INTRODUCTION TO CONTINUOUS CONTROL SYSTEMS
COLUMBIA UNIVERSITY MECHANICAL AND ELECTRICAL ENGINEERING DEPARTMENTS: E3601

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Homework 4

Problem 1 (Laplace Transform – Differentiation)**.** *Show that the Laplace transform of the n_{th} derivative of a function $f(t)$ is given by the following identity:*

$$\begin{aligned}\mathscr{L}\left\{\frac{d^n f(t)}{dt^n}\right\} &= s^n F(s) \\ &- \lim_{t\to 0}\left[s^{n-1}f(t) + s^{n-2}\frac{df(t)}{dt} + \cdots \right. \\ &\qquad \left. + \frac{d^{n-1}f(t)}{dt^{n-1}}\right] \\ &= s^n F(s) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \cdots - f^{(n-1)}(0) \end{aligned} \tag{1}$$

Solution

Consider the Laplace transform of the first derivate of a function,

$$L\{\frac{df(t)}{dt}\} = \int_0^\infty \frac{df(t)}{dt} e^{-st} dt \tag{2}$$

using integration by parts, $\int_0^\infty uv' = [uv]_0^\infty - \int_0^\infty u'v$
Assuming $u = e^{-st}$, $v' = \frac{df(t)}{dt}$, *and* $u' = -se^{-st}$ *then,*

$$L\{\frac{df(t)}{dt}\} = \left[e^{-st}f(t)\right]_0^\infty + s\int_0^\infty f(t)e^{-st}dt = 0 - f(0) + sF(s) \tag{3}$$

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Now let us consider the second derivative, where $u = e^{-st}$, $v' = \frac{d^2 f(t)}{dt^2}$ *and* $u = -se^{-st}$,

$$L\{\frac{d^2 f(t)}{dt^2}\} = \left[e^{-st}\frac{df(t)}{dt}\right]_0^\infty + s\int_0^\infty \frac{df(t)}{dt}e^{-st}dt = 0 - \left[\frac{df(t)}{dt}\right]_{t=0} + sL\{\frac{df(t)}{dt}\} \tag{4}$$

Using the results for $L\{\frac{df(t)}{dt}\}$ *derived above, we get*

$$L\{\frac{d^2 f(t)}{dt^2}\} = -\left[\frac{df(t)}{dt}\right]_{t=0} + s(-f(0) + sF(s)) \tag{5}$$

By induction from the past process, we have,

$$L\{\frac{d^n f(t)}{dt^n}\} = s^n F(s) - \lim_{t\to 0}\left[s^{n-1}f(t) + s^{n-2}\frac{df(t)}{dt} + \cdots + \frac{d^{n-1}f(t)}{dt^{n-1}}\right] \tag{6}$$