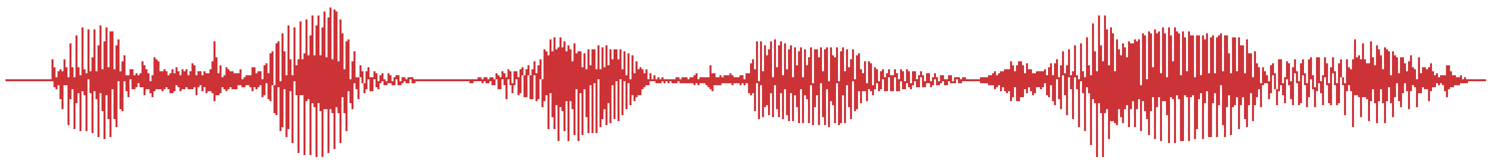
Introduction to Continuous Control Systems

EEME E3601



Week 13

Homayoon Beigi

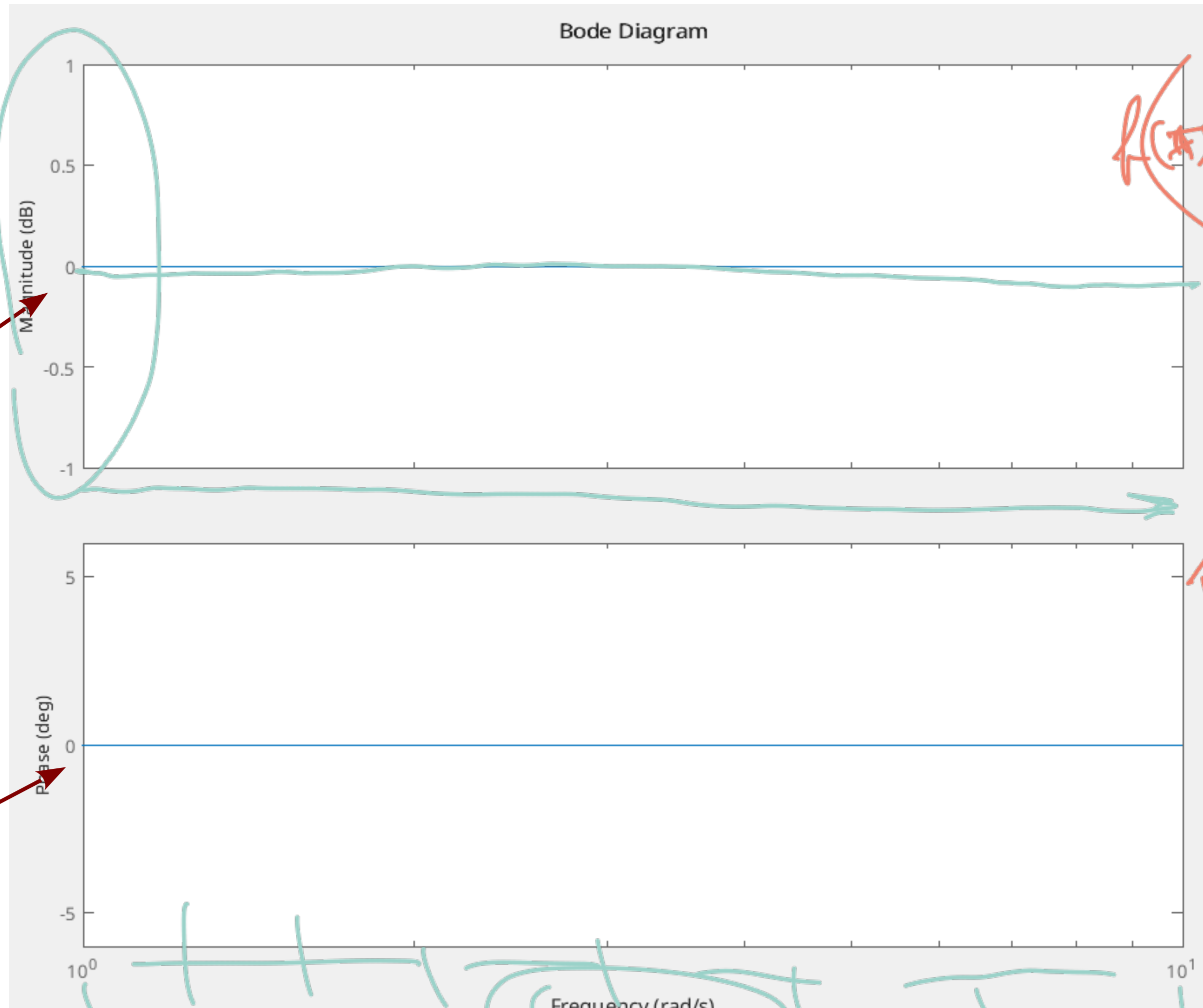
Homayoon.Beigi@columbia.edu
https://www.RecoTechnologies.com/beigi

Mechanical Engineering dept.
&
Electrical Engineering dept.

Columbia University, NYC, NY, U.S.A.

homayoon.beigi@columbia.edu

Bode Plot



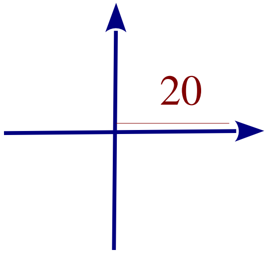
20 log 1 = 0

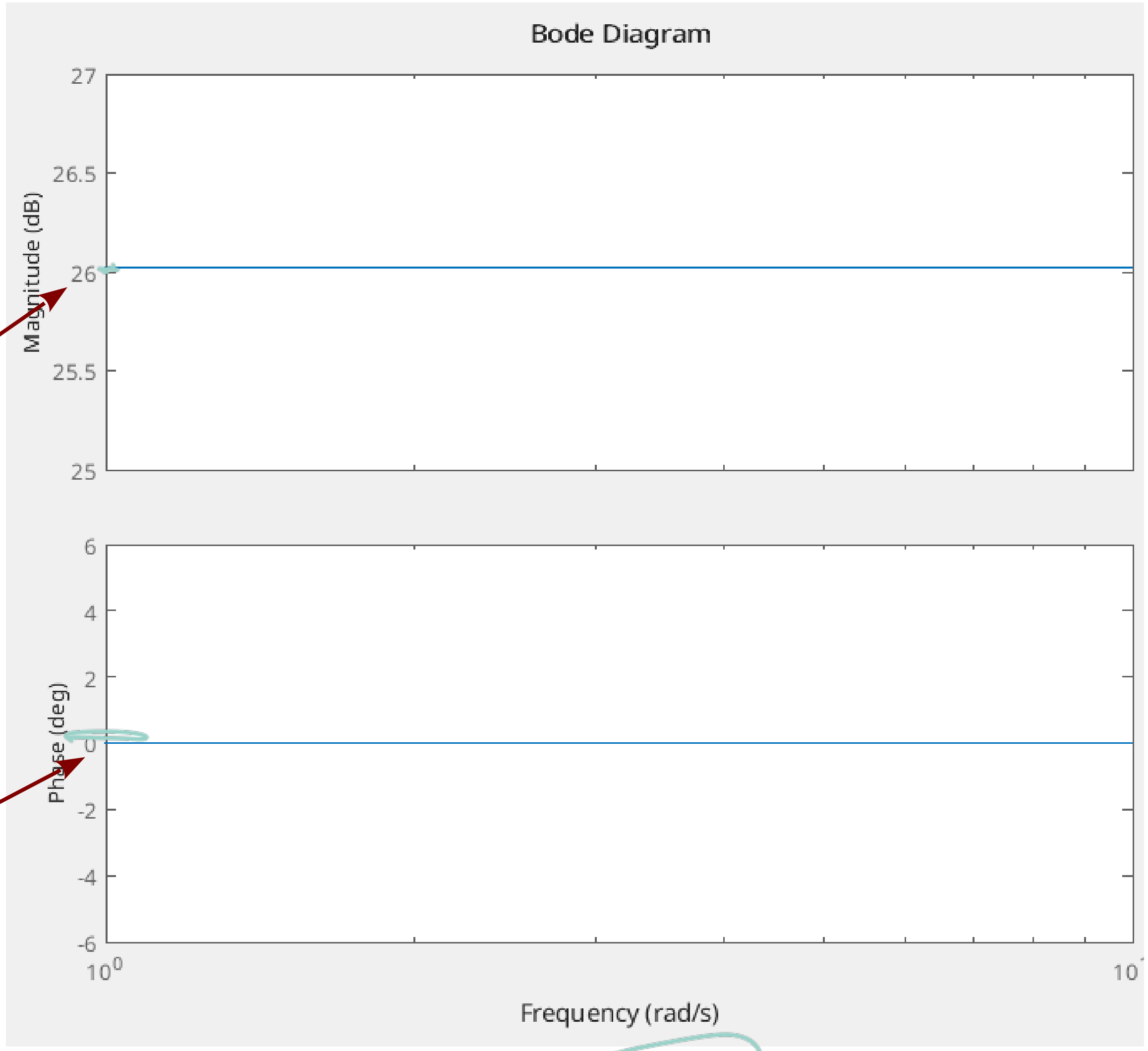
0 phase shift for positive gains

$G = 1$

homayoon.beigi@columbia.edu

Bode Plot



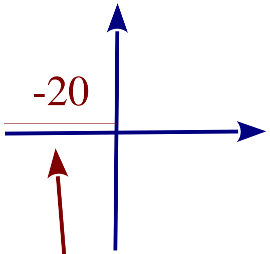


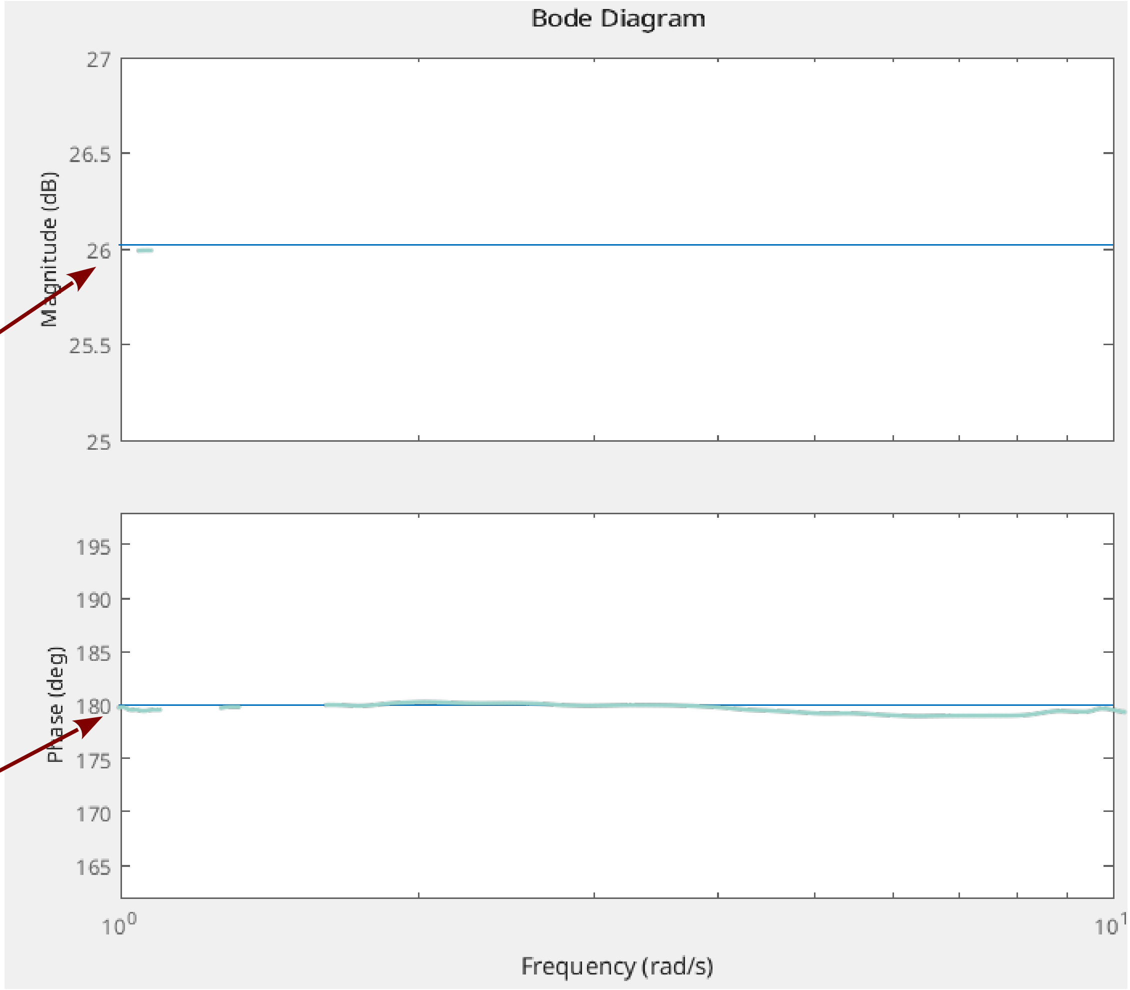
20 log 20

0 phase shift for positive gains

$G = 20$

Bode Plot





20 log 20

Opposite (180
degree phase)
For negative gains

$$G = -20$$

homayoon.beigi@columbia.edu

Bode Plot (Integrator)

