INTRODUCTION TO CONTINUOUS CONTROL SYSTEMS
COLUMBIA UNIVERSITY MECHANICAL AND ELECTRICAL ENGINEERING DEPARTMENTS: E3601

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Homework 3

Problem 1 (Analyticity of the Trigonometric Functions)**.**
Prove thet the following functions are analytic everywhere in the $\mathbb{C}$ plane, except at some special points, and that $\sin(s)$ *and* $\cos(s)$ *are periodic with period* 2π*.*

Trigonometric Functions of complex variable s are defined in terms of the Exponential Function, e^s*, as follows,*

$$\sin(s) \triangleq \frac{e^{is} - e^{-is}}{2i} \tag{1}$$

$$\cos(s) \triangleq \frac{e^{is} + e^{-is}}{2} \tag{2}$$

$$\csc(s) \triangleq \frac{1}{\sin(s)} \tag{3}$$

$$\sec(s) \triangleq \frac{1}{\cos(s)} \tag{4}$$

$$\tan(s) \triangleq \frac{\sin(s)}{\cos(s)} \tag{5}$$

$$\cot(s) \triangleq \frac{\cos(s)}{\sin(s)} \tag{6}$$

Solution

1. *Analyticity of sin*(s)

Solution 1. *Using Euler's formula* $(e^{\imath\sigma} = \cos\sigma + \imath\sin\sigma)$

$$sin(s) = \frac{e^{\imath s} - e^{-\imath s}}{2\imath}$$

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Since, $s \in \mathbb{C}$
We can replace s with $\sigma + \iota\omega$*, where* $\iota = \sqrt{-1}$ *and* $\sigma, \omega \in \mathbb{R}$

So,

$$sin(s) = \frac{e^{\iota(\sigma+\iota\omega)} - e^{-\iota(\sigma+\iota\omega)}}{2\iota}$$

$$sin(s) = \frac{e^{\iota\sigma}.e^{-\omega} - e^{-\iota\sigma}.e^{\omega}}{2\iota}$$

Using Euler's formula $(e^{\iota\sigma} = \cos\sigma + \iota\sin\sigma)$ *to simplify,*

$$sin(s) = \frac{e^{-\omega}}{2\iota}(\cos\sigma + \iota\sin\sigma) - \frac{e^{\omega}}{2\iota}(\cos\sigma - \iota\sin\sigma)$$

$$sin(s) = \frac{\sin\sigma}{2}\left(e^{\omega} + e^{-\omega}\right) + \frac{\iota\cos\sigma}{2}\left(e^{\omega} - e^{-\omega}\right)$$

So, $Re[sin(s)] = \frac{\sin\sigma}{2}\left(e^{\omega} + e^{-\omega}\right)$ *and* $Im[sin(s)] = \frac{\cos\sigma}{2}\left(e^{\omega} - e^{-\omega}\right)$
To prove analyticity, the partial derivatives, U_σ*,* U_ω*,* V_σ*,* V_ω *must exist and satisfy the Cauchy-Reimann conditions.*

$$U_\sigma = \frac{\partial U}{\partial\sigma} = \frac{\cos\sigma}{2}\left(e^{\omega} + e^{-\omega}\right)$$

$$U_\omega = \frac{\partial U}{\partial\omega} = \frac{\sin\sigma}{2}\left(e^{\omega} - e^{-\omega}\right)$$

$$V_\sigma = \frac{\partial V}{\partial\sigma} = \frac{-\sin\sigma}{2}\left(e^{\omega} - e^{-\omega}\right)$$

$$V_\omega = \frac{\partial V}{\partial\omega} = \frac{\cos\sigma}{2}\left(e^{\omega} + e^{-\omega}\right)$$

$\forall\sigma, \omega \in \mathbb{R}$*,* U_σ*,* U_ω*,* V_σ*,* V_ω *exit and satisfy the Cauchy-Reimann conditions, i.e.*

$$U_\sigma = V_\omega$$

$$U_\omega = -V_\sigma$$

Hence, proved that sin(s) *is analytic. QED*

To, calculate the period of sin(s) *:*
Lets, assume that the period is $s_1 = \sigma_1 + \iota\omega_1$.
Using the expression for sin(s)*, we derived above,*

$$sin(s+s_1) = \frac{\sin\sigma+\sigma_1}{2}\left(e^{\omega+\omega_1} + e^{-\omega-\omega_1}\right) + \frac{\iota\cos\sigma+\sigma_1}{2}\left(e^{\omega+\omega_1} - e^{-\omega-\omega_1}\right)$$

