INTRODUCTION TO CONTINUOUS CONTROL SYSTEMS
COLUMBIA UNIVERSITY MECHANICAL AND ELECTRICAL ENGINEERING DEPARTMENTS: E3601

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Homework 5

Problem 1 (Controllable Canonical Form)**.**
Show that the general n^{th} order differential equation with constant coefficients given by equation 1 may be represented as an array of first order differential equations given by equations 2 and 3.

$$\frac{d^n y(t)}{dt^n} + p_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + p_1\frac{dy(t)}{dt} + p_0 y(t) = q_{n-1}\frac{d^{n-1}u(t)}{dt^{n-1}} + \cdots + q_1\frac{du(t)}{dt} + q_0 u(t) \tag{1}$$

$$\dot{\vec{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & \dots & . \\ -p_0 & -p_1 & \dots & -p_{n-2} & -p_{n-1} \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ . \\ 1 \end{bmatrix} u(t) \tag{2}$$

$$y(t) = \begin{bmatrix} q_0 & q_1 & \dots & q_{n-2} & q_{n-1} \end{bmatrix} \vec{x}(t) \tag{3}$$

Solution

Multiplying out the matrices of equations 2 and 3 we will have the following:

$$\dot{x}_1 = x_2 \tag{4}$$

$$\dot{x}_2 = x_3 \tag{5}$$

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$$\dot{x}_n = -p_0 x_1 - p_1 x_2 - p_2 x_3 \cdots - p_{n-1} x_n + u(t) \tag{6}$$

and

$$y(t) = q_0 x_1 + q_1 x_2 + \cdots + q_{n-1} x_n \tag{7}$$

Rewriting everything in terms of x_1, we will have the following two equations

$$u(t) = \frac{d^n x_1(t)}{dt^n} + p_{n-1} \frac{d^{n-1} x_1(t)}{dt^{n-1}} + \cdots + p_0 x_1(t) \tag{8}$$

and

$$y(t) = q_{n-1} \frac{d^{n-1} x_1(t)}{dt^{n-1}} + \cdots + q_0 x_1(t) \tag{9}$$

Let us take the Laplace transforms of the above two equations to yield,

$$U(s) = (s^n p_{n-1} s^{n-1} + \cdots + p_0) X_1(s) \tag{10}$$

and

$$Y(s) = (q_{n-1} s^{n-1} + \cdots + q_1 s + q_0) X_1(s) \tag{11}$$

In other words,

$$X_1(s) = \frac{U(s)}{(s^n p_{n-1} s^{n-1} + \cdots + p_0)} \tag{12}$$

and

$$X_1(s) = \frac{Y(s)}{(q_{n-1} s^{n-1} + \cdots + q_1 s + q_0)} \tag{13}$$

Equating the right hand sides of equations 12 and 13 we have

$$(s^n + p_{n-1} s^{n-1} + \cdots + p_0) Y(s) = (q_{n-1} s^{n-1} + \cdots + q_1 s + q_0) U(s) \tag{14}$$

Rewriting equation 14 in the time domain through inversion,

$$\frac{d^n y(t)}{dt^n} + p_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + p_0 y(t) = q_{n-1} \frac{d^{n-1} u(t)}{dt^{n-1}} + \cdots + q_0 u(t) \qquad Q.E.D. \tag{15}$$