INTRODUCTION TO CONTINUOUS CONTROL SYSTEMS
COLUMBIA UNIVERSITY MECHANICAL AND ELECTRICAL ENGINEERING DEPARTMENTS: E3601

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Homework 6

Problem 1 (Laplace Transform). *Show the following:*

$$\int_0^\infty t^2 e^{-at} e^{-st} dt = \frac{2}{(s+a)^3} \tag{1}$$

Solution

Using integration by parts, with,

$$\begin{aligned} f &= t^2 \\ g' &= e^{-(s+a)t} \\ g &= -\frac{1}{(s+a)} e^{-(s+a)t} \\ f' &= 2t \end{aligned}$$

then the integral may be written as,

$$\int_0^\infty t^2 e^{-at} e^{-st} dt \tag{2}$$

$$= [fg]_0^\infty - \int_0^\infty f' g \tag{3}$$

$$= \left[-\frac{t^2}{(s+a)} e^{-(s+a)t} \right]_0^\infty - \int_0^\infty 2t \left(-\frac{1}{(s+a)} e^{-(s+a)t} \right) dt \tag{4}$$

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$$= \frac{2}{(s+a)} \int_0^\infty t e^{-(s+a)t} dt \tag{5}$$

Now, we can use integration by parts again using the following definitions,

$$\begin{aligned} f &= t \\ g' &= e^{-(s+a)t} \\ f' &= 1 \\ g &= -\frac{1}{(s+a)} e^{-(s+a)t} \end{aligned}$$

Then,

$$\frac{2}{(s+a)} \int_0^\infty t e^{-(s+a)t} dt \tag{6}$$

$$= \frac{2}{(s+a)} \{ \left[-\frac{t}{(s+a)} e^{-(s+a)t} \right]_0^\infty - \int_0^\infty 1 \left(-\frac{1}{(s+a)} e^{-(s+a)t} \right) dt \} \tag{7}$$

$$= \frac{2}{(s+a)^2} \left[\int_0^\infty e^{-(s+a)t} dt \right] \tag{8}$$

$$= \frac{2}{(s+a)^2} \left[-\frac{1}{(s+a)} e^{-(s+a)t} \right]_0^\infty \tag{9}$$

$$= \frac{2}{(s+a)^3} \tag{10}$$

Problem 2 (Transfer Function)**.** *Consider,*

$$\begin{aligned} X(s) &= \frac{{\omega_n}^2}{s(s^2 + 2\zeta\omega_n s + {\omega_n}^2)} \\ &= \frac{K_0}{s} + \frac{K_{-\alpha+i\omega}}{s+\alpha-i\omega} + \frac{K_{-\alpha-i\omega}}{s+\alpha+i\omega} \end{aligned} \tag{11}$$

We showed that

$$K_{-\alpha+i\omega} = [(s+\alpha-i\omega)X(s)]_{s=-\alpha+i\omega} = \frac{{\omega_n}^2}{2i\omega(-\alpha+i\omega)} \tag{12}$$

Please show that

$$K_{-\alpha+i\omega} = \frac{\omega_n}{2\omega} e^{-i(\theta+\frac{\pi}{2})} \tag{13}$$

where $\theta = tan^{-1}(\frac{\omega}{-\alpha})$

Solution

$$\begin{aligned}
\frac{{\omega_n}^2}{2i\omega(-\alpha+i\omega)} &= \frac{{\omega_n}^2}{-2\omega(\omega+i\alpha)} \\
&= \frac{{\omega_n}^2}{2\omega}.\frac{1}{-(\omega+i\alpha)}.\frac{\omega-i\alpha}{\omega-i\alpha} \\
&= \frac{{\omega_n}^2}{2\omega}.\frac{\omega-i\alpha}{-(\omega^2+\alpha^2)}
\end{aligned} \tag{14}$$