Since, $sin(s) = sin(s+s_1)$*, we get,* $\sigma_1 = 2\pi$ *and* $\omega_1 = 0$

This is because e^x and e^{-x} are monotonically increasing and decreasing functions respectively, and periodicity of $sin(x)$ and $cos(x)$ is 2π, where $x \in \mathbb{R}$
Hence, proved that $sin(s)$ has a periodicity of 2π. QED

2. *Analyticity of $cos(s)$*

Solution 2. *Using Euler's formula ($e^{\iota\sigma} = \cos\sigma + \iota\sin\sigma$)*

$$cos(s) = \frac{e^{\iota s} + e^{-\iota s}}{2}$$

Since, $s \in \mathbb{C}$
We can replace s with $\sigma + \iota\omega$, where $\iota = \sqrt{-1}$ and $\sigma, \omega \in \mathbb{R}$

So,

$$cos(s) = \frac{e^{\iota(\sigma+\iota\omega)} + e^{-\iota(\sigma+\iota\omega)}}{2}$$

$$cos(s) = \frac{e^{\iota\sigma}.e^{-\omega} + e^{-\iota\sigma}.e^{\omega}}{2}$$

Using Euler's formula ($e^{\iota\sigma} = \cos\sigma + \iota\sin\sigma$) to simplify,

$$cos(s) = \frac{e^{-\omega}}{2}(\cos\sigma + \iota\sin\sigma) + \frac{e^{\omega}}{2}(\cos\sigma - \iota\sin\sigma)$$

$$cos(s) = \frac{\cos\sigma}{2}\left(e^{\omega} + e^{-\omega}\right) - \frac{\iota\sin\sigma}{2}\left(e^{\omega} - e^{-\omega}\right)$$

So, $Re[cos(s)] = \frac{\cos\sigma}{2}\left(e^{\omega} + e^{-\omega}\right)$ and $Im[cos(s)] = -\frac{\sin\sigma}{2}\left(e^{\omega} - e^{-\omega}\right)$
To prove analyticity, the partial derivatives, U_σ, U_ω, V_σ, V_ω must exist and satisfy the Cauchy-Reimann conditions.

$$U_\sigma = \frac{\partial U}{\partial\sigma} = \frac{-\sin\sigma}{2}\left(e^{\omega} + e^{-\omega}\right)$$

$$U_\omega = \frac{\partial U}{\partial\omega} = \frac{\cos\sigma}{2}\left(e^{\omega} - e^{-\omega}\right)$$

$$V_\sigma = \frac{\partial V}{\partial\sigma} = \frac{-\cos\sigma}{2}\left(e^{\omega} - e^{-\omega}\right)$$

$$V_\omega = \frac{\partial V}{\partial\omega} = \frac{-\sin\sigma}{2}\left(e^{\omega} + e^{-\omega}\right)$$

$\forall\sigma, \omega \in \mathbb{R}$, U_σ, U_ω, V_σ, V_ω exit and satisfy the Cauchy-Reimann conditions, i.e.

$$U_\sigma = V_\omega$$

$U_\omega = -V_\sigma$

Hence, proved that cos(s) is analytic. QED

To, calculate the period of cos(s) :
Lets, assume that the period is $s_1 = \sigma_1 + \iota\omega_1$.
Using the expression for cos(s), we derived above,

$$cos(s+s_1) = \frac{\cos\sigma+\sigma_1}{2}\left(e^{\omega+\omega_1}+e^{-\omega-\omega_1}\right) - \frac{\iota\sin\sigma+\sigma_1}{2}\left(e^{\omega+\omega_1}-e^{-\omega-\omega_1}\right)$$

Since, $cos(s) = cos(s+s_1)$*, we get,* $\sigma_1 = 2\pi$ *and* $\omega_1 = 0$
This is because e^x *and* e^{-x} *are monotonically increasing and decreasing functions respectively, and periodicity of* $sin(x)$ *and* $cos(x)$ *is* 2π*, where* $x \in \mathbb{R}$
Hence, proved that cos(s) has a periodicity of 2π*. QED*

3. *Analyticity of cosec(s)*

Solution 3. *Using Euler's formula* $(e^{\iota\sigma} = \cos\sigma + \iota\sin\sigma)$

$$cosec(s) = \frac{2\iota}{e^{\iota s}-e^{-\iota s}}$$

Domain of this complex valued function is $s \in \mathbb{C} - n\pi$ *where* $n \in \mathbb{Z}$
Since, $s \in \mathbb{C}$
We can replace s with $\sigma + \iota\omega$*, where* $\iota = \sqrt{-1}$ *and* $\sigma, \omega \in \mathbb{R}$

