INTRODUCTION TO CONTINUOUS CONTROL SYSTEMS
COLUMBIA UNIVERSITY MECHANICAL AND ELECTRICAL ENGINEERING DEPARTMENTS: E3601

Homayoon Beigi†

1340 Mudd Building
Columbia University, New York City, NY 10027
hb87@columbia.edu

Homework 7

Problem 1 (Characteristic Equation and Eigenvalues)**.**

Write the characteristic equations, Eigenvalues, and Eigenvectors of the following matrices.

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 4 & 9 \end{bmatrix} \tag{1}$$

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 5 \\ 0 & -7 & 3 \\ -1 & -6 & -2 \end{bmatrix} \tag{2}$$

solution

$$\begin{aligned} |\mathbf{A} - \lambda \mathbf{I}| &= \begin{vmatrix} 2-\lambda & 1 \\ 4 & 9-\lambda \end{vmatrix} \\ &= (2-\lambda)(9-\lambda) - 4 \\ &= \lambda^2 - 11\lambda + 18 - 4 \\ &= \lambda^2 - 11\lambda + 14 \\ &= 0 \end{aligned} \tag{3}$$

$$\begin{aligned} \lambda_{1,2} &= \frac{11 \pm \sqrt{121-56}}{2} \\ &= \frac{11 \pm 8.0623}{2} \\ &= 9.53, 1.47 \end{aligned} \tag{4}$$

†Homayoon Beigi is Professor of Professional Pratice in the department of mecanical engineering and in the department of electrical engineering at Columbia University

$A = \begin{bmatrix} 2 & 1 \\ 4 & 9 \end{bmatrix}$ $\lambda_{1,2} = 9.53, 1.47$

$\{A - \lambda_1 I\} \vec{r}_1 = 0$

$\begin{bmatrix} 2-9.53 & 1 \\ 4 & 9-9.53 \end{bmatrix} \vec{r}_1 = \begin{bmatrix} -7.53 & 1 \\ 4 & -.53 \end{bmatrix} \begin{bmatrix} r_{1,1} \\ r_{1,2} \end{bmatrix} = 0$

$-7.53 r_{1,1} + r_{1,2} = 0$ if $r_{1,1} = 1 \Rightarrow r_{1,2} = 7.53$

$\lambda_1 = 9.53$ $\vec{r}_1 = \begin{bmatrix} 1 \\ 7.53 \end{bmatrix}$

$\{A - \lambda_2 I\} \vec{r}_2 = 0$

$\begin{bmatrix} 2-1.47 & 1 \\ 4 & 9-1.47 \end{bmatrix} \vec{r}_2 = \begin{bmatrix} 0.53 & 1 \\ 4 & 7.53 \end{bmatrix} \begin{bmatrix} r_{2,1} \\ r_{2,2} \end{bmatrix} = 0$

$0.53 r_{2,1} + r_{2,2} = 0$ if $r_{2,1} = 1 \Rightarrow r_{2,2} = -0.53$

$\lambda_2 = 1.47$ $\vec{r}_2 = \begin{bmatrix} 1 \\ -0.53 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & -1 & 5 \\ 0 & -7 & 3 \\ -1 & -6 & -2 \end{bmatrix} \qquad |A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 & 5 \\ 0 & -7-\lambda & 3 \\ -1 & -6 & -2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(-7-\lambda)(-2-\lambda) + 18\right] + 1\left[0 + 3\right] + 5\left\{0 - (7+\lambda)\right\} = 0$$

$$(1-\lambda)\left[(\lambda+7)(\lambda+2) + 18\right] + 3 - 5\lambda - 35 = 0$$

$$(1-\lambda)\left\{\lambda^2 + 9\lambda + 14 + 18\right] - 5\lambda - 32 = 0$$

$$(1-\lambda)\left[\lambda^2 + 9\lambda + 32\right] - 5\lambda - 32 = 0$$

$$\lambda^2 + 9\lambda + 32 - \lambda^3 - 9\lambda^2 - 32\lambda - 5\lambda - 32 = 0$$