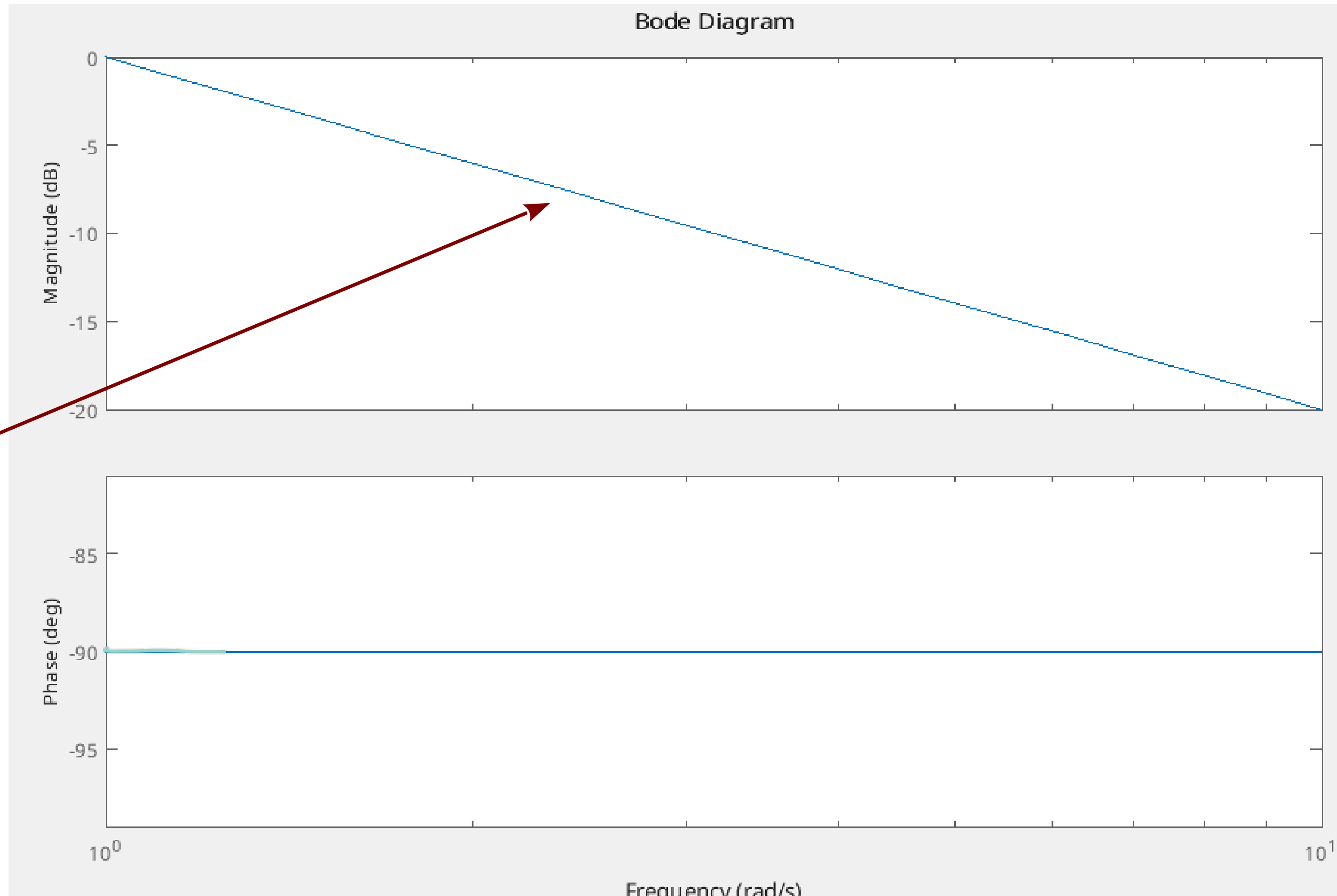
$U(s) \to \boxed{\frac{1}{s}} \to Y(s)$

$$
\begin{aligned}
G(i\omega) &= \frac{1}{i\omega} \\
&= -\frac{1}{\omega} i
\end{aligned}
$$

$$
\begin{aligned}
|G(i\omega)| &= \left| -\frac{1}{\omega} i \right| \\
&= \frac{1}{\omega}
\end{aligned}
$$

$$\angle G(i\omega) = -90^\circ$$

Slope=20db/decade



$$G = \frac{1}{s}$$

homayoon.beigi@columbia.edu

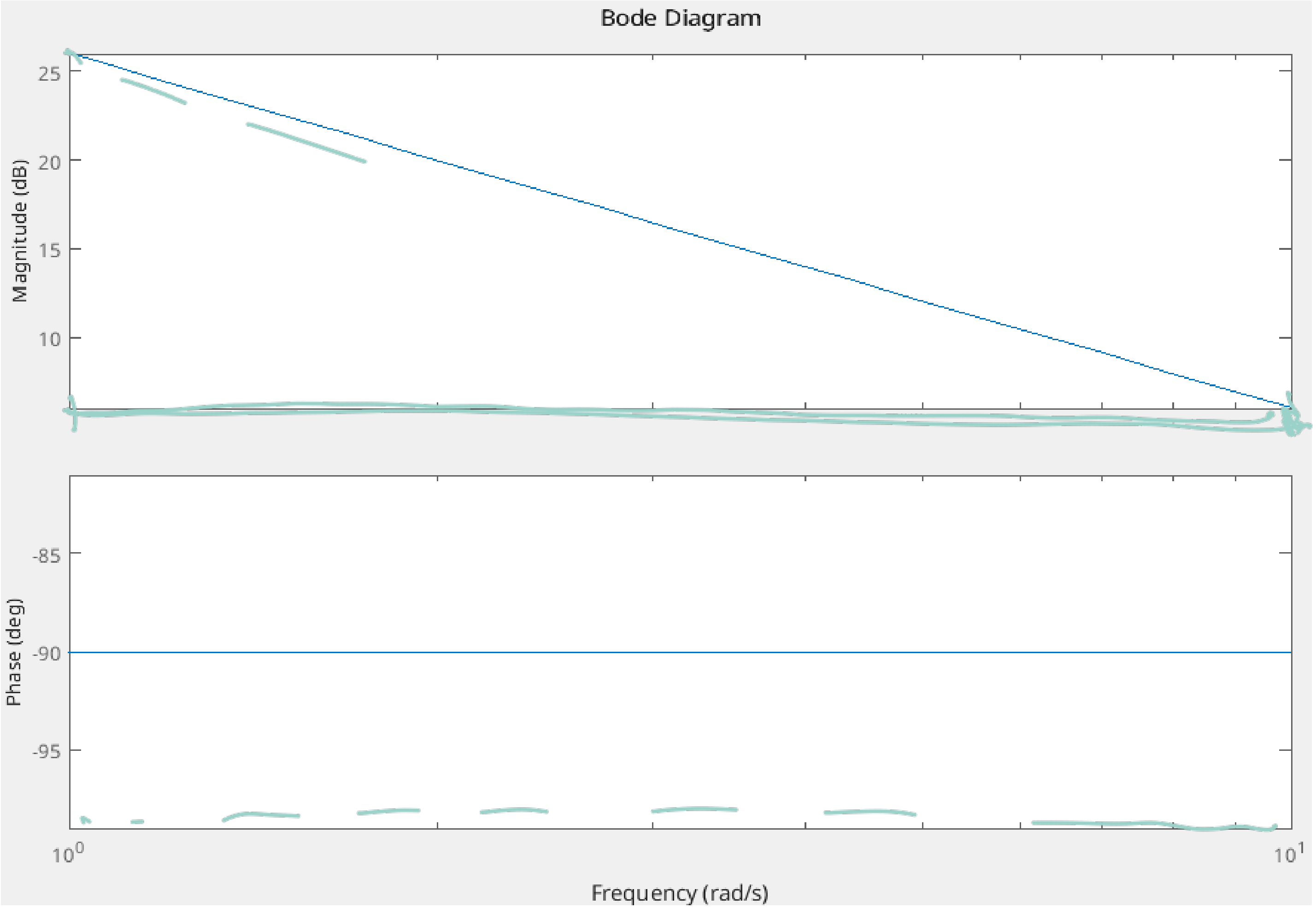
Bode Plot

$$G(s) = \frac{20}{s}$$
$$= 20\frac{1}{s}$$

$$|G(i\omega)| = |20|\left|-\frac{1}{\omega}i\right|$$

$$20\log(|20|\frac{1}{\omega}) =$$
$$20\log(|20|) + 20\log(\frac{1}{\omega})$$

$$\angle G(i\omega) = \angle 20 + \angle -\frac{1}{\omega}i$$
$$= 0^\circ - 90^\circ$$
$$= -90^\circ$$



$$G(s) = \frac{20}{s}$$

Bode Plot

$$
\begin{aligned}
G(s) &= \frac{20}{s^2} \\
&= 20\frac{1}{s}\frac{1}{s}
\end{aligned}
$$

$$
|G(i\omega)| = |20|\left|-\frac{1}{\omega}i\right|\left|-\frac{1}{\omega}i\right|
$$

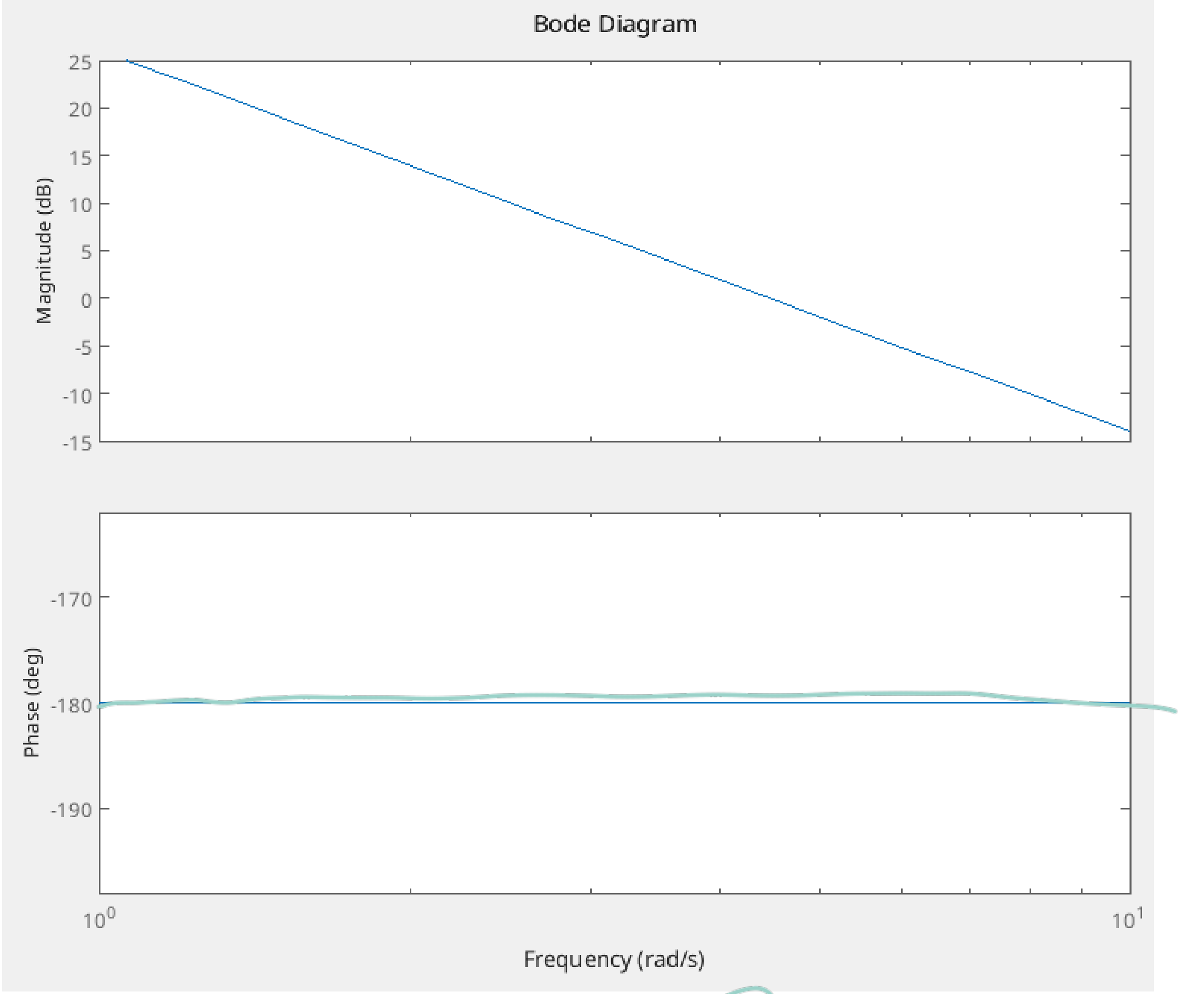
$$
20\log(|20|\frac{1}{\omega}\frac{1}{\omega}) =
$$