So,

$$cosec(s) = \frac{2\iota}{e^{\iota(\sigma+\iota\omega)}-e^{-\iota(\sigma+\iota\omega)}}$$

Using Euler's formula $(e^{\iota\sigma} = \cos\sigma + \iota\sin\sigma)$

$$cosec(s) = \frac{2}{(e^{\omega}+e^{-\omega})\sin\sigma + \iota\,(e^{\omega}-e^{-\omega})\cos\sigma}$$

Using $|z|^2 = zz'$ *to simplify,*

$$cosec(s) = \frac{2\left(\left(e^{\omega}+e^{-\omega}\right)\sin\sigma - \iota\left(e^{\omega}-e^{-\omega}\right)\cos\sigma\right)}{(e^{\omega}+e^{-\omega})^2\sin^2\sigma + (e^{\omega}-e^{-\omega})^2\cos^2\sigma}$$

$$cosec(s) = \frac{\left(e^{\omega}+e^{-\omega}\right)\sin\sigma}{\frac{e^{2\omega}+e^{-2\omega}}{2}-\cos 2\sigma} - \frac{\iota\left(e^{\omega}-e^{-\omega}\right)\cos\sigma}{\frac{e^{2\omega}+e^{-2\omega}}{2}-\cos 2\sigma}$$

So, $Re\left[cosec(s)\right] = \frac{(e^{\omega}+e^{-\omega})\sin\sigma}{\frac{e^{2\omega}+e^{-2\omega}}{2}-\cos 2\sigma}$ *and* $Im\left[cosec(s)\right] = \frac{(e^{\omega}-e^{-\omega})\cos\sigma}{\frac{e^{2\omega}+e^{-2\omega}}{2}-\cos 2\sigma}$

To prove analyticity, the partial derivatives, U_σ*,* U_ω*,* V_σ*,* V_ω *must exist in the domain and satisfy the Cauchy-Reimann conditions.*
Below the steps have been left for brevity,

Use the quotient rule and trigonometric identities to get the results below,

$$U_\sigma = \frac{\partial U}{\partial \sigma} = \frac{\cos\sigma\left(e^\omega + e^{-\omega}\right)\left(\frac{e^{2\omega}+e^{-2\omega}}{2} - \cos 2\sigma - 4\sin^2\sigma\right)}{\left(\frac{e^{2\omega}+e^{-2\omega}}{2} - \cos 2\sigma\right)^2}$$

$$U_\omega = \frac{\partial U}{\partial \omega} = -\frac{\sin\sigma\left(e^\omega - e^{-\omega}\right)\left(\frac{e^{2\omega}+e^{-2\omega}}{2} + \cos 2\sigma + 2\right)}{\left(\frac{e^{2\omega}+e^{-2\omega}}{2} - \cos 2\sigma\right)^2}$$

$$V_\sigma = \frac{\partial V}{\partial \sigma} = \frac{\sin\sigma\left(e^\omega - e^{-\omega}\right)\left(\frac{e^{2\omega}+e^{-2\omega}}{2} + \cos 2\sigma + 2\right)}{\left(\frac{e^{2\omega}+e^{-2\omega}}{2} - \cos 2\sigma\right)^2}$$

$$V_\omega = \frac{\partial V}{\partial \omega} = \frac{\cos\sigma\left(e^\omega + e^{-\omega}\right)\left(\frac{e^{2\omega}+e^{-2\omega}}{2} - \cos 2\sigma - 4\sin^2\sigma\right)}{\left(\frac{e^{2\omega}+e^{-2\omega}}{2} - \cos 2\sigma\right)^2}$$

$\forall \sigma, \omega \in \mathbb{R}$, *$U_\sigma$, U_ω, V_σ, V_ω exit and satisfy the Cauchy-Reimann conditions, i.e.*

$U_\sigma = V_\omega$

$U_\omega = -V_\sigma$

The Cauchy-Riemann conditions are satisfied, but do the partial derivatives exist everywhere in the domain? For that check the denominators. The denominator vanishes only in the case where $\omega = 0$ and $\sigma = n\pi$, where $n \in \mathbb{Z}$. We have already excluded these points from the domain. Hence, proved that cosec(s) is analytic. QED