$$-\lambda^3 - 8\lambda^2 - 28\lambda = 0$$

$$\lambda\left[\lambda^2 + 8\lambda + 28\right] = 0 \qquad \lambda_1 = 0$$

$$\lambda_{2,3} = \frac{-8 \pm \sqrt{64 - 112}}{2}$$

$$= -4 \pm 2\sqrt{3}$$

$$\boxed{\lambda_{1,2,3} = 0,\ -4 \pm 2\sqrt{3}\, i}$$

48	2
24	2
12	2
6	2
3	3
1	

$$A = \begin{bmatrix} 1 & -1 & 5 \\ 0 & -7 & 3 \\ -1 & -6 & -2 \end{bmatrix} \qquad \lambda_{1,2,3} = 0,\ -4 \pm 2\sqrt{3}\, i$$

$\lambda_1 = 0$

$$(A - \lambda_1 I)\vec{r}_1 = 0$$

$$\begin{bmatrix} 1 & -1 & 5 \\ 0 & -7 & 3 \\ -1 & -6 & -2 \end{bmatrix} \begin{bmatrix} r_{11} \\ r_{12} \\ r_{13} \end{bmatrix} = 0$$

① $r_{11} - r_{12} + 5 r_{13} = 0$

② $-7 r_{12} + 3 r_{13} = 0 \Rightarrow r_{12} = \frac{3}{7} r_{13}$

③ $r_{11} + 6 r_{12} + 2 r_{13} = 0$

$$③ - ① = 6 r_{12} + 2 r_{13} + r_{12} - 5 r_{13}$$
$$= 7 r_{12} - 3 r_{13} = 0 \Rightarrow r_{12} = \frac{3}{7} r_{13}$$

plug in r_{12} in ①: $r_{11} - \frac{3}{7} r_{13} + \frac{35}{7} r_{13} = 0$

if $\boxed{r_{11} = 1} \Rightarrow 1 + \frac{32}{7} r_{13} = 0$

$$\boxed{r_{13} = -\frac{7}{32}}$$

$$r_{12} = \frac{3}{7} r_{13} = \frac{3}{7}\left(-\frac{7}{32}\right) = -\frac{3}{32}$$

$$\boxed{\lambda = 0, \quad \vec{r}_1 = \begin{bmatrix} 1 \\ -\frac{3}{32} \\ -\frac{7}{32} \end{bmatrix}}$$

$r_{11} - r_{12} + 5 r_{13} = 0$

$-7 r_{12} + 3 r_{13} = 0$

$-r_{11} - 6 r_{12} - 2 r_{13} = 0$

$-7 r_{12} + 3 r_{13} = 0$

$\boxed{r_{12} = \frac{3}{7} r_{13}}$

$1 - \frac{3}{7} r_{13} + 5 r_{13} = 0$

$1 + \frac{32}{7} r_{13} = 0$

$\boxed{r_{13} = -\frac{7}{32}}$

$r_{12} = \frac{3}{7}\left(-\frac{7}{32}\right) = -\frac{3}{32}$

same outcome

$$A = \begin{bmatrix} 1 & -1 & 5 \\ 0 & -7 & 3 \\ -1 & -6 & -2 \end{bmatrix} \qquad \lambda_{1,2,3} = 0,\ -4 \pm 2\sqrt{3}\,i$$

$$(A - \lambda_2 I)\vec{r}_2 = 0$$

$$= \begin{bmatrix} 1+4-2\sqrt{3}i & -1 & 5 \\ 0 & -7+4-2\sqrt{3}i & 3 \\ -1 & -6 & -2+4-2\sqrt{3}i \end{bmatrix} \begin{bmatrix} r_{21} \\ r_{22} \\ r_{23} \end{bmatrix} = \begin{bmatrix} 5-2\sqrt{3}i & -1 & 5 \\ 0 & -3-2\sqrt{3}i & 3 \\ -1 & -6 & 2-2\sqrt{3}i \end{bmatrix} \begin{bmatrix} r_{21} \\ r_{22} \\ r_{23} \end{bmatrix}$$