$$
20\log(|20|) + 20\log(\frac{1}{\omega}) + 20\log(\frac{1}{\omega})
$$

$$
=
$$

$$
20\log(|20|) + 40\log(\frac{1}{\omega})
$$

$$
\begin{aligned}
\angle G(i\omega) &= \angle 20 + \angle -\frac{1}{\omega}i + \angle -\frac{1}{\omega}i \\
&= 0^\circ - 90^\circ - 90^\circ \\
&= -180^\circ
\end{aligned}
$$

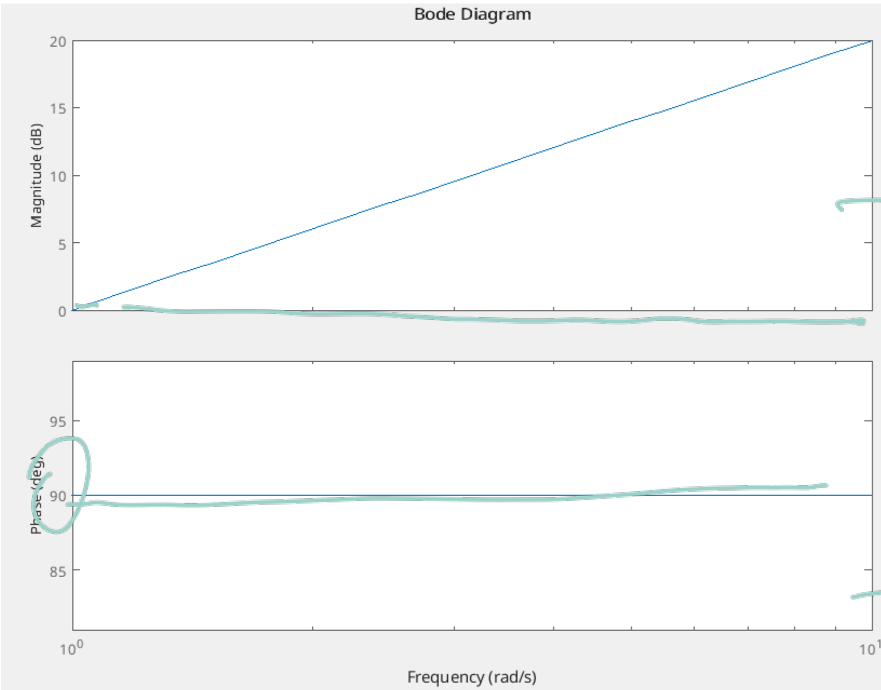


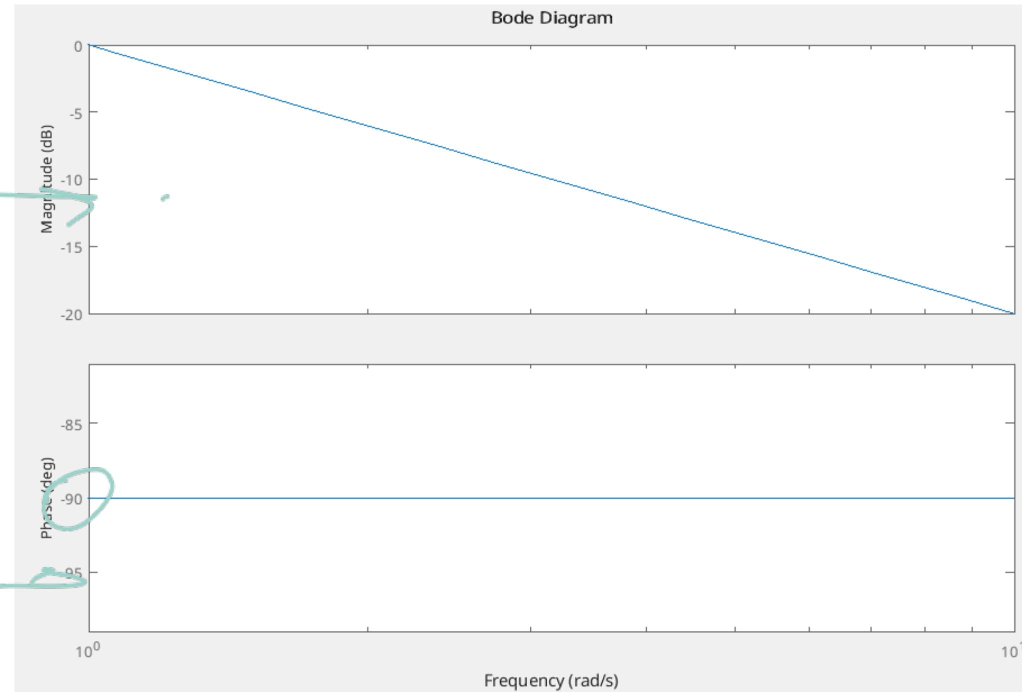
$$
G(s) = \frac{20}{s^2}
$$

Bode Plot

$Y(s) \rightarrow \boxed{s} \rightarrow Y_1(s)$

$Y_1(s) \rightarrow \boxed{\frac{1}{s}} \rightarrow Y_2(s)$





$$G = s$$

Single Zero

Positive phase
Positive slope

$$G = \frac{1}{s}$$

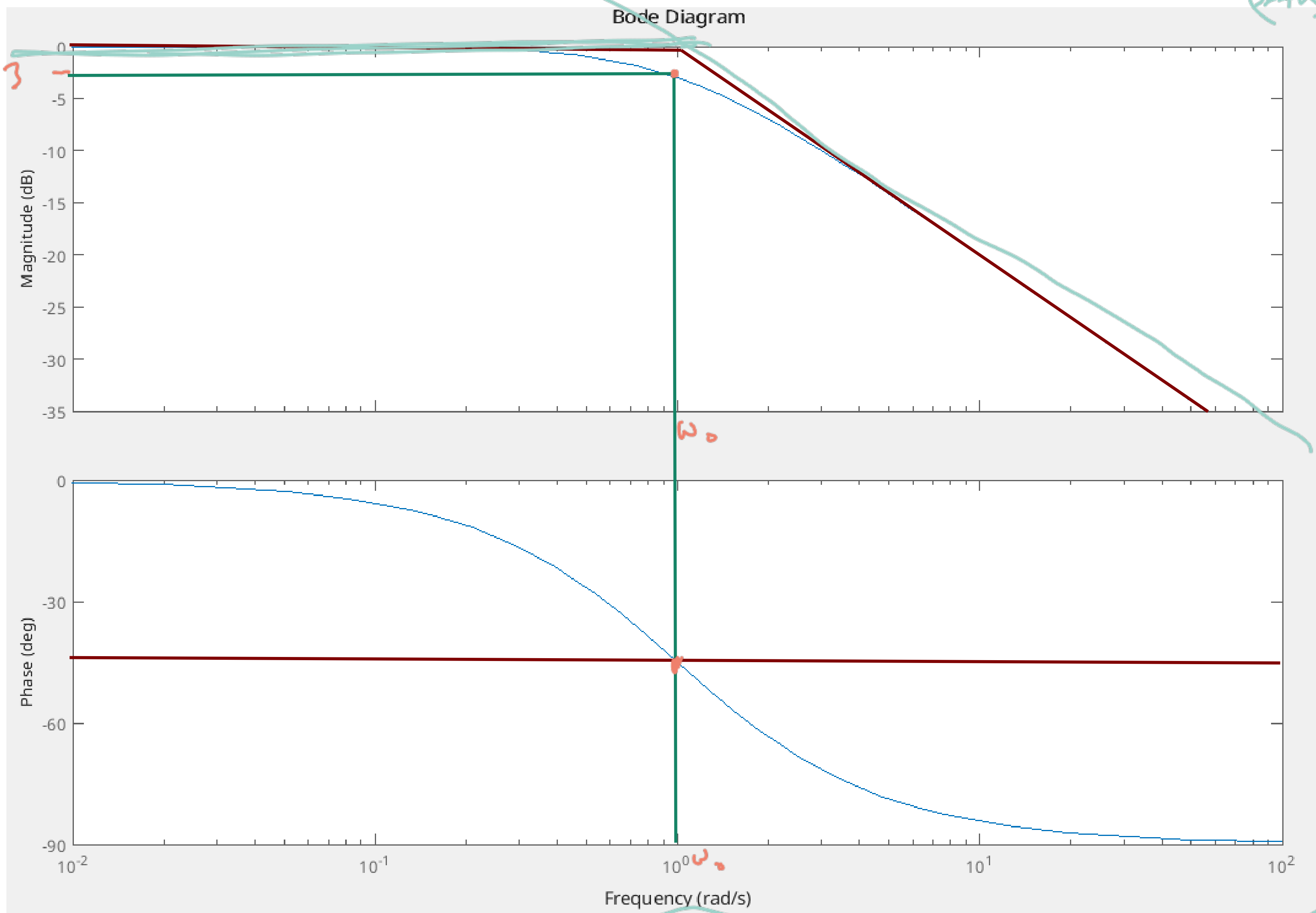
Single Pole

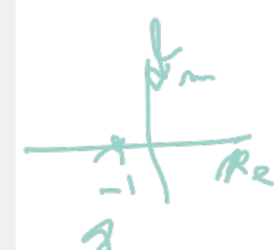
Negative phase
Negative slope

homayoon.beigi@columbia.edu

Bode Plot

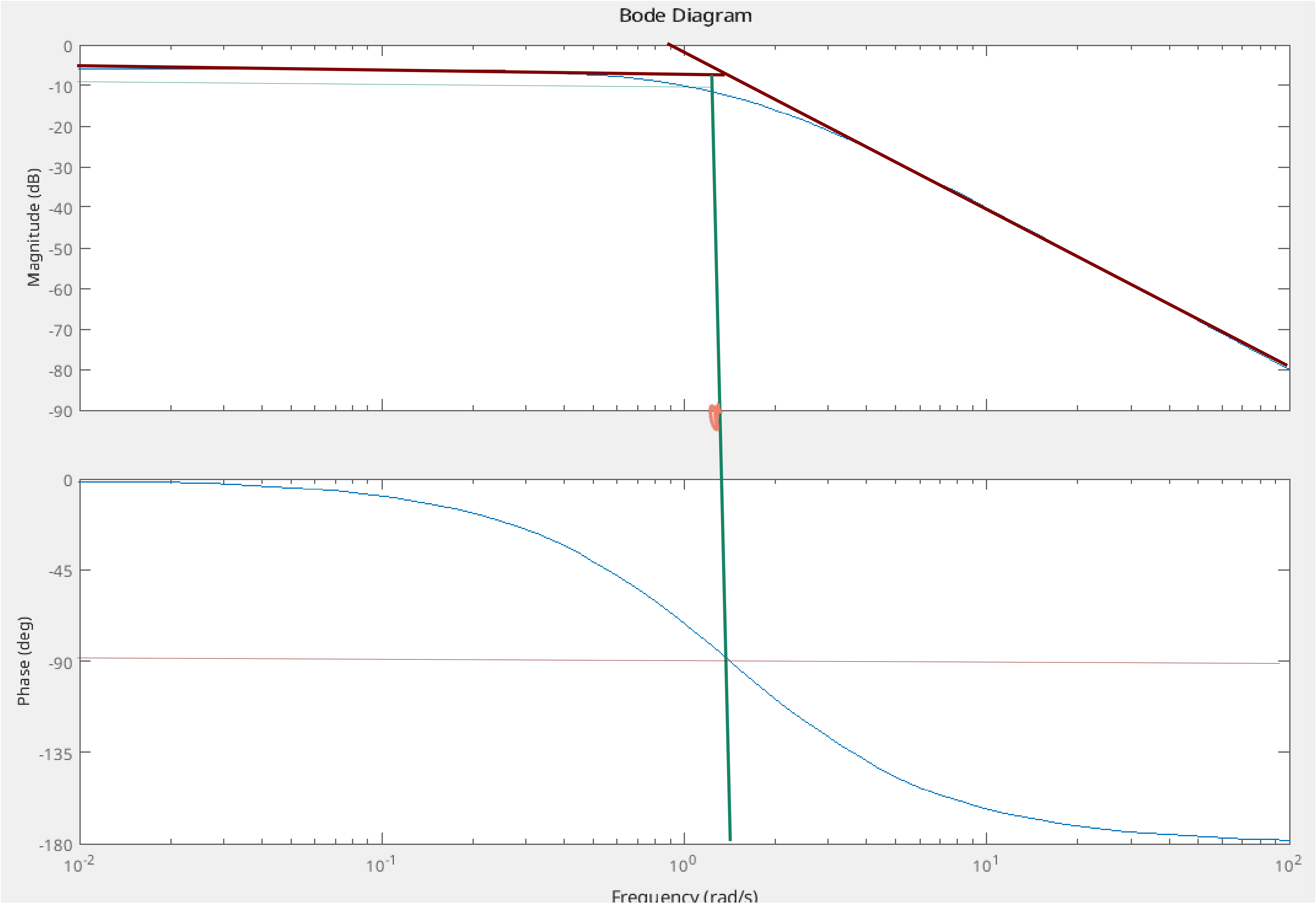
$$\frac{1}{(s+\alpha)(s+\beta)}$$





$$G = \frac{1}{s+1}$$

Bode Plot



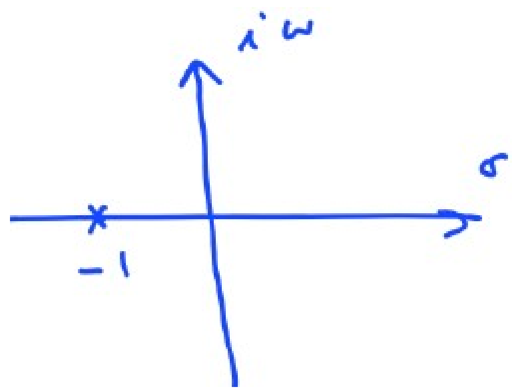
$$G = \frac{1}{s^2 + 3s + 2}$$

Bode Plot

$$\dot{y}(t) = -\lambda y(t) + u(t)$$

$$\tau \triangleq \frac{1}{\lambda}$$

time constant



$$(s+\lambda)\, Y(s) = U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s+\lambda}$$

$$= \frac{1}{\lambda + s}$$

$$= \frac{1}{\lambda}\,\frac{1}{1+\frac{s}{\lambda}}$$

$$\boxed{\frac{1}{\lambda}\,\frac{1}{\frac{\lambda+s}{\lambda}} = \frac{1}{\cancel{\lambda}}\,\frac{\cancel{\lambda}}{\lambda+s} = \frac{1}{\lambda+s}}$$

$$\tau\,\frac{1}{1+\tau s}$$

Bode Plot

use the following $G(s) = \dfrac{1}{1 + \dfrac{s}{\omega_0}}$

set $s = i\omega$ $\quad G(i\omega) = \dfrac{1}{1 + i\dfrac{\omega}{\omega_0}} \cdot \dfrac{1 - i\dfrac{\omega}{\omega_0}}{1 - i\dfrac{\omega}{\omega_0}}$

$$= \frac{1 - i\frac{\omega}{\omega_0}}{1 + \left(\frac{\omega}{\omega_0}\right)^2}$$

gain $\quad |G(i\omega)| = \sqrt{\mathrm{Re}\{G(i\omega)\}^2 + \mathrm{Im}\{G(i\omega)\}^2}$

$$= \frac{1}{1 + \left(\frac{\omega}{\omega_0}\right)^2} \sqrt{\left[1 + \left(\frac{\omega}{\omega_0}\right)^2\right]}$$

$$= \boxed{\frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}}}$$

homayoon.beigi@columbia.edu

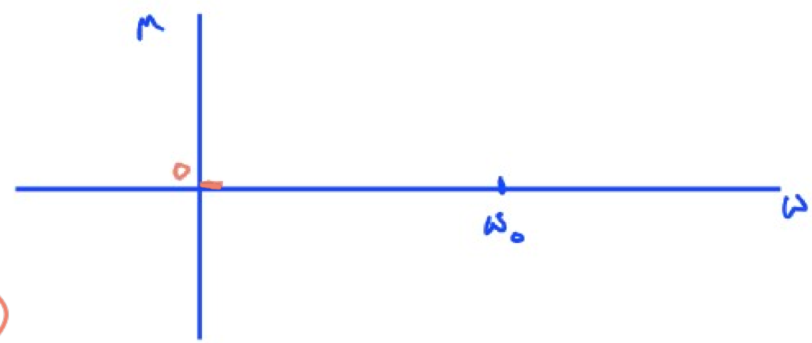
Bode Plot

$$|G(i\omega)| = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_0}\right)^2}}$$

gain $m = 20 \log |G(i\omega)| = -20 \log \left(\sqrt{1+\left(\frac{\omega}{\omega_0}\right)^2} \right)$

phase $\varphi = \tan^{-1} \left[\frac{\mathrm{Im}\{G(i\omega)\}}{\mathrm{Re}\{G(i\omega)\}} \right] = \tan^{-1} \left[-\frac{\omega}{\omega_0} \right]$