4. *Analyticity of sec(s)*

Solution 4. *Using Euler's formula* $(e^{\iota\sigma} = \cos\sigma + \iota\sin\sigma)$

$sec(s) = \dfrac{2}{e^{\iota s} + e^{-\iota s}}$

Domain of this complex valued function is $s \in \mathbb{C} - \left(n + \frac{1}{2}\right)\pi$ where $n \in \mathbb{Z}$
Since, $s \in \mathbb{C}$
We can replace s with $\sigma + \iota\omega$, where $\iota = \sqrt{-1}$ and $\sigma, \omega \in \mathbb{R}$

So,

$sec(s) = \dfrac{2}{e^{\iota(\sigma+\iota\omega)} + e^{-\iota(\sigma+\iota\omega)}}$

Using Euler's formula $(e^{\iota\sigma} = \cos\sigma + \iota\sin\sigma)$

$sec(s) = \dfrac{2}{\left(e^\omega + e^{-\omega}\right)\cos\sigma - \iota\left(e^\omega - e^{-\omega}\right)\sin\sigma}$

Using $|z|^2 = zz'$ to simplify,

$$sec(s) = \frac{2\left(\left(e^{\omega}+e^{-\omega}\right)\cos\sigma + \iota\left(e^{\omega}-e^{-\omega}\right)\sin\sigma\right)}{\left(e^{\omega}+e^{-\omega}\right)^2\cos^2\sigma + \left(e^{\omega}-e^{-\omega}\right)^2\sin^2\sigma}$$

$$sec(s) = \frac{\left(e^{\omega}+e^{-\omega}\right)\cos\sigma}{\frac{e^{2\omega}+e^{-2\omega}}{2}+\cos 2\sigma} + \frac{\iota\left(e^{\omega}-e^{-\omega}\right)\sin\sigma}{\frac{e^{2\omega}+e^{-2\omega}}{2}+\cos 2\sigma}$$

So, $Re\left[sec(s)\right] = \frac{\left(e^{\omega}+e^{-\omega}\right)\cos\sigma}{\frac{e^{2\omega}+e^{-2\omega}}{2}+\cos 2\sigma}$ *and* $Im\left[sec(s)\right] = \frac{\left(e^{\omega}-e^{-\omega}\right)\sin\sigma}{\frac{e^{2\omega}+e^{-2\omega}}{2}+\cos 2\sigma}$

To prove analyticity, the partial derivatives, U_σ, U_ω, V_σ, V_ω *must exist in the domain and satisfy the Cauchy-Reimann conditions.*

Below the steps have been left for brevity,

Use the quotient rule and trigonometric identities to get the results below,

$$U_\sigma = \frac{\partial U}{\partial \sigma} = -\frac{\sin\sigma\left(e^{\omega}+e^{-\omega}\right)\left(\frac{e^{2\omega}+e^{-2\omega}}{2}+\cos 2\sigma - 4\cos^2\sigma\right)}{\left(\frac{e^{2\omega}+e^{-2\omega}}{2}+\cos 2\sigma\right)^2}$$

$$U_\omega = \frac{\partial U}{\partial \omega} = \frac{\cos\sigma\left(e^{\omega}-e^{-\omega}\right)\left(\frac{e^{2\omega}+e^{-2\omega}}{2}+\cos 2\sigma - 2\right)}{\left(\frac{e^{2\omega}+e^{-2\omega}}{2}+\cos 2\sigma\right)^2}$$

$$V_\sigma = \frac{\partial V}{\partial \sigma} = -\frac{\cos\sigma\left(e^{\omega}-e^{-\omega}\right)\left(\frac{e^{2\omega}+e^{-2\omega}}{2}+\cos 2\sigma - 2\right)}{\left(\frac{e^{2\omega}+e^{-2\omega}}{2}+\cos 2\sigma\right)^2}$$

$$V_\omega = \frac{\partial V}{\partial \omega} = -\frac{\sin\sigma\left(e^{\omega}+e^{-\omega}\right)\left(\frac{e^{2\omega}+e^{-2\omega}}{2}+\cos 2\sigma - 4\cos^2\sigma\right)}{\left(\frac{e^{2\omega}+e^{-2\omega}}{2}+\cos 2\sigma\right)^2}$$