$$-3-2\sqrt{3}\,i\, r_{22} = -3 r_{23} \Rightarrow r_{23} = \left[1 + \frac{2}{3}\sqrt{3}i\right] r_{22}$$

$$-r_{21} - 6r_{22} + (2 - 2\sqrt{3}i)\, r_{23} = 0$$

$$\text{if } \boxed{r_{21} = 1} \Rightarrow -1 - 6r_{22} + (2 - 2\sqrt{3}i)\left(1 + \frac{2}{3}\sqrt{3}i\right) r_{22} = 0$$

$$-1 - 6r_{22} + \left[2 + \frac{4}{3}\sqrt{3}i - \frac{6}{3}\sqrt{3}i + 4\right] r_{22} = 0$$

$$-1 - \cancel{6r_{22}} + \cancel{6r_{22}} - \frac{2}{3}\sqrt{3}\,i\, r_{22} = 0$$

$$-\frac{2}{3}\sqrt{3}\,i\, r_{22} = 1$$

$$r_{22} = -\frac{1}{\frac{2}{3}\sqrt{3}i}$$

$$= -\frac{3}{2\sqrt{3}i} \cdot \frac{\sqrt{3}i}{\sqrt{3}i} = \cancel{-}\frac{\cancel{3}\sqrt{3}i}{\cancel{-6}\ 2}$$

$$\boxed{r_{22} = \frac{\sqrt{3}i}{2}}$$

$$r_{23} = \left[1 + \frac{2}{3}\sqrt{3}i\right] r_{22}$$

$$= \left[1 + \frac{2}{3}\sqrt{3}i\right] \frac{\sqrt{3}i}{2}$$

$$\boxed{r_{23} = -1 + \frac{\sqrt{3}}{2}i}$$

$$\boxed{\lambda_2 \quad -4 + 2\sqrt{3}i}$$

$$\boxed{\vec{r}_2 = \begin{bmatrix} 1 \\ \frac{\sqrt{3}i}{2} \\ -1 + \frac{\sqrt{3}}{2}i \end{bmatrix}}$$

$$\lambda_3 = \overline{\lambda_2} = -4 - 2\sqrt{3}\,i$$

$$\vec{v}_3 = \overline{\vec{v}_2} = \begin{bmatrix} 1 \\ -\frac{\sqrt{3}}{2} i \\ -1 - \frac{\sqrt{3}}{2} i \end{bmatrix}$$

Problem 2 (Similarity Transform)**.**

Find the Eigenvalues and Eigenvectors of the following matrix and convert the matrices into diagonal or block diagonal form, whichever is approprate.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} \tag{3}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \tag{4}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 0 \\ 2 & -4 & 3 \end{bmatrix} \tag{5}$$

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} \quad |A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 5 & 4-\lambda \end{vmatrix}$$

$$= (1-\lambda)(4-\lambda) - 10$$

$$= \lambda^2 - 5\lambda + 4 - 10$$

$$= \lambda^2 - 5\lambda - 6$$

$$= 0$$

$$\lambda_{1,2} = \frac{5 \pm \sqrt{25+24}}{2}$$

$$= \frac{5 \pm \sqrt{49}}{2}$$

$$= \frac{5 \pm 7}{2}$$

$$= 6, -1$$

$$\lambda_1 = 6 \Rightarrow (A - \lambda_1 I)\vec{v}_1 = \begin{bmatrix} -5 & 2 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$$

if $v_{11} = 1 \Rightarrow v_{12} = 5/2$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 5/2 \end{bmatrix}$$

$$\lambda_2 = -1 \Rightarrow (A - \lambda I)\vec{v}_2 = \begin{bmatrix} 2 & 2 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix}$$