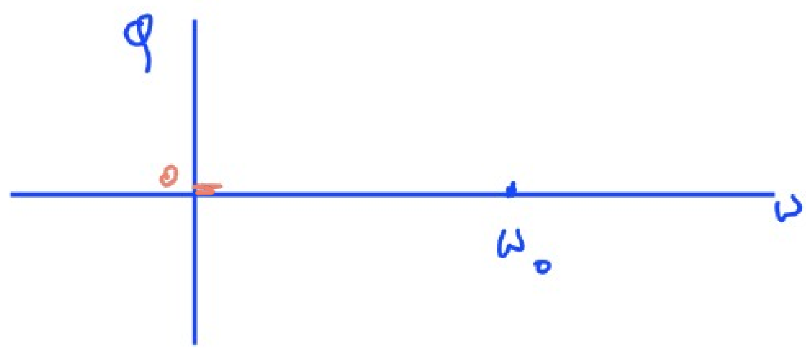
mind the quadrant



if $\omega \ll \omega_0$

$m \to 0$

$\varphi \to 0$



homayoon.beigi@columbia.edu

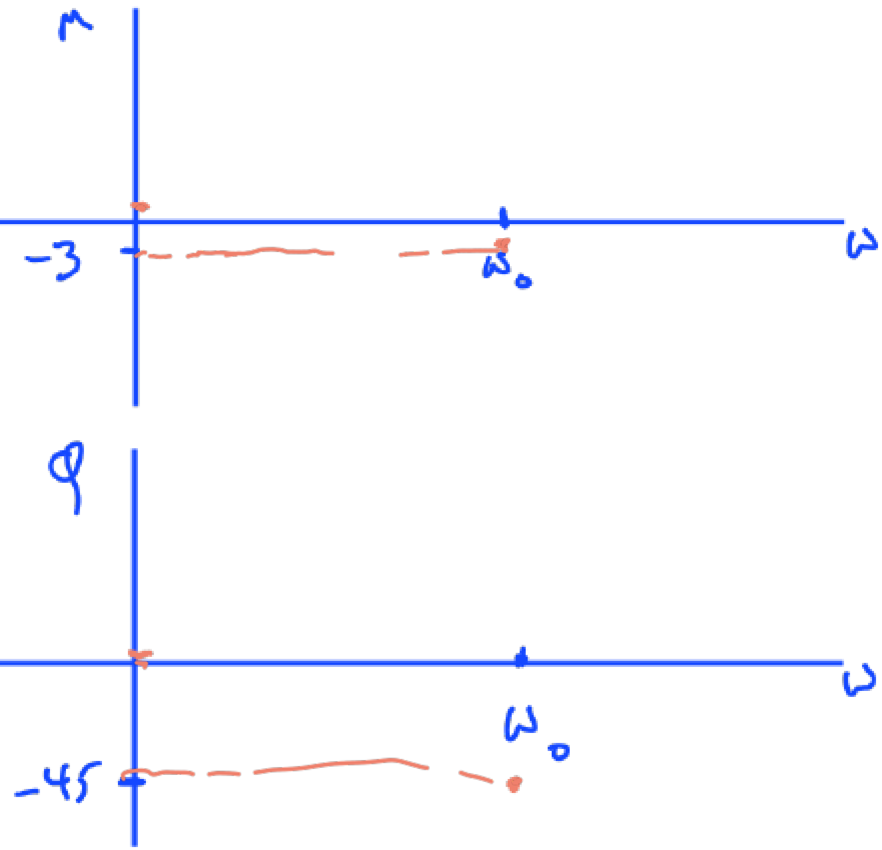
Bode Plot

if

$\boxed{\omega = \omega_0}$

$M = -20 \log \sqrt{2}$

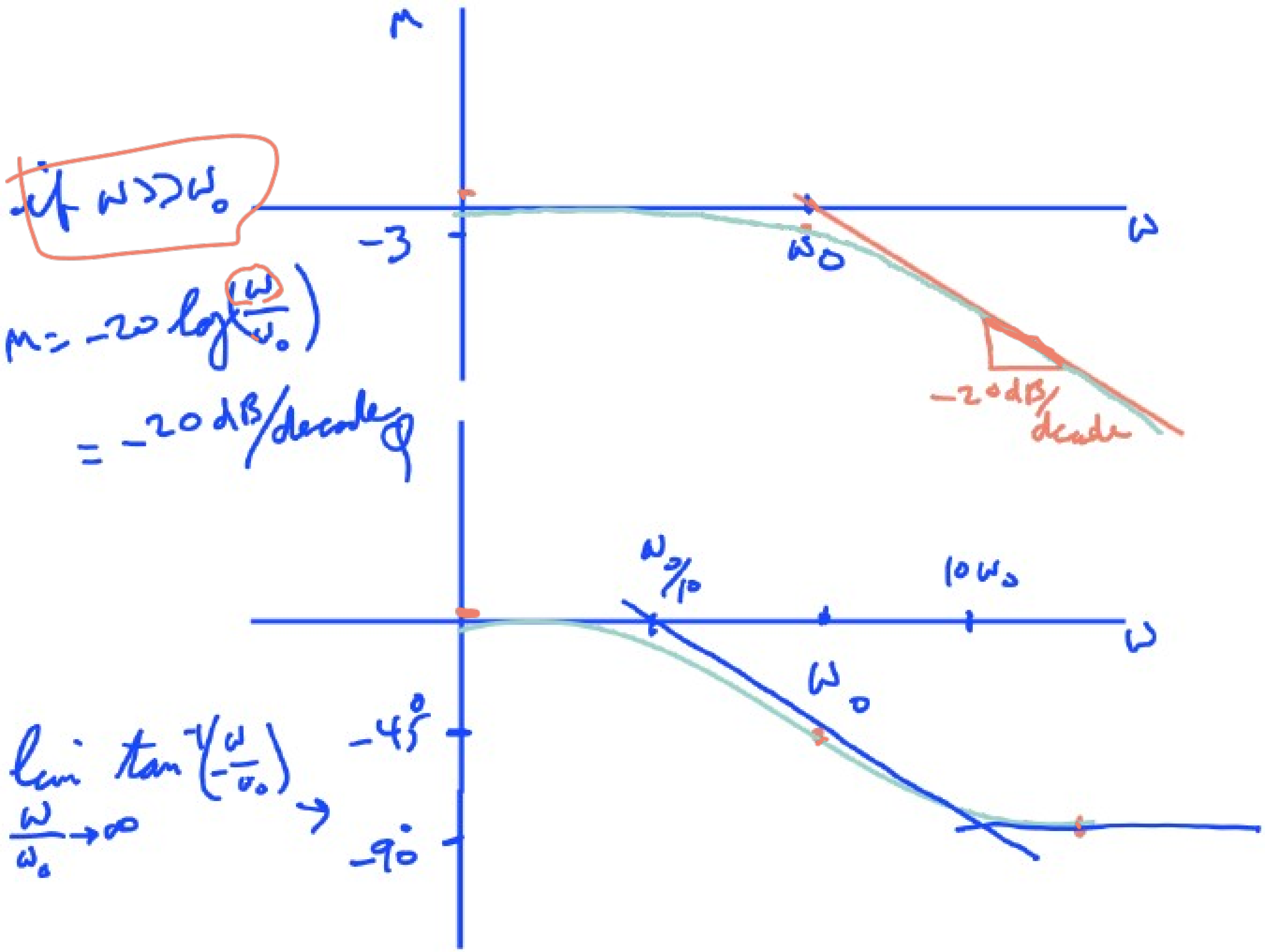
$= -3 dB$



$\varphi = \tan^{-1}(-1)$

$= -45°$

Bode Plot


M
if ω >> ω0
-3
ω0
ω
M = -20 log(ω/ω0)
= -20 dB/decade
-20 dB/decade
φ
ω0/10
10ω0
ω
ω0
-45°
lim tan-1(-ω/ω0) →
ω/ω0 → ∞
-90°

Bode Plot

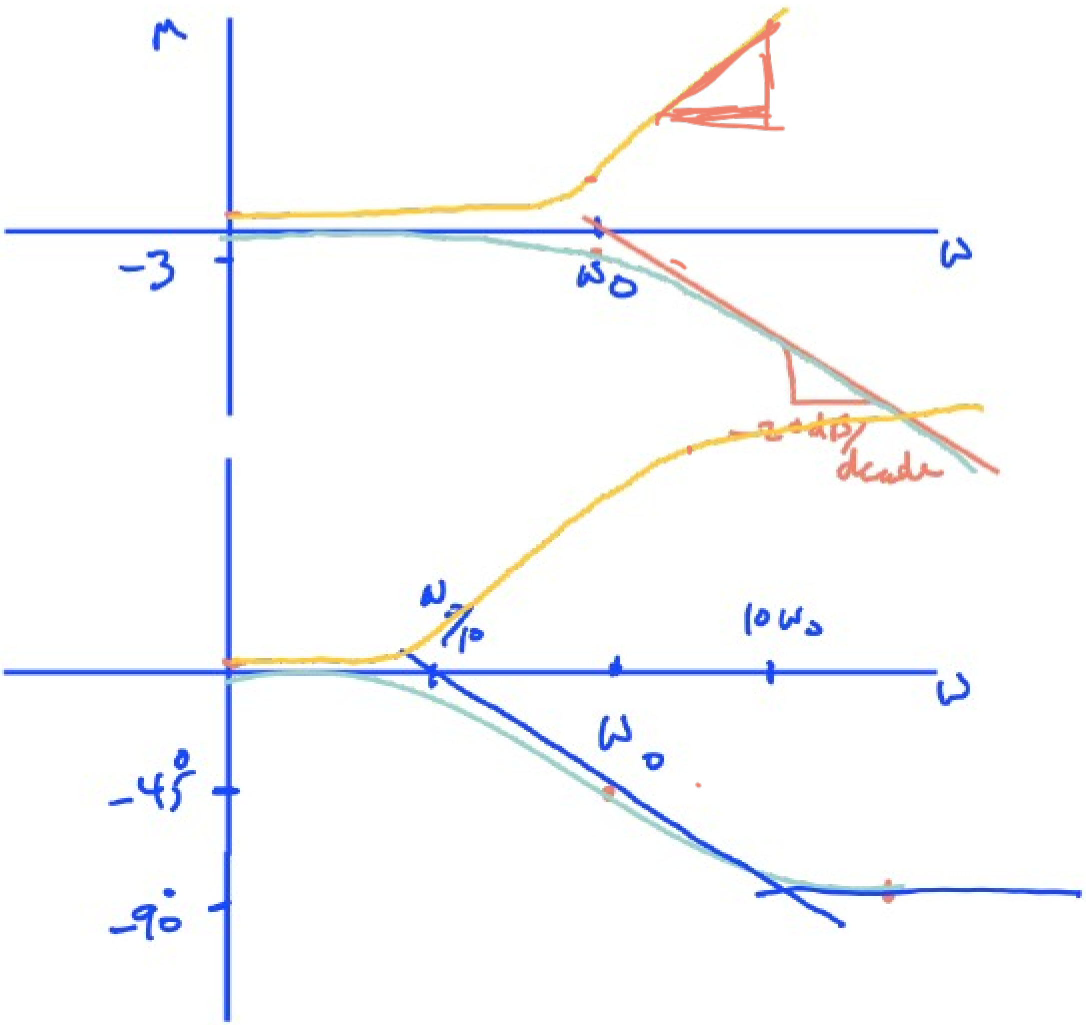
for a simple zero

$$G(s) = 1 + \frac{s}{\omega_0} \qquad (s+\omega_0)$$

$$G(j\omega) = 1 + \frac{j\omega}{\omega_0}$$

$$M = 20\log\left(\sqrt{1+\left(\frac{\omega}{\omega_0}\right)^2}\right)$$

$$\varphi = \tan^{-1}\left(\frac{\omega}{\omega_0}\right)$$



Bode Plot

$\omega_0 = \omega_n$

complex conjugate poles

recall the form

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

$$= \frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\frac{s}{\omega_0} + 1}$$

$$ms^2 + cs + k = 0$$

$$\frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

$c^2 < 4km \Rightarrow$ complex conjugate

$$\zeta \triangleq \sqrt{\frac{c^2}{4km}} = \frac{c}{2\sqrt{km}}$$

damping ratio

$$G(i\omega) = \frac{1}{\left(\frac{i\omega}{\omega_0}\right)^2 + 2\zeta\frac{\omega}{\omega_0}i + 1}$$

$$= \frac{1}{-\left(\frac{\omega}{\omega_0}\right)^2 + 2\zeta\frac{\omega}{\omega_0} + 1} = \frac{1}{1 - \left(\frac{\omega}{\omega_0}\right)^2 + 2\zeta\frac{\omega}{\omega_0}i}$$

homayoon.beigi@columbia.edu

Bode Plot

$$G(i\omega) = \frac{1}{1-\left(\frac{\omega}{\omega_0}\right)^2 + 2\zeta\frac{\omega}{\omega_0} i} \cdot \frac{1-\left(\frac{\omega}{\omega_0}\right)^2 - 2\zeta\frac{\omega}{\omega_0} i}{1-\left(\frac{\omega}{\omega_0}\right)^2 - 2\zeta\frac{\omega}{\omega_0} i}$$

$$= \frac{1-\left(\frac{\omega}{\omega_0}\right)^2 - 2\zeta\frac{\omega}{\omega_0} i}{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_0}\right]^2}$$

$$|G(i\omega)| = \frac{1}{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_0}\right]^2}\left[\sqrt{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_0}\right)^2}\right]$$

$$= \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2}}$$

Bode Plot