$\forall \sigma, \omega \in \mathbb{R}$, U_σ, U_ω, V_σ, V_ω *exit and satisfy the Cauchy-Reimann conditions, i.e.*

$$U_\sigma = V_\omega$$

$$U_\omega = -V_\sigma$$

The Cauchy-Riemann conditions are satisfied, but do the partial derivatives exist everywhere in the domain? For that check the denominators. The denominator vanishes only in the case where $\omega = 0$ *and* $\sigma = \left(n+\frac{1}{2}\right)\pi$, *where* $n \in \mathbb{Z}$. *We have already excluded these points from the domain. Hence, proved that* $sec(s)$ *is analytic. QED*

5. *Analyticity of* $tan(s)$

Solution 5. *Using Euler's formula* ($e^{\iota\sigma} = \cos\sigma + \iota\sin\sigma$)

$$tan(s) = -\iota\frac{e^{\iota s}-e^{-\iota s}}{e^{\iota s}+e^{-\iota s}}$$

Domain of this complex valued function is $s \in \mathbb{C} - \left(n+\frac{1}{2}\right)\pi$ *where* $n \in \mathbb{Z}$

Since, $s \in \mathbb{C}$
We can replace s with $\sigma + \imath\omega$, where $\imath = \sqrt{-1}$ and $\sigma, \omega \in \mathbb{R}$

So,

$$tan(s) = -\imath \frac{e^{\imath(\sigma+\imath\omega)} - e^{-\imath(\sigma+\imath\omega)}}{e^{\imath(\sigma+\imath\omega)} + e^{-\imath(\sigma+\imath\omega)}}$$

Using Euler's formula ($e^{\imath\sigma} = \cos\sigma + \imath\sin\sigma$)

$$tan(s) = \frac{\left(e^{\omega} + e^{-\omega}\right)\sin\sigma + \imath\left(e^{\omega} - e^{-\omega}\right)\cos\sigma}{\left(e^{\omega} + e^{-\omega}\right)\cos\sigma - \imath\left(e^{\omega} - e^{-\omega}\right)\sin\sigma}$$

Using $|z|^2 = zz'$ to simplify,

$$tan(s) = \frac{2\sin 2\sigma + \imath\left(e^{2\omega} - e^{-2\omega}\right)}{\left(e^{\omega} + e^{-\omega}\right)^2\cos^2\sigma + \left(e^{\omega} - e^{-\omega}\right)^2\sin^2\sigma}$$

So, $Re\left[tan(s)\right] = \frac{\sin 2\sigma}{\frac{e^{2\omega}+e^{-2\omega}}{2} + \cos 2\sigma}$ and $Im\left[tan(s)\right] = \frac{\frac{\left(e^{2\omega}-e^{-2\omega}\right)}{2}}{\frac{e^{2\omega}+e^{-2\omega}}{2} + \cos 2\sigma}$
To prove analyticity, the partial derivatives, U_σ, U_ω, V_σ, V_ω must exist in the domain and satisfy the Cauchy-Reimann conditions.
Below the steps have been left for brevity,
Use the quotient rule and trigonometric identities to get the results below,

$$U_\sigma = \frac{\partial U}{\partial \sigma} = \frac{\cos 2\sigma\left(e^{2\omega} + e^{-2\omega}\right) + 2}{\left(\frac{e^{2\omega}+e^{-2\omega}}{2} + \cos 2\sigma\right)^2}$$

$$U_\omega = \frac{\partial U}{\partial \omega} = -\frac{\sin 2\sigma\left(e^{2\omega} + e^{-2\omega}\right)}{\left(\frac{e^{2\omega}+e^{-2\omega}}{2} + \cos 2\sigma\right)^2}$$

$$V_\sigma = \frac{\partial V}{\partial \sigma} = \frac{\sin 2\sigma\left(e^{2\omega} + e^{-2\omega}\right)}{\left(\frac{e^{2\omega}+e^{-2\omega}}{2} + \cos 2\sigma\right)^2}$$

$$V_\omega = \frac{\partial V}{\partial \omega} = \frac{\cos 2\sigma\left(e^{2\omega} + e^{-2\omega}\right) + 2}{\left(\frac{e^{2\omega}+e^{-2\omega}}{2} + \cos 2\sigma\right)^2}$$

$\forall \sigma, \omega \in$ the domain, U_σ, U_ω, V_σ, V_ω exist and satisfy the Cauchy-Reimann conditions, i.e.