However,

$$\alpha = \zeta\omega_n \tag{15}$$

and

$$\omega \triangleq \omega_n\sqrt{1-\zeta^2} \tag{16}$$

Therefore,

$$\begin{aligned}
\omega^2 &= {\omega_n}^2(1-\zeta^2) \\
&= {\omega_n}^2 - {\omega_n}^2\zeta^2 \\
&= {\omega_n}^2 - \alpha^2
\end{aligned} \tag{17}$$

or,

$${\omega_n}^2 = \omega^2 + \alpha^2 \tag{18}$$

Using equation 18 in equation 14 we have,

$$\begin{aligned}
\frac{{\omega_n}^2}{2i\omega(-\alpha+i\omega)} &= \frac{\omega_n}{2\omega}.\frac{(\omega-i\alpha)}{-\sqrt{\omega^2+\alpha^2}} \\
&= \frac{\omega_n}{2\omega}.\frac{(-\omega+i\alpha)}{\sqrt{\omega^2+\alpha^2}} \qquad (19)
\end{aligned}$$

(20)

Now, let's define θ as prescribed in the problem statement,

$$\begin{aligned}
\theta &\triangleq tan^{-1}\left(\frac{\omega}{-\alpha}\right) \\
\Rightarrow &\quad \sin(\theta) = \frac{\omega}{\sqrt{(\omega^2+\alpha^2)}} \qquad (21) \\
&\quad \cos(\theta) = \frac{-\alpha}{\sqrt{(\omega^2+\alpha^2)}} \qquad (22)
\end{aligned}$$

(23)

Problem 3 (Temporal Response)**.** *Show that equation 24 may be rewritten as 25:*

$$x(t) = u(t) + \frac{\omega_n}{2\omega}\left(e^{-i(\theta+\frac{\pi}{2})}e^{(-\alpha+i\omega)t} + e^{i(\theta+\frac{\pi}{2})}e^{(-\alpha-i\omega)t}\right) \tag{24}$$

$$x(t) = u(t) + \frac{\omega_n}{\omega}e^{-\alpha t} sin(\omega t - \theta) \tag{25}$$

Solution

$$\begin{aligned} e^{-i(\theta+\frac{\pi}{2})}e^{(-\alpha+i\omega)t} + e^{i(\theta+\frac{\pi}{2})}e^{(-\alpha-i\omega)t} &= e^{-\alpha t}\left(e^{i(\omega t-\theta+\frac{\pi}{2}} + e^{-i(\omega t-\theta-\frac{\pi}{2}}\right) \\ &= e^{-\alpha t} i\left(e^{i\omega t-\theta} - e^{-i(\omega t-\theta)}\right) \\ &= 2e^{-\alpha t}\sin(\omega t-\theta) \end{aligned} \tag{26}$$

$$x(t) = u(t) + \frac{\omega_n}{\omega}e^{-\alpha t} sin(\omega t - \theta) \tag{27}$$

Problem 4 (Use Laplace Transform and Partial Fractions to solve these ODEs)**.**

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \sin(3t) \tag{28}$$

$$\begin{aligned} y(0) &= -1 \\ \dot{y}(0) &= 0 \end{aligned} \tag{29}$$

Solution

Take the Laplace transform of both sides:

$\mathscr{L}\{\ddot{y}(t) + 3\dot{y}(t) + 2y(t)\} = \mathscr{L}\{\sin(3t)\}$

$$s^2Y(s) - sy(0) - \dot{y}(0) + 3(sY(s) - y(0)) + 2Y(s) = \frac{3}{s^2+9}$$

Substitute initial conditions $y(0) = -1, \dot{y}(0) = 0$*:*

$$s^2Y(s) + s + 3sY(s) + 3 + 2Y(s) = \frac{3}{s^2+9}$$

Combine like terms:

$$(s^2+3s+2)Y(s) = \frac{3}{s^2+9} - (s+3)$$

$$Y(s) = \frac{3-(s+3)(s^2+9)}{(s^2+9)(s+1)(s+2)}$$

$(s+3)(s^2+9) = s^3+3s^2+9s+27 \Rightarrow Numerator = -s^3-3s^2-9s-24$

$$Y(s) = \frac{-s^3-3s^2-9s-24}{(s^2+9)(s+1)(s+2)}$$

$$Y(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{Cs+D}{s^2+9}$$

Solving for constants (e.g., using symbolic computation), we get:

$$Y(s) = \frac{10}{13(s+2)} - \frac{17}{10(s+1)} - \frac{3(3s+7)}{130(s^2+9)}$$

Using Laplace inverse formulas:

$$\mathscr{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}, \quad \mathscr{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} = \cos(\omega t), \quad \mathscr{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} = \frac{1}{\omega}\sin(\omega t)$$