$$|G(i\omega)| = \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2+\left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2}}$$

$$M = 20\log|G(i\omega)| = -20\log\sqrt{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2+\left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2}$$

$$\varphi = \tan^{-1}\left[\frac{-2\zeta\frac{\omega}{\omega_0}}{1-\left(\frac{\omega}{\omega_0}\right)^2}\right]$$

if $\omega \ll \omega_0$ $\left.\begin{array}{l} Re\{G(i\omega)\} \to 1 \\ Im\{G(i\omega)\} \to 0 \end{array}\right\} \Longrightarrow |G(i\omega)| \to 1$

$\Rightarrow M \to 0$

$$\varphi = \tan^{-1}\left(\frac{Im(G(i\omega))}{Re(G(i\omega))}\right) \to 0^\circ$$

homayoon.beigi@columbia.edu

Bode Plot

if $\omega \gg \omega_0$

$$\mathrm{Re}\{G(i\omega)\} \rightarrow \frac{-\left(\frac{\omega}{\omega_0}\right)^2}{\left(\frac{\omega}{\omega_0}\right)^4 + \epsilon} = -\left(\frac{\omega}{\omega_0}\right)^{-2}$$

$$\mathrm{Im}\{G(i\omega)\} \rightarrow \frac{-2\zeta\frac{\omega}{\omega_0}}{\left(\frac{\omega}{\omega_0}\right)^4} = -2\zeta\left(\frac{\omega}{\omega_0}\right)^{-3}$$

$$|G(i\omega)| \rightarrow \sqrt{\left[-\left(\frac{\omega}{\omega_0}\right)^{-2}\right]^2 + \underbrace{\left[-2\zeta\left(\frac{\omega}{\omega_0}\right)^{-3}\right]^2}_{\rightarrow\,\epsilon}}$$

$$\approx \sqrt{\left(\left(\frac{\omega}{\omega_0}\right)^{-2}\right)^2}$$

$$= \left(\frac{\omega}{\omega_0}\right)^{-2}$$

$$M = 20\log|G(i\omega)| = 20\log\left(\frac{\omega}{\omega_0}\right)^{-2} = -40\log\left(\frac{\omega}{\omega_0}\right)$$

-40 dB/decade

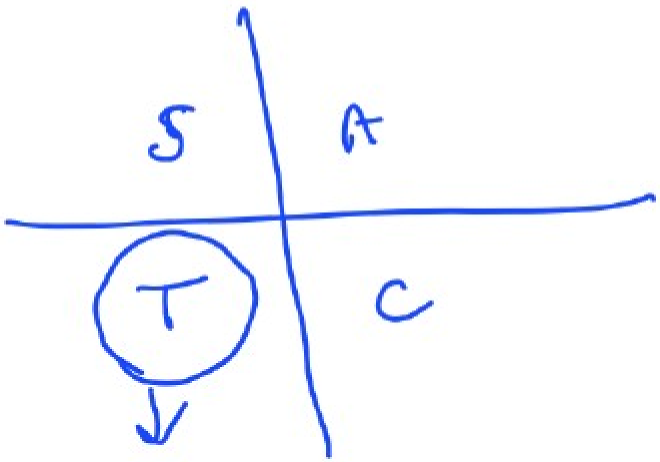
Bode Plot

$$\phi = \tan^{-1}\left[\frac{-25\left(\frac{\omega}{\omega_0}\right)^{-3}}{-\left(\frac{\omega}{\omega_0}\right)^{-2}}\right]$$

$$= \tan^{-1}\left(25\left(\frac{\omega}{\omega_0}\right)^{-1}\right) - 180^\circ$$

$$= 0 - 180^\circ$$

$$= -180^\circ$$



$\tan^{-1} - 180^\circ$

Bode Plot

If $\omega = \omega_0$

$$Re\{G(i\omega)\} = \frac{\cancel{1-\left(\frac{\omega}{\omega_0}\right)^2} \to 0}{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left[2\zeta\frac{\omega}{\omega_0}\right]^2} \cong 0$$

$$Im\{G(i\omega)\} = \frac{-2\zeta\frac{\omega}{\omega_0} \to -2\zeta}{\cancel{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2}^{\,0} + \cancel{\left[2\zeta\frac{\omega}{\omega_0}\right]^2}\,(2\zeta)^2} \cong -\frac{1}{2\zeta}$$