$$U_\sigma = V_\omega$$

$$U_\omega = -V_\sigma$$

The Cauchy-Riemann conditions are satisfied, but do the partial derivatives exist everywhere in the domain? For that check the denominators. The denominator vanishes only in the case where $\omega = 0$ and $\sigma = \left(n + \frac{1}{2}\right)\pi$, where $n \in \mathbb{Z}$. We have already excluded these points from the domain. Hence, proved that $tan(s)$ is analytic. QED

6. *Analyticity of* $cot(s)$

Solution 6. *Using Euler's formula* $(e^{\iota\sigma} = \cos\sigma + \iota\sin\sigma)$

$$cot(s) = \iota\frac{e^{\iota s} + e^{-\iota s}}{e^{\iota s} - e^{-\iota s}}$$

Domain of this complex valued function is $s \in \mathbb{C} - \{n\pi\}$ *where* $n \in \mathbb{Z}$
Since, $s \in \mathbb{C}$
We can replace s with $\sigma + \iota\omega$*, where* $\iota = \sqrt{-1}$ *and* $\sigma, \omega \in \mathbb{R}$

So,

$$cot(s) = \iota\frac{e^{\iota(\sigma+\iota\omega)} + e^{-\iota(\sigma+\iota\omega)}}{e^{\iota(\sigma+\iota\omega)} - e^{-\iota(\sigma+\iota\omega)}}$$

Using Euler's formula $(e^{\iota\sigma} = \cos\sigma + \iota\sin\sigma)$

$$cot(s) = \frac{\left(e^{\omega} + e^{-\omega}\right)\cos\sigma - \iota\left(e^{\omega} - e^{-\omega}\right)\sin\sigma}{\left(e^{\omega} + e^{-\omega}\right)\sin\sigma + \iota\left(e^{\omega} - e^{-\omega}\right)\cos\sigma}$$

Using $|z|^2 = zz'$ *to simplify,*

$$cot(s) = \frac{2\sin 2\sigma - \iota\left(e^{2\omega} - e^{-2\omega}\right)}{\left(e^{\omega} - e^{-\omega}\right)^2\cos^2\sigma + \left(e^{\omega} + e^{-\omega}\right)^2\sin^2\sigma}$$

So, $Re[tan(s)] = \frac{\sin 2\sigma}{\frac{e^{2\omega}+e^{-2\omega}}{2} - \cos 2\sigma}$ *and* $Im[tan(s)] = -\frac{\frac{\left(e^{2\omega}-e^{-2\omega}\right)}{2}}{\frac{e^{2\omega}+e^{-2\omega}}{2} - \cos 2\sigma}$

To prove analyticity, the partial derivatives, U_σ*,* U_ω*,* V_σ*,* V_ω *must exist in the domain and satisfy the Cauchy-Reimann conditions.*
Below the steps have been left for brevity,
Use the quotient rule and trigonometric identities to get the results below,

$$U_\sigma = \frac{\partial U}{\partial\sigma} = \frac{\cos 2\sigma\left(e^{2\omega} + e^{-2\omega}\right) - 2}{\left(\frac{e^{2\omega}+e^{-2\omega}}{2} - \cos 2\sigma\right)^2}$$

$$U_\omega = \frac{\partial U}{\partial\omega} = -\frac{\sin 2\sigma\left(e^{2\omega} - e^{-2\omega}\right)}{\left(\frac{e^{2\omega}+e^{-2\omega}}{2} - \cos 2\sigma\right)^2}$$

$$V_\sigma = \frac{\partial V}{\partial\sigma} = \frac{\sin 2\sigma\left(e^{2\omega} - e^{-2\omega}\right)}{\left(\frac{e^{2\omega}+e^{-2\omega}}{2} - \cos 2\sigma\right)^2}$$

$$V_\omega = \frac{\partial V}{\partial\omega} = \frac{\cos 2\sigma\left(e^{2\omega} + e^{-2\omega}\right) - 2}{\left(\frac{e^{2\omega}+e^{-2\omega}}{2} - \cos 2\sigma\right)^2}$$

$\forall \sigma, \omega \in$ the domain, U_σ, U_ω, V_σ, V_ω exist and satisfy the Cauchy-Reimann conditions, i.e.

$U_\sigma = V_\omega$

$U_\omega = -V_\sigma$

The Cauchy-Riemann conditions are satisfied, but do the partial derivatives exist everywhere in the domain? For that check the denominators. The denominator vanishes only in the case where $\omega = 0$ and $\sigma = n\pi$, where $n \in \mathbb{Z}$. We have already excluded these points from the domain. Hence, proved that $cot(s)$ is analytic. QED