Break down:

$$\mathscr{L}^{-1}\left\{\frac{10}{13(s+2)}\right\} = \frac{10}{13}e^{-2t}$$
$$\mathscr{L}^{-1}\left\{\frac{17}{10(s+1)}\right\} = \frac{17}{10}e^{-t}$$
$$\mathscr{L}^{-1}\left\{\frac{-9s}{130(s^2+9)}\right\} = -\frac{9}{130}\cos(3t)$$
$$\mathscr{L}^{-1}\left\{\frac{-21}{130(s^2+9)}\right\} = -\frac{7}{130}\sin(3t)$$

Final Answer

$$\boxed{y(t) = \frac{10}{13}e^{-2t} - \frac{17}{10}e^{-t} - \frac{9}{130}\cos(3t) - \frac{7}{130}\sin(3t)}$$

Problem 5.

$$\ddot{y}(t) + 4\dot{y}(t) + 4y(t) = \cos(4t) \tag{30}$$

$$\begin{aligned} y(0) &= 1 \\ \dot{y}(0) &= 1 \end{aligned} \tag{31}$$

Solution

Apply the Laplace transform to both sides:

$$s^2Y(s) - sy(0) - \dot{y}(0) + 4(sY(s) - y(0)) + 4Y(s) = \frac{s}{s^2+16}$$

Substitute the initial conditions:

$$s^2Y(s) - s - 1 + 4sY(s) - 4 + 4Y(s) = \frac{s}{s^2+16}$$

$$(s^2+4s+4)Y(s) = \frac{s}{s^2+16} + s + 5 \Rightarrow Y(s) = \frac{\frac{s}{s^2+16} + s + 5}{(s+2)^2}$$

Combine the terms over a common denominator:

$$Y(s) = \frac{s}{(s^2+16)(s+2)^2} + \frac{s+5}{(s+2)^2}$$

$$Y(s) = \frac{(s+5)(s^2+16)+s}{(s^2+16)(s+2)^2} = \frac{s^3+5s^2+17s+80}{(s^2+16)(s+2)^2}$$

$$Y(s) = \frac{-3s+16}{100(s^2+16)} + \frac{103}{100(s+2)} + \frac{29}{10(s+2)^2}$$

Using standard Laplace pairs:

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2+16}\right\} = \cos(4t), \quad \mathscr{L}^{-1}\left\{\frac{1}{s^2+16}\right\} = \frac{1}{4}\sin(4t)$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}, \quad \mathscr{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} = te^{-2t}$$

Apply term by term:

$$\mathscr{L}^{-1}\left\{\frac{-3s}{100(s^2+16)}\right\} = -\frac{3}{100}\cos(4t)$$

$$\mathscr{L}^{-1}\left\{\frac{16}{100(s^2+16)}\right\} = \frac{1}{25}\sin(4t)$$

$$\mathscr{L}^{-1}\left\{\frac{103}{100(s+2)}\right\} = \frac{103}{100}e^{-2t}, \quad \mathscr{L}^{-1}\left\{\frac{29}{10(s+2)^2}\right\} = \frac{29}{10}te^{-2t}$$

Final Answer

$$\boxed{y(t) = \frac{103}{100}e^{-2t} + \frac{29}{10}te^{-2t} - \frac{3}{100}\cos(4t) + \frac{1}{25}\sin(4t)}$$

Problem 6.

$$\ddot{y}(t) + \dot{y}(t) + y(t) = 1 + \cos(6t) \tag{32}$$

$$\begin{aligned} y(0) &= 0 \\ \dot{y}(0) &= 0 \end{aligned} \tag{33}$$

Solution

Taking the Laplace transform of both sides:

$\mathscr{L}\{\ddot{y}(t)\} + \mathscr{L}\{\dot{y}(t)\} + \mathscr{L}\{y(t)\} = \mathscr{L}\{1\} + \mathscr{L}\{\cos(6t)\}$

$(s^2 Y(s) - sy(0) - \dot{y}(0)) + (sY(s) - y(0)) + Y(s) = \dfrac{1}{s} + \dfrac{s}{s^2+36}$

Using initial conditions $y(0) = 0$, $\dot{y}(0) = 0$, we simplify:

$(s^2 + s + 1)Y(s) = \dfrac{1}{s} + \dfrac{s}{s^2+36}$

Combine the right-hand side over a common denominator:

$Y(s) = \dfrac{2(s^2+18)}{s(s^2+36)(s^2+s+1)}$

Using symbolic computation, we obtain:

$Y(s) = \dfrac{1}{s} - \dfrac{35s-36}{1261(s^2+36)} - \dfrac{2(613s+631)}{1261(s^2+s+1)}$

Use known Laplace inverse pairs:

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$\mathscr{L}^{-1}\left\{\frac{-35s+36}{1261(s^2+36)}\right\} = -\frac{35}{1261}\cos(6t) + \frac{6}{1261}\sin(6t)$$

$$\mathscr{L}^{-1}\left\{\frac{-2(613s+631)}{1261(s^2+s+1)}\right\} = -\frac{2}{1261}e^{-t/2}\left(613\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1262}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right)$$

Final Answer

$$\boxed{y(t) = 1 - \frac{35}{1261}\cos(6t) + \frac{6}{1261}\sin(6t) - \frac{2}{1261}e^{-t/2}\left(613\cos\left(\tfrac{\sqrt{3}}{2}t\right) + \tfrac{1262}{\sqrt{3}}\sin\left(\tfrac{\sqrt{3}}{2}t\right)\right)}$$

Problem 7.

$$\ddot{y}(t) + y(t) = 1 + e^{(-2t)} \tag{34}$$

$$\begin{aligned} y(0) &= 0 \\ \dot{y}(0) &= 0 \end{aligned} \tag{35}$$

Solution

Taking the Laplace transform of both sides:

$\mathscr{L}\{\ddot{y}(t)\} + \mathscr{L}\{y(t)\} = \mathscr{L}\{1\} + \mathscr{L}\{e^{-2t}\}$

$$(s^2Y(s) - sy(0) - \dot{y}(0)) + Y(s) = \frac{1}{s} + \frac{1}{s+2}$$

Using the initial conditions $y(0) = 0,\ \dot{y}(0) = 0$, we simplify:

$$(s^2+1)Y(s) = \frac{1}{s} + \frac{1}{s+2} \Rightarrow Y(s) = \frac{1}{s(s^2+1)} + \frac{1}{(s+2)(s^2+1)}$$

Combine the terms:

$$Y(s) = \frac{2(s+1)}{s(s+2)(s^2+1)}$$

Using partial fraction decomposition:

$$Y(s) = \frac{1}{s} + \frac{1}{5(s+2)} - \frac{2(3s-1)}{5(s^2+1)}$$

Apply known Laplace inverse formulas:

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$
$$\mathscr{L}^{-1}\left\{\frac{1}{5(s+2)}\right\} = \frac{1}{5}e^{-2t}$$
$$\mathscr{L}^{-1}\left\{\frac{6s}{5(s^2+1)}\right\} = \frac{6}{5}\cos(t)$$
$$\mathscr{L}^{-1}\left\{\frac{2}{5(s^2+1)}\right\} = \frac{2}{5}\sin(t)$$

So:

$$\mathscr{L}^{-1}\{Y(s)\} = 1 + \frac{1}{5}e^{-2t} - \frac{6}{5}\cos(t) + \frac{2}{5}\sin(t)$$

Final Answer

$$\boxed{y(t) = 1 + \frac{1}{5}e^{-2t} - \frac{6}{5}\cos(t) + \frac{2}{5}\sin(t)}$$