$$|G(i\omega)| = \sqrt{0^2 + \left(-\frac{1}{2\zeta}\right)^2}$$

$$= \frac{1}{2\zeta}$$

$$M = 20\log|G(i\omega)| = 20\log\frac{1}{2\zeta}$$

$$= -20\log(2\zeta)$$

$$\Phi = Tan^{-1}(\infty) = -90$$

3rd quadrant

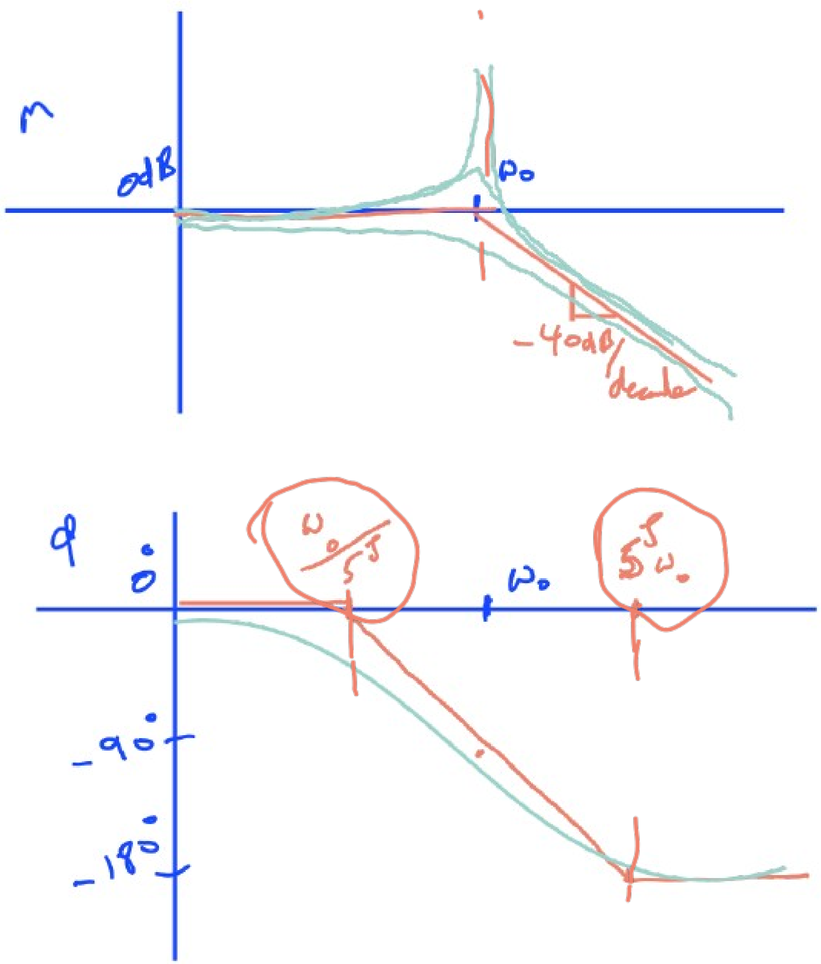
if $\zeta = 0 \Rightarrow M \to \infty$

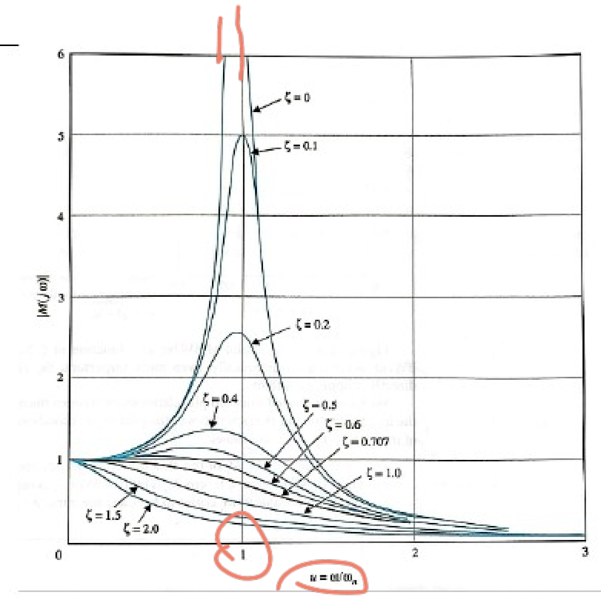
if $\zeta = 0.5 \Rightarrow M = 0$

if $\zeta > 0.5 \Rightarrow M \downarrow$ (negative)

if $\zeta > 0.5 \Rightarrow$ max. at ω_0 $= -20\log(2\zeta)$

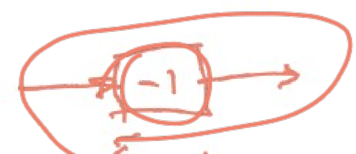
Bode Plot





- trouble spot: $M = 0$ & $\varphi = -180$

$CGH = -1$

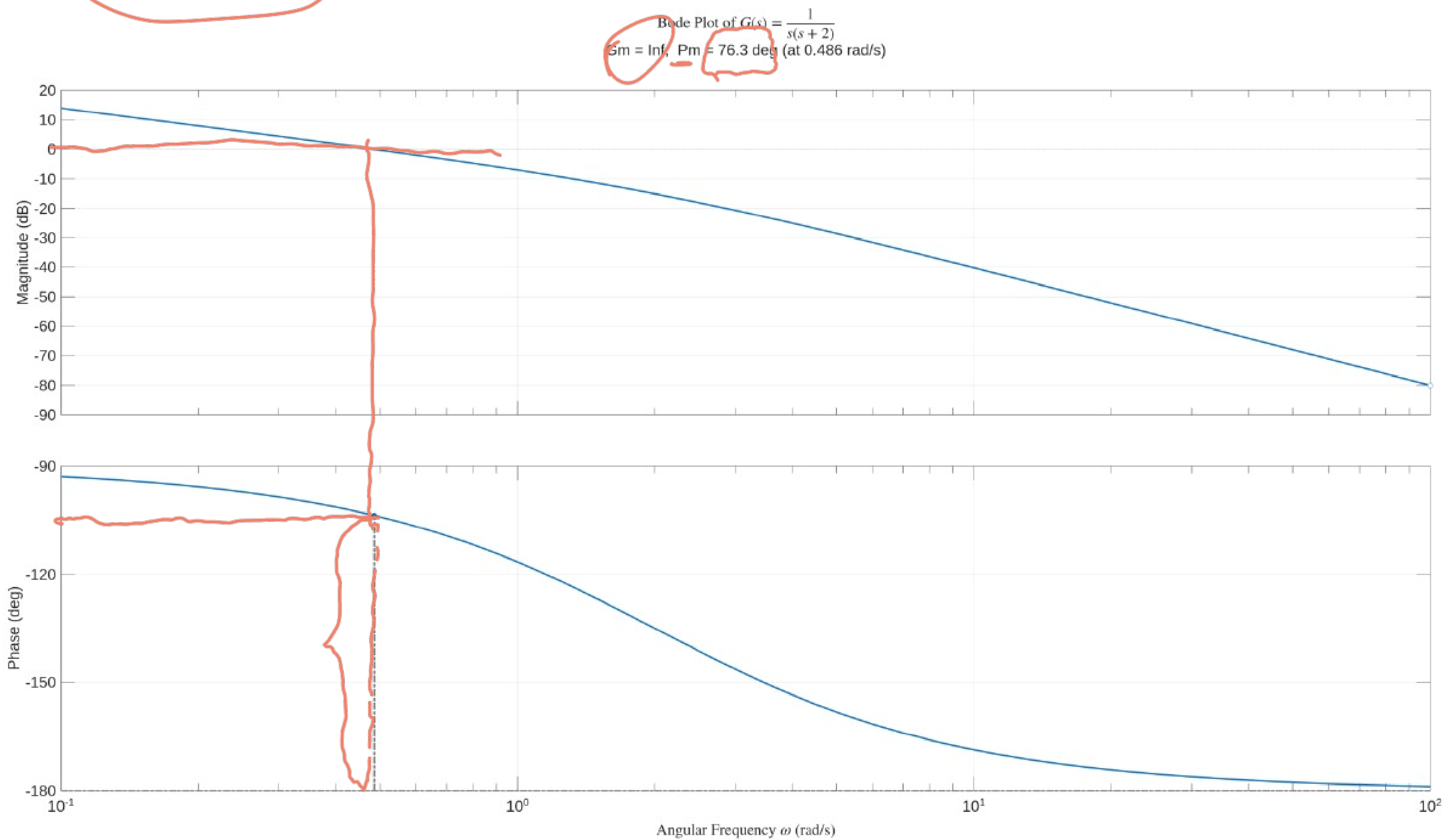


$\tilde{G} = -1$

should not happen simultaneously $\left(\begin{array}{l} |\tilde{G}| = 1 \quad 20 \log |\tilde{G}| = 0 \\ \angle \tilde{G} = -180° \end{array} \right.$