if $v_{21} = 1 \Rightarrow v_{22} = -v_{21} = -1 \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\lambda_1 = 6 \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 5/2 \end{bmatrix}$$

$$\lambda_2 = -1 \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 1 \\ 5/2 & -1 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} -.28571 & .28571 \\ .71429 & -0.28571 \end{bmatrix}$$

$$\tilde{A} = M^{-1} A M$$

$$= \begin{bmatrix} -.28571 & .28571 \\ .71429 & -.28571 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 5/4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \quad \left|(A - \lambda I)\right| = \left|\begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}\right|$$

$$= (1-\lambda)(4-\lambda) - 4$$

$$= \lambda^2 - 5\lambda + 4 - 4$$

$$= \lambda(\lambda - 5)$$

$$\lambda_{1,2} = 0, 5$$

$$\lambda_1 = 0 \quad [A - \lambda_1 I]\vec{v}_1 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \vec{0}$$

$$v_{11} + 2v_{12} = 0$$

$$\text{if } v_{11} = 1 \Rightarrow v_{12} = -\frac{1}{2}$$

$$\lambda_1 = 0 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$$

$$\lambda_2 = 5 \quad (A - \lambda_2 I)\vec{v}_2 = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$$

$$2v_{21} = v_{22}$$

$$\text{if } v_{21} = 1 \Rightarrow v_{22} = 2$$

$$\lambda_2 = 5 \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 1 \\ -\frac{1}{2} & 2 \end{bmatrix} \quad M^{-1} = \begin{bmatrix} .8 & -.4 \\ .2 & .4 \end{bmatrix}$$

$$\tilde{A} = M^{-1} A M = \begin{bmatrix} .8 & -.4 \\ .2 & .4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -\frac{1}{2} & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 0 \\ 2 & -4 & 3 \end{bmatrix} \qquad |(A - \lambda I)| = \begin{vmatrix} 1-\lambda & 2 & 0 \\ -1 & 3-\lambda & 0 \\ 2 & -4 & 3-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left[(3-\lambda)^2\right] - 2\left[\lambda - 3\right]$$

$$= \lambda^3 - 7\lambda^2 + 17\lambda - 15$$

$$\lambda_{1,2,3} = 3,\ 2 \pm 1i$$

$$\lambda_1 = 3 \quad (A - \lambda_1 I)\vec{r}_1 = \begin{bmatrix} -2 & 2 & 0 \\ -1 & 0 & 0 \\ 2 & -4 & 0 \end{bmatrix} \begin{pmatrix} r_{11} \\ r_{12} \\ r_{13} \end{pmatrix} = 0$$

$$\Rightarrow \vec{r}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 2 + i \quad (A - \lambda_2 I)\vec{r}_2 = \begin{bmatrix} -1-i & 2 & 0 \\ -1 & 1-i & 0 \\ 2 & -4 & 1-i \end{bmatrix} \begin{pmatrix} r_{21} \\ r_{22} \\ r_{23} \end{pmatrix}$$

$$2r_{22} = (1+i)r_{21} \Rightarrow r_{22} = \frac{(1+i)r_{21}}{2}$$

$$(1-i)r_{22} = r_{21} \Rightarrow r_{22} = \frac{(1+i)r_{21}}{2}$$

$$2r_{21} - 2(1+i)r_{21} + (1-i)r_{23} = 0$$

$$\text{if } r_{21} = 1 \Rightarrow 2 - 2(1+i) + (1-i)r_{23} = 0$$

$$(1-i)r_{23} - 2i = 0$$

$$r_{23} = \frac{2i}{1-i} \cdot \frac{1+i}{1+i}$$

$$= \frac{-2+2i}{2}$$

$$= -1+i$$

$$\lambda_2 = 2 + i \quad \vec{r}_2 = \begin{pmatrix} 1 \\ \frac{1+i}{2} \\ -1+i \end{pmatrix}$$

