{27}
$$

if $v_{1,1}=1 \implies 1-4v_{1,2}=0,\ v_{1,2}=\frac{1}{4}$

$$
\boxed{\lambda_1 = 2 \quad \vec{v}_1 = \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}} \tag{28}
$$

$$
\begin{aligned}
(\mathbf{A}-(-1)\mathbf{I})\vec{v}_2 &= \left[\begin{bmatrix} 3 & -4 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]\vec{v}_2 && (29)\\
&= \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix}\vec{v}_2 && (30)\\
&= \vec{0} && (31)
\end{aligned}
$$

$$
-v_{2,1}+v_{2,2}=0 \tag{32}
$$

if $v_{2,1}=1 \implies v_{2,2}=1$

$$
\boxed{\lambda_2 = -1 \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \tag{33}
$$

$$
\begin{aligned}
\mathbf{M} &= \left[\vec{v}_1|\vec{v}_2\right] && (34)\\
&= \begin{bmatrix} 1 & 1 \\ \frac{1}{4} & 1 \end{bmatrix} && (35)
\end{aligned}
$$

$$
\mathbf{M}^{-1} = \frac{1}{3}\begin{bmatrix} 4 & -4 \\ -1 & 4 \end{bmatrix} \tag{36}
$$

$$
\vec{x}(t) = C_1\begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}e^{2t} + C_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-t} \tag{37}
$$

$$\begin{aligned}
\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} &= \mathbf{M}^{-1}\vec{x}(0) && (38)\\
&= \frac{1}{3}\begin{bmatrix} 4 & -4 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} && (39)\\
&= \frac{1}{3}\begin{bmatrix} 0 \\ 3 \end{bmatrix} && (40)\\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix} && (41)
\end{aligned}$$

$$\begin{aligned}
\vec{x}(t) &= \mathbf{M}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix}\mathbf{M}^{-1}\vec{x}_0 && (42)\\
&= \begin{bmatrix} 1 & 1 \\ \frac{1}{4} & 1 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} && (43)\\
&= \begin{bmatrix} e^{2t} & e^{-t} \\ \frac{1}{4}e^{2t} & e^{-t} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} && (44)\\
&= \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix} && (45)
\end{aligned}$$

$$\begin{aligned}
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix}\vec{x}(t) && (46)\\
&= e^{-t} && (47)
\end{aligned}$$