Bode Plot

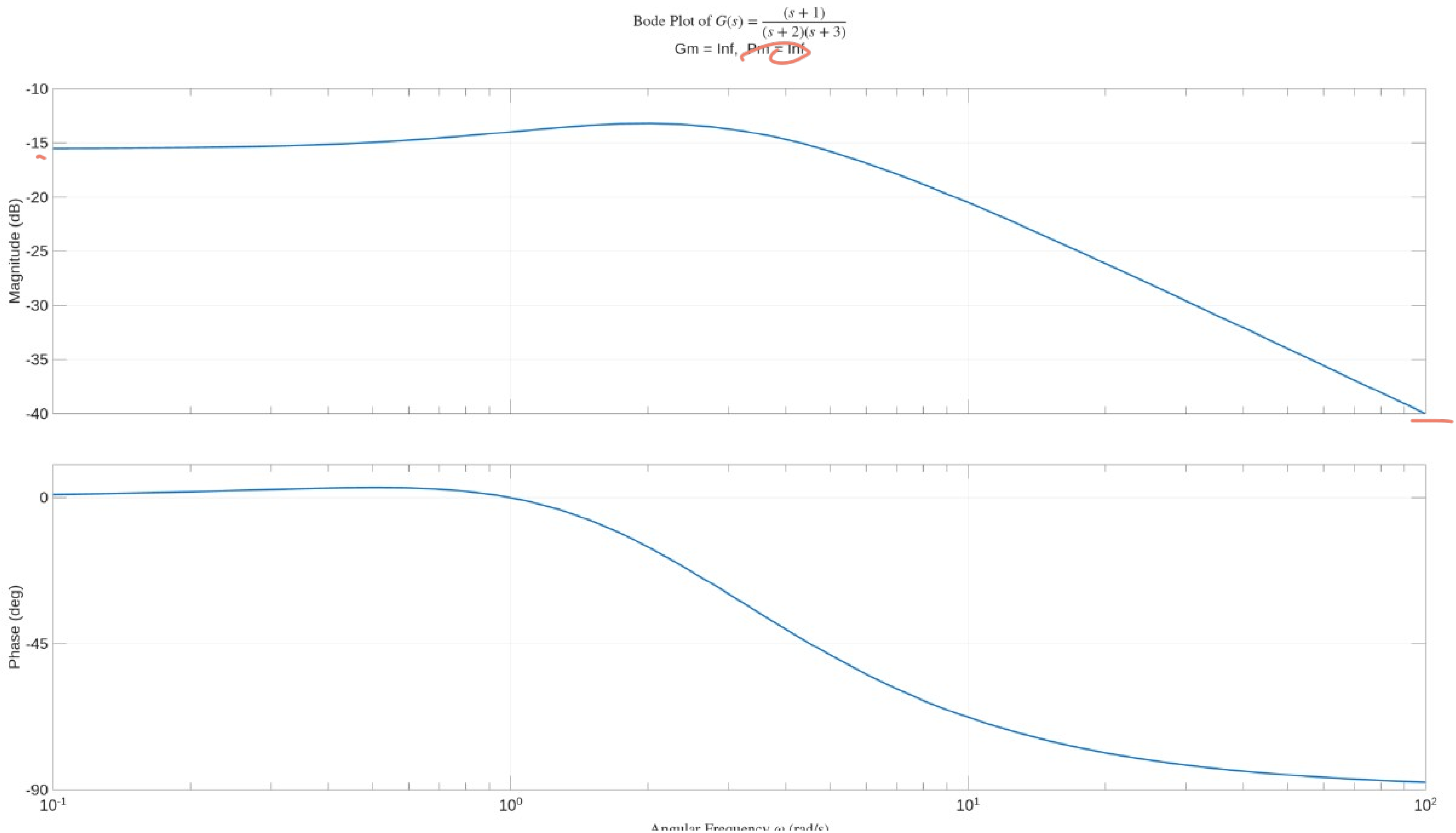
$$G(s) = \frac{1}{s(s+2)}$$



homayoon.beigi@columbia.edu

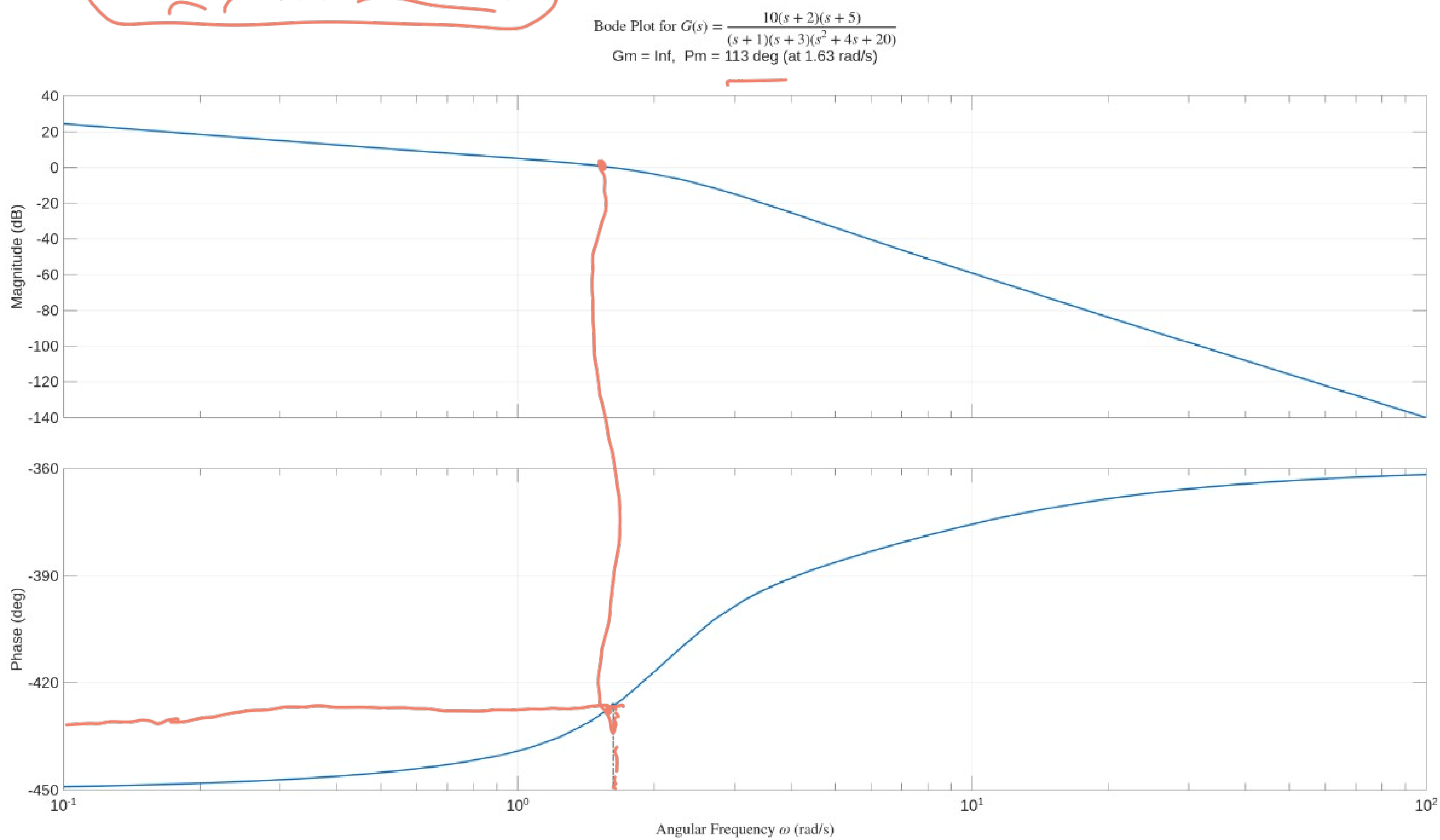
Bode Plot

$$G(s) = \frac{(s+1)}{(s+2)(s+3)}$$



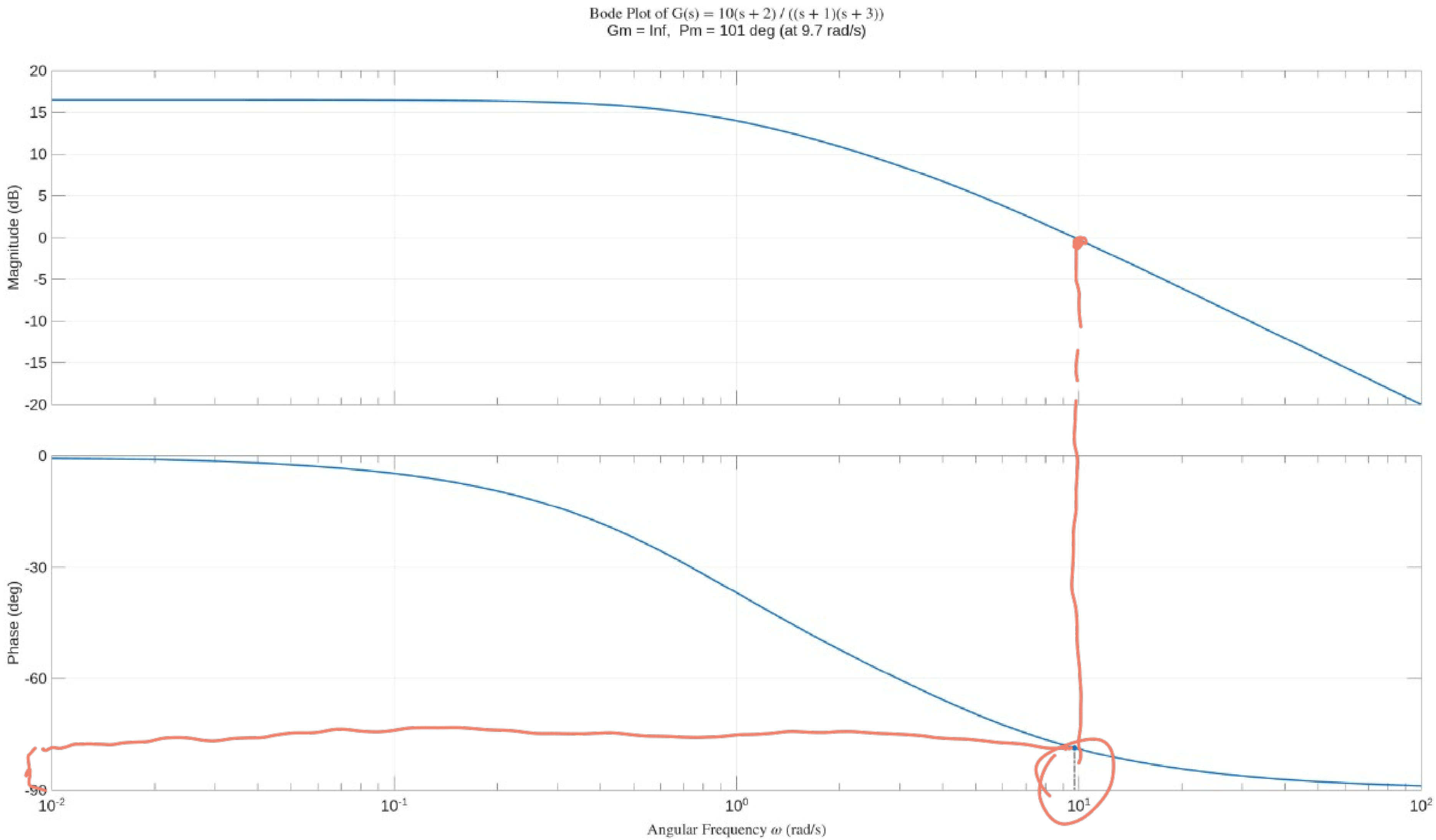
Bode Plot

$$G(s) = \frac{10(s+2)(s+5)}{(s+3)(s+1)(s^2+4s+20)}$$



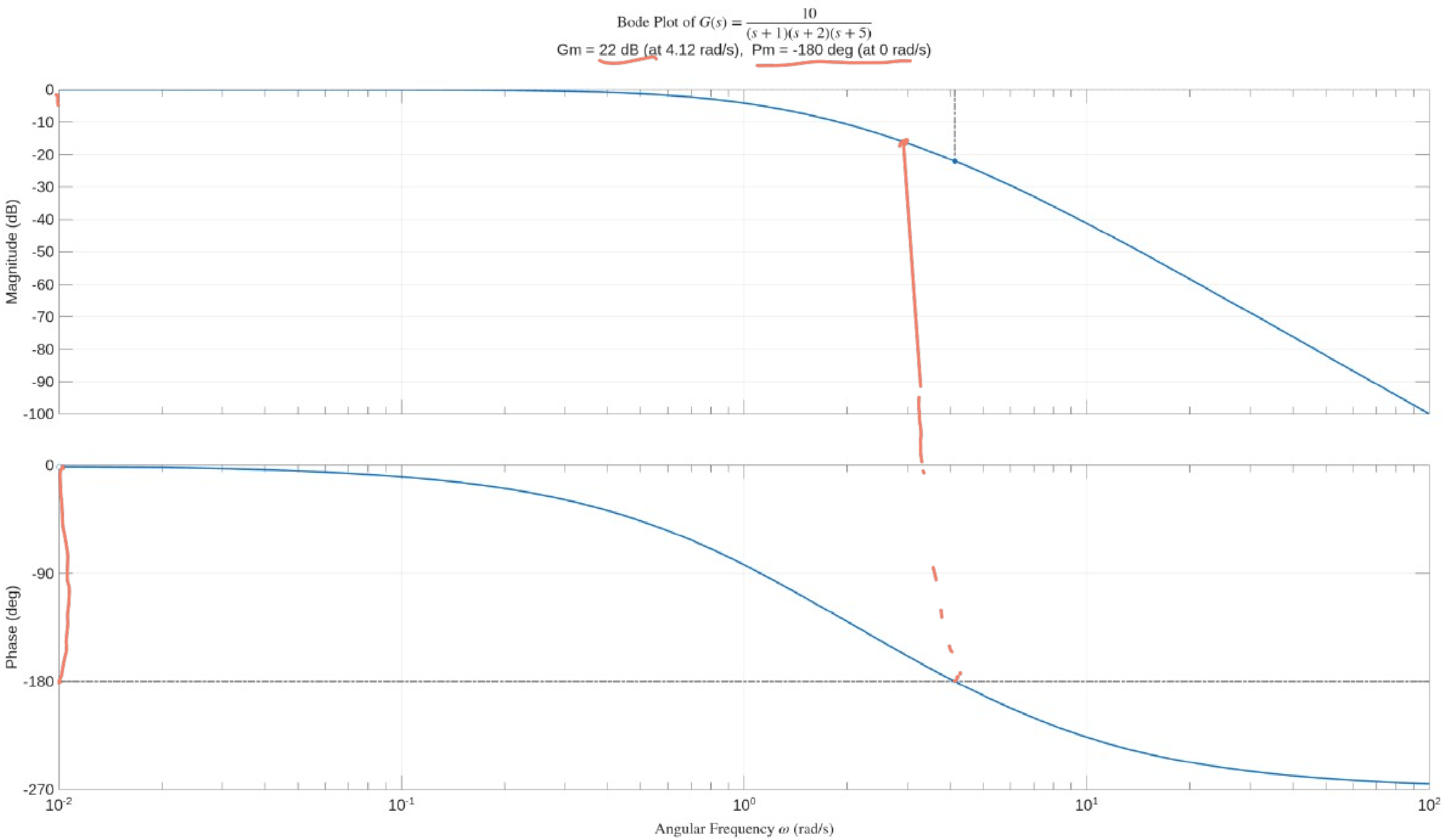
$$G(s) = \frac{10(s+2)}{(s+3)(s+1)}$$

Bode Plot

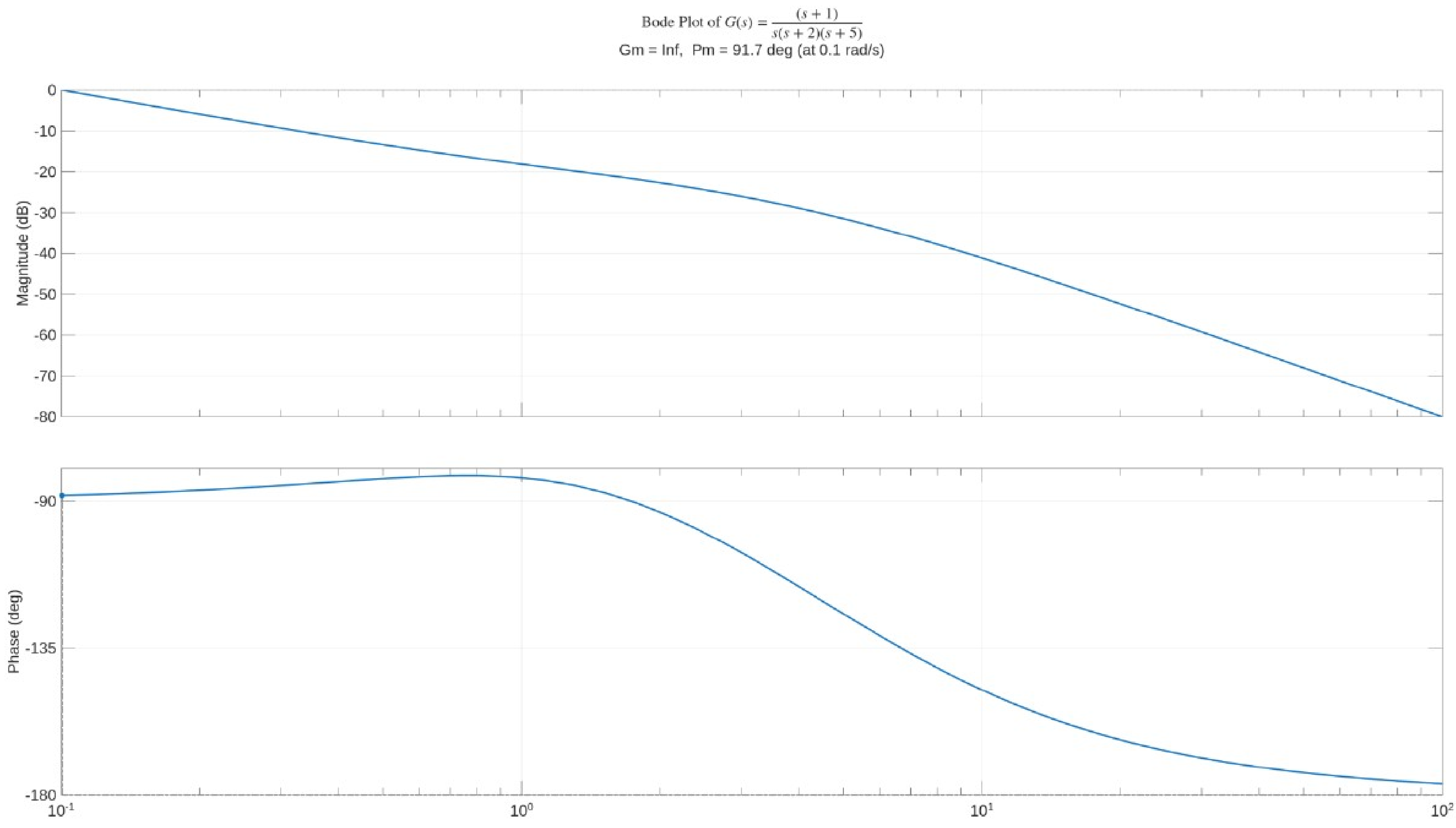