$$\lambda_3 = \bar{\lambda}_2 = 2 - i \quad \vec{r}_3 = \bar{\vec{r}}_2 = \begin{pmatrix} 1 \\ \frac{1-i}{2} \\ -1-i \end{pmatrix}$$

$$\tilde{M} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & \frac{1+i}{2} & \frac{1-i}{2} \\ 1 & -1+i & -1-i \end{bmatrix} \qquad u_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad v = \begin{bmatrix} 1 \\ 1/2 \\ -1 \end{bmatrix} \quad w = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

$$M = [u_1 \mid v \mid w] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} 2 & -2 & 1 \\ 1 & 0 & 0 \\ -1 & 2 & 0 \end{bmatrix}$$

$$\tilde{A} = M^{-1} A M$$

$$= \begin{bmatrix} 2 & -2 & 1 \\ 1 & 0 & 0 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 0 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

Problem 3. *Prove the following Theorem:*

Theorem 1 (Complex Conjugate Eigenvalues)**.** *Suppose* **A** *has the following eigenvalues,*

$$\lambda_i = \sigma_i + i\omega_i \quad (6)$$
$$\lambda_{i+1} = \sigma_i - i\omega_i = \bar{\lambda}_i \quad (7)$$
$$(8)$$

for $i = \{1, 3, 5, \cdots, m-1\}$ *and*

$$\lambda_i = \bar{\lambda}_i \quad (9)$$

for $i = \{m+1, m+2, \cdots, n\}$
and a linealy independent set of eigenvectors

$$\mathbf{u}_i = \mathbf{v}_i + i\mathbf{w}_i \quad (10)$$
$$\mathbf{u}_{i+1} = \mathbf{v}_i - i\mathbf{w}_i = \bar{\mathbf{u}}_i \quad (11)$$
$$(12)$$

for $i = \{1, 3, 5, \cdots, m-1\}$ *and*

$$\mathbf{u}_i = \bar{\mathbf{u}}_i \quad (13)$$

for $i = \{m+1, m+2, \cdots, n\}$
The, the real-valued matrix,

$$\mathbf{U} = [\mathbf{v}_1 \ \ \mathbf{w}_1 \ \ \mathbf{v}_3 \ \ \mathbf{w}_3 \ \ \cdots \ \ \mathbf{v}_{m-1} \ \ \mathbf{w}_{m-1} \ \ \mathbf{u}_{m+1} \ \ \mathbf{u}_{n-1} \ \ \cdots \ \ \mathbf{u}_n] \quad (14)$$

is nonsingular and may be used to transform **A** *into the block-diagram form,*

$$\mathbf{U}^{-1}\mathbf{A}\mathbf{U} = \begin{bmatrix} \mathbf{\Lambda}_1 & 0 & \cdots & 0 & 0 \\ 0 & \mathbf{\Lambda}_3 & \cdots & 0 & 0 \\ \ldots & \ldots & \ldots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{\Lambda}_{m-1} & 0 \\ 0 & 0 & 0 & \cdots & \mathbf{\Lambda}_{m+1} \end{bmatrix} \quad (15)$$

where,

$$\mathbf{\Lambda}_i = \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix} \quad (16)$$

for $i = \{1, 3, 5, \cdots, m-1\}$ *and*

$$\mathbf{\Lambda}_{m+1} = \begin{bmatrix} \lambda_{m+1} & 0 & \cdots & 0 & 0 \\ 0 & \lambda_{m+2} & \cdots & 0 & 0 \\ \ldots & \ldots & \ldots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \quad (17)$$

This result also holds for non-distinct eigenvalues, provided that the eigenvectors are linearly independent.