Problem 8.

$$\ddot{y}(t) = e^{-5t} \tag{36}$$

$$\begin{aligned} y0 &= 0 \\ \dot{y}(0) &= 0 \end{aligned} \tag{37}$$

Solution

Taking the Laplace transform of both sides:

$$\mathscr{L}\{\ddot{y}(t)\} = \mathscr{L}\{e^{-5t}\}$$

$s^2Y(s) - sy(0) - \dot{y}(0) = \frac{1}{s+5}$

Using the initial conditions $y(0) = 0$, $\dot{y}(0) = 0$, we simplify:

$$s^2Y(s) = \frac{1}{s+5} \Rightarrow Y(s) = \frac{1}{s^2(s+5)}$$

We decompose:

$$\frac{1}{s^2(s+5)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+5}$$

Solving, we find:

$$Y(s) = \frac{1}{25(s+5)} - \frac{1}{25s} + \frac{1}{5s^2}$$

Using standard Laplace inverse formulas:

$$\begin{aligned} \mathscr{L}^{-1}\left\{\frac{1}{s+5}\right\} &= e^{-5t} \\ \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} &= 1 \\ \mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} &= t \end{aligned}$$

Apply them:

$$\mathscr{L}^{-1}\{Y(s)\} = \frac{1}{25}e^{-5t} - \frac{1}{25} + \frac{1}{5}t$$

Final Answer

$$\boxed{y(t) = \frac{1}{25}e^{-5t} - \frac{1}{25} + \frac{1}{5}t}$$

Problem 9 (Transition Matrix)**.**
Starting with the series definition of $e^{\mathbf{A}t}$, compute $e^{\mathbf{A}t}$ for the following matrices:

1A

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \tag{38}$$

Solution

$$\begin{aligned} e^{\mathbf{A}t} &\triangleq \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \cdots && (39) \\ &= \mathbf{I} + \begin{bmatrix} t & 0 \\ 0 & 2t \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} t^2 & 0 \\ 0 & 4t^2 \end{bmatrix} + \cdots && (40) \\ &= \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} && (41) \end{aligned}$$

1B

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \tag{42}$$

Solution

$$\begin{aligned} e^{\mathbf{A}t} &\triangleq \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \cdots && (43) \\ &= \mathbf{I} + \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} t + \frac{1}{2!}\begin{bmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{bmatrix} t^2 + \frac{1}{3!}\begin{bmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{bmatrix} t^3 + \frac{1}{4!}\begin{bmatrix} \lambda^4 & 4\lambda^3 \\ 0 & \lambda^4 \end{bmatrix} t^4 + \cdots && (44) \\ &= \begin{bmatrix} \sum_{n=0}^{\infty} \frac{\lambda^n t^n}{n!} & \sum_{n=1}^{\infty} \frac{1}{n!} n\lambda^{n-1} t^n \\ 0 & \sum_{n=0}^{\infty} \frac{\lambda^n t^n}{n!} \end{bmatrix} && (45) \end{aligned}$$

But

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n!} n\lambda^{n-1} t^n &= t\sum_{n=1}^{\infty} \frac{1}{(n-1)!}\lambda^{n-1} t^{(n-1)} && (46) \\ &= t\sum_{n=0}^{\infty} \frac{1}{(n)!}\lambda^n t^n && (47) \\ &= te^{\lambda t} && (48) \end{aligned}$$

Therefore,

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} \tag{49}$$

Problem 10.