Bode Plot

$$G(s) = \frac{10}{(s+1)(s+2)(s+3)}$$

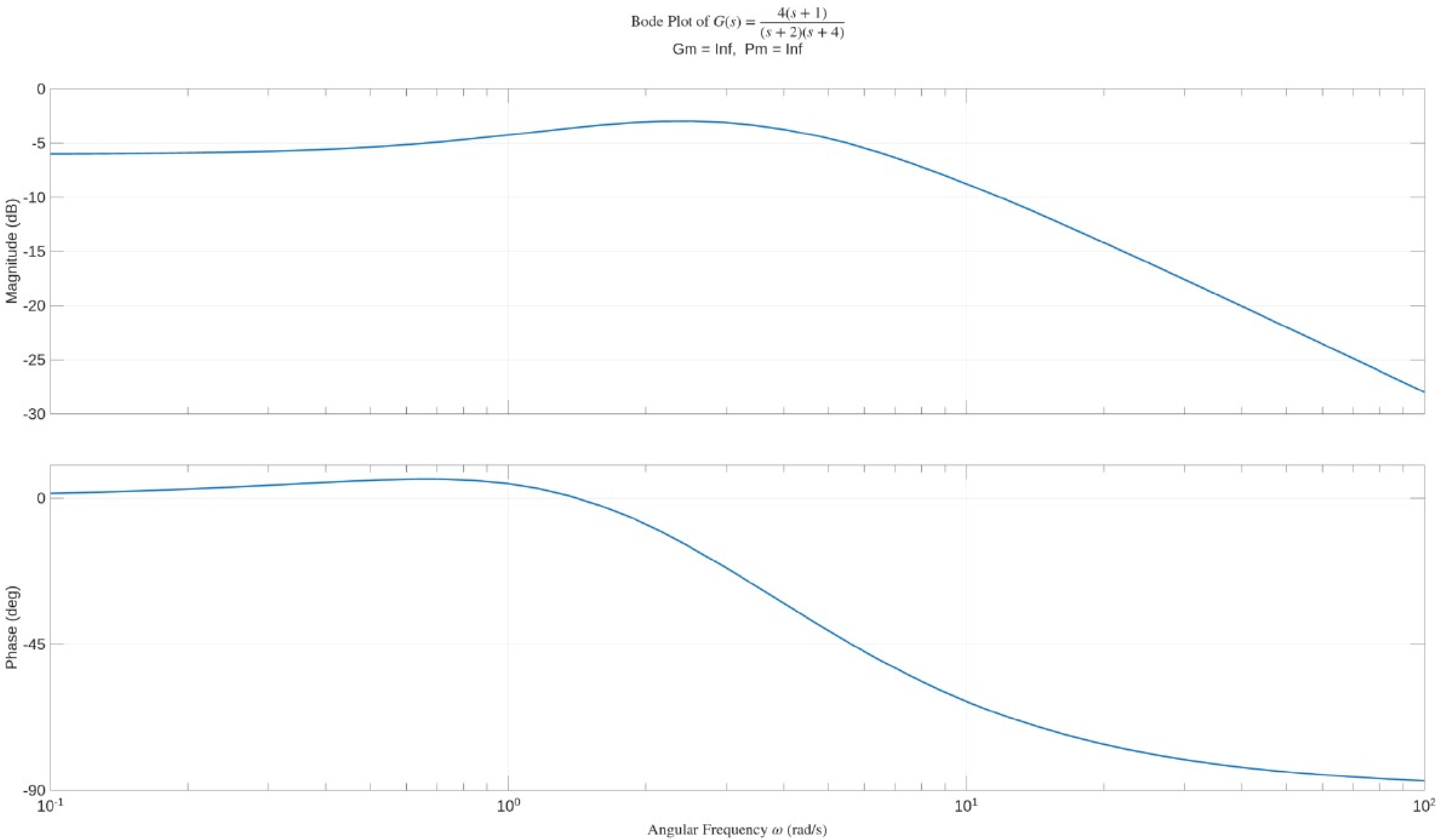


Bode Plot


Bode Plot of $G(s) = \frac{(s+1)}{s(s+2)(s+5)}$
Gm = Inf, Pm = 91.7 deg (at 0.1 rad/s)
Magnitude (dB)
0
-10
-20
-30
-40
-50
-60
-70
-80
Phase (deg)
-90
-135
-180
10^{-1}
10^{0}
10^{1}
10^{2}

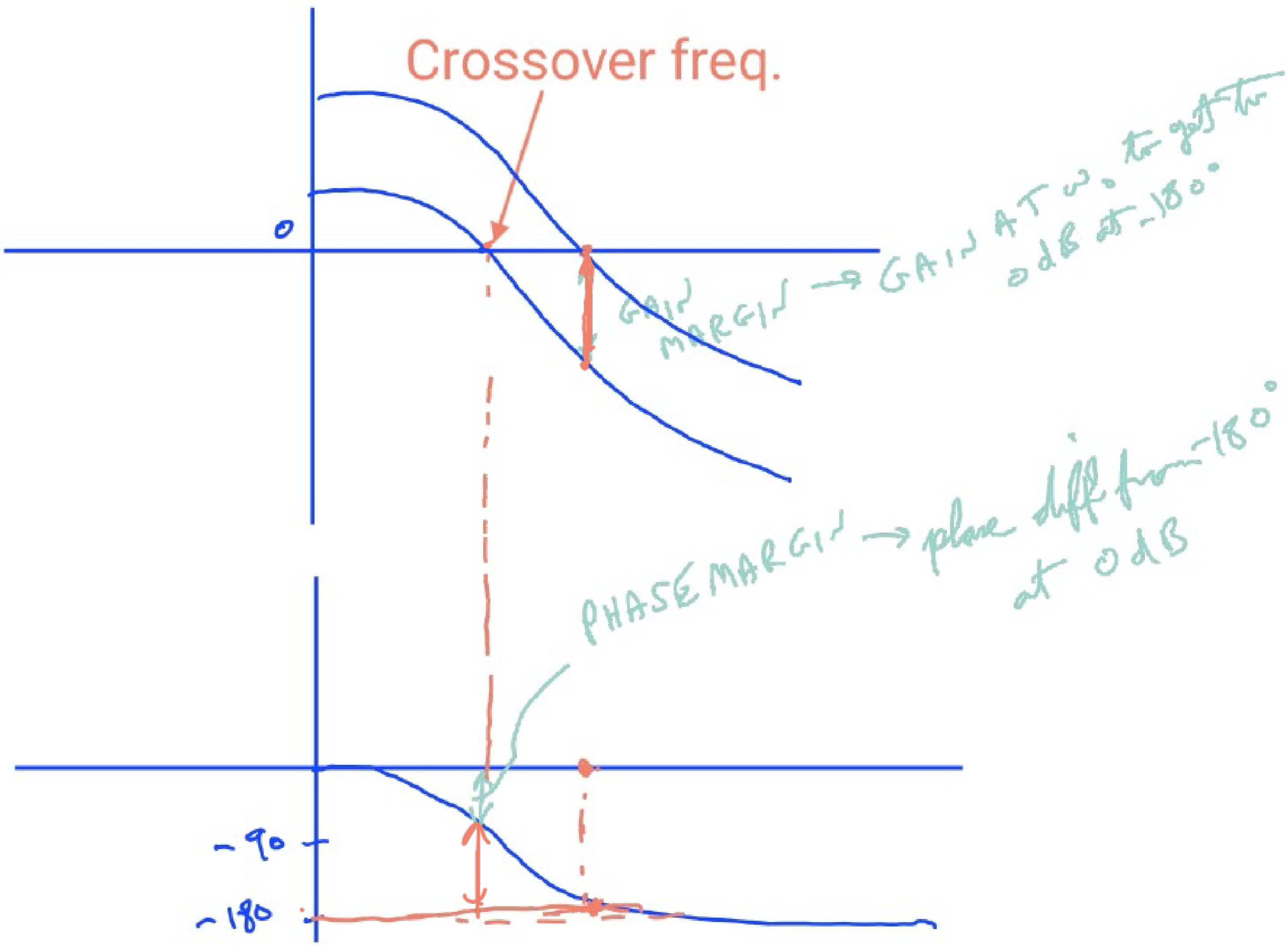
homayoon.beigi@columbia.edu

Bode Plot



homayoon.beigi@columbia.edu

Bode Margins