Hint:
First show that

$$\mathbf{A}\mathbf{v}_i = \sigma_i \mathbf{v}_i - \omega_i \mathbf{w}_i \quad (18)$$
$$\mathbf{A}\mathbf{w}_i = \omega_i \mathbf{v}_i + \sigma_i \mathbf{w}_i \quad (19)$$

$A u_i = \lambda_i u_i \qquad \lambda = \sigma + i\omega$

$u_i = v_i + i w_i$

$A(v_i + i w_i) = (\sigma_i + i\omega_i)(v_i + i w_i)$ match real & imaginary parts

$A(v_i + i w_i) = \sigma_i v_i + i\sigma_i w_i + i\omega_i v_i - \omega_i w_i$

Real: $A v_i = \sigma_i v_i - \omega_i w_i$

Imag.: $A w_i = \omega_i v_i + \sigma_i w_i$

for $i = \{1, 3, 5, \cdots, m-1\}$ *and then proceed as if you have all diagonal elements.*

matrix form

$$A \begin{bmatrix} v_i & w_i \end{bmatrix} = \begin{bmatrix} v_i & w_i \end{bmatrix} \underbrace{\begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix}}_{\to \Lambda_i} \quad i = 1, 3, \cdots, m-1$$

for real Eigenvalues, $A v_i = v_i \lambda_i$

$\therefore$ For each partition we can write the above resulting in

$$A \left[\vec{v}_1 \ \vec{w}_1 \,\vdots\, \vec{v}_3 \ \vec{w}_3 \,\vdots\, \cdots \,\vdots\, \vec{v}_{m-1} \ \vec{w}_{m-1} \,\vdots\, \vec{u}_{m+1} \,\vdots \cdots \vdots\, \vec{u}_{m-1} \,\vdots\, u_m \right] = \left[\vec{v}_1 \ \vec{w}_1 \,\vdots\, \vec{v}_3 \ \vec{w}_3 \,\vdots\, \cdots \,\vdots\, \vec{v}_{m-1} \ \vec{w}_{m-1} \,\vdots\, \vec{u}_{m+1} \,\vdots \cdots \vdots\, u_{m-1} \,\vdots\, u_m \right] \begin{bmatrix} \Lambda_1 & & & 0 \\ & \Lambda_3 & & \\ & & \ddots & \\ 0 & & & \lambda_{m+1} \\ & & & & \lambda_{m-1} \\ & & & & & \lambda_m \end{bmatrix}$$

Each of the above partitions may be multiplied independently.

Problem 4. *If* $\mathbf{A} : \mathscr{R}^n \mapsto \mathscr{R}^n$ *and* $m \geq n$*, show that* $\mathbf{A}^m$ *may be written as,*

$$\mathbf{A}^m = \lambda_0 \mathbf{I} + \lambda_1 \mathbf{A} + \lambda_2 \mathbf{A}^2 + \cdots + \lambda_{n-1} \mathbf{A}^{n-1} \tag{20}$$

for some coefficients λ_i.

Hint:

Use the Cayley-Hamilton theorem recursively.

Characteristic eqn:

$$s^n + \alpha_{n-1} s^{n-1} + \dots + \alpha_0 = 0$$

By Cayley Hamilton,

$$A^n + \alpha_{n-1} A^{n-1} + \dots + \alpha_0 I = 0$$

If m = n then we may solve for $A^m = A^n$ in terms of powers of A^{n-1} and lower

If m = n+1 then A^m is written in terms of powers of $A^{n-1} \to A^0$

$$A^n = \sum_{i=0}^{n-1} -\alpha_i A^i$$

$$A^{n+1} = A \sum_{i=0}^{n-1} -\alpha A^i$$

$$= A\left[-\alpha_0 I - \alpha_1 A \dots - \alpha_{n-1} A^{n-1}\right]$$

$$= -\alpha_0 A - \alpha_1 A^2 - \dots \alpha_{n-2} A^{n-1} + \alpha_{n-1} A^n$$

but $A^n = \sum_{i=0}^{n-1} -\alpha_i A^i \Rightarrow A^{n+1} = \sum_{i=0}^{n-1} \hat{\alpha}_i A^i$

By the same token

$$A^{n+2} = A A^{n+1} = \sum_{i=0}^{n-1} \hat{\hat{\alpha}}_i A^{i}$$

Continue to any $m \geq n$