Part A
Solve for $\mathbf{x}(t)$ *in the following:*

$$\begin{aligned} \dot{\vec{x}}(t) &= \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \vec{x}(t) && (50) \\ \vec{x}(0) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} && (51) \end{aligned}$$

Solution

$$\mathbf{x}(t) = C\mathbf{v}e^{st} \tag{52}$$

$$\cancel{C}s\mathbf{v}\cancel{e^{st}} = \mathbf{A}\cancel{C}\mathbf{v}\cancel{e^{st}} \tag{53}$$

$$\mathbf{A}\mathbf{v} = s\mathbf{v} \tag{54}$$

$$(\mathbf{A} - s\mathbf{I})\mathbf{v} = 0 \tag{55}$$

$$\left|\mathbf{A}-s\mathbf{I}\right|=0 \tag{56}$$

$$\left|\left[\begin{array}{cc}-1 & 2\\ -2 & -1\end{array}\right]-\left[\begin{array}{cc}s & 0\\ 0 & s\end{array}\right]\right|=0 \tag{57}$$

$$\left|\begin{array}{cc}-1-s & 2\\ -2 & -1-s\end{array}\right|=0 \tag{58}$$

$$s^2+2s+5=0 \tag{59}$$

$$\begin{aligned} s_{1,2} &= \frac{-2\pm\sqrt{4-20}}{2} \qquad (60)\\ &= -1\pm 2i \qquad (61)\end{aligned}$$

$$\left(\left[\begin{array}{cc}-1 & 2\\ -2 & -1\end{array}\right]-\left[\begin{array}{cc}-1+2i & 0\\ 0 & -1+2i\end{array}\right]\right)\left[\begin{array}{c}v_{1,1}\\ v_{1,2}\end{array}\right]=\left[\begin{array}{c}0\\ 0\end{array}\right] \tag{62}$$

$$\left[\begin{array}{cc}-2i & 2\\ -2 & -2i\end{array}\right]\left[\begin{array}{c}v_{1,1}\\ v_{1,2}\end{array}\right]=\left[\begin{array}{c}0\\ 0\end{array}\right] \tag{63}$$

$$\left[\begin{array}{c}v_{1,1}\\ v_{1,2}\end{array}\right]=\left[\begin{array}{c}-i\\ 1\end{array}\right] \tag{64}$$

$$\left(\left[\begin{array}{cc}-1 & 2\\ -2 & -1\end{array}\right]-\left[\begin{array}{cc}-1-2i & 0\\ 0 & -1-2i\end{array}\right]\right)\left[\begin{array}{c}v_{2,1}\\ v_{2,2}\end{array}\right]=\left[\begin{array}{c}0\\ 0\end{array}\right] \tag{65}$$

$$\left[\begin{array}{cc}2i & 2\\ -2 & 2i\end{array}\right]\left[\begin{array}{c}v_{2,1}\\ v_{2,2}\end{array}\right]=\left[\begin{array}{c}0\\ 0\end{array}\right] \tag{66}$$

$$\left[\begin{array}{c}v_{2,1}\\ v_{2,2}\end{array}\right]=\left[\begin{array}{c}1\\ -i\end{array}\right] \tag{67}$$

$$\begin{aligned}\vec{x}(t) &= c_1\begin{pmatrix}-i\\ 1\end{pmatrix}e^{(-1+2i)t}+c_2\begin{pmatrix}1\\ -i\end{pmatrix}e^{(-1-2i)t}\\ &= e^{-t}\left[c_1\begin{pmatrix}-i\\ 1\end{pmatrix}e^{2it}+c_2\begin{pmatrix}1\\ -i\end{pmatrix}e^{-2it}\right] \qquad (68)\end{aligned}$$

$$\vec{x}(0)=\begin{pmatrix}0\\1\end{pmatrix} \implies c_1\begin{pmatrix}-i\\1\end{pmatrix}+c_2\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix} \tag{69}$$

$$i\left[-ic_1+c_2=0\right] \implies c_1+ic_2=0 \tag{70}$$

also,

$$c_1-ic_2=1 \tag{71}$$

Therefore,

$$2c_1=1 \implies \boxed{c_1=\frac{1}{2}} \tag{72}$$

$$\frac{1}{2}+ic_2=0 \implies \boxed{c_2=-\frac{1}{2i}=i\frac{1}{2}} \tag{73}$$

$$\begin{aligned}
\vec{x}(t) &= e^{-t}\left[c_1\begin{pmatrix}-i\\1\end{pmatrix}e^{2it}+c_2\begin{pmatrix}1\\-i\end{pmatrix}e^{-2it}\right]\\
&= e^{-t}\left[\frac{1}{2}\begin{pmatrix}-i\\1\end{pmatrix}e^{2it}+\frac{1}{2}i\begin{pmatrix}1\\-i\end{pmatrix}e^{-2it}\right]\\
&= e^{-t}\left[\frac{1}{2}\begin{pmatrix}-i\cos(2t)+\sin(t)\\ \cos(2t)+i\sin(2t)\end{pmatrix}+\frac{1}{2}i\begin{pmatrix}\cos(2t)-i\sin(2t)\\ -i\cos(2t)-\sin(2t)\end{pmatrix}\right]\\
&= e^{-t}\begin{bmatrix}\cancel{-\frac{1}{2}i\cos(2t)}+\frac{1}{2}\sin(2t)+\cancel{\frac{1}{2}i\cos(2t)}+\frac{1}{2}\sin(2t)\\ \frac{1}{2}\cos(2t)+\cancel{\frac{1}{2}i\sin(2t)}+\frac{1}{2}\cos(2t)-\cancel{\frac{1}{2}i\sin(2t)}\end{bmatrix}\\
&= e^{-t}\begin{pmatrix}\sin(2t)\\ \cos(2t)\end{pmatrix}
\end{aligned} \tag{74}$$

Part B

Compute $e^{\mathbf{A}t}$ for

$$\mathbf{A} = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \tag{75}$$

using the series definition of $e^{\mathbf{A}t}$.

Solution

$$e^{\mathbf{A}t} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} \tag{76}$$

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \end{aligned}$$

where, $\sigma = -1$ and $\omega = 2$.

$$\begin{aligned} e^{\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} t} &= e^{\left(\sigma \mathbf{I} t + \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}\right) t} \\ &= e^{\sigma \mathbf{I} t} e^{\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} t} \end{aligned} \tag{77}$$

$$\begin{aligned} e^{\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} t} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} t + \frac{1}{2!} \begin{bmatrix} \omega^2 & 0 \\ 0 & -\omega^2 \end{bmatrix} t^2 \\ &+ \frac{1}{3!} \begin{bmatrix} 0 & -\omega^3 \\ \omega^3 & 0 \end{bmatrix} t^3 + \frac{1}{4!} \begin{bmatrix} \omega^4 & 0 \\ 0 & \omega^4 \end{bmatrix} t^4 \\ &+ \frac{1}{5!} \begin{bmatrix} 0 & \omega^5 \\ -\omega^5 & 0 \end{bmatrix} t^5 + \cdots \end{aligned} \tag{78}$$

$$\begin{aligned} e^{\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} t} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2!} \begin{bmatrix} \omega^2 & 0 \\ 0 & -\omega^2 \end{bmatrix} t^2 + \frac{1}{4!} \begin{bmatrix} \omega^4 & 0 \\ 0 & \omega^4 \end{bmatrix} t^4 \\ &+ \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} t + \frac{1}{3!} \begin{bmatrix} 0 & -\omega^3 \\ \omega^3 & 0 \end{bmatrix} t^3 + \frac{1}{5!} \begin{bmatrix} 0 & \omega^5 \\ -\omega^5 & 0 \end{bmatrix} t^5 + \cdots \end{aligned} \tag{79}$$

$$\begin{aligned} e^{\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} t} &= \begin{bmatrix} \cos(\omega t) & 0 \\ 07\cos(\omega t) & \end{bmatrix} + \begin{bmatrix} 0 & \sin(\omega t) \\ -\sin(\omega t) & 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix} \end{aligned} \tag{80}$$

$$
\begin{aligned}
e^{\mathbf{A}t} &= \begin{pmatrix} e^{\sigma t} & 0 \\ 0 & e^{\sigma t} \end{pmatrix} \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix} \\
&= \begin{bmatrix} e^{\sigma t}\cos(\omega t) & e^{\sigma t}\sin(\omega t) \\ -e^{\sigma t}\sin(\omega t) & e^{\sigma t}\cos(\omega t) \end{bmatrix}
\end{aligned} \tag{81}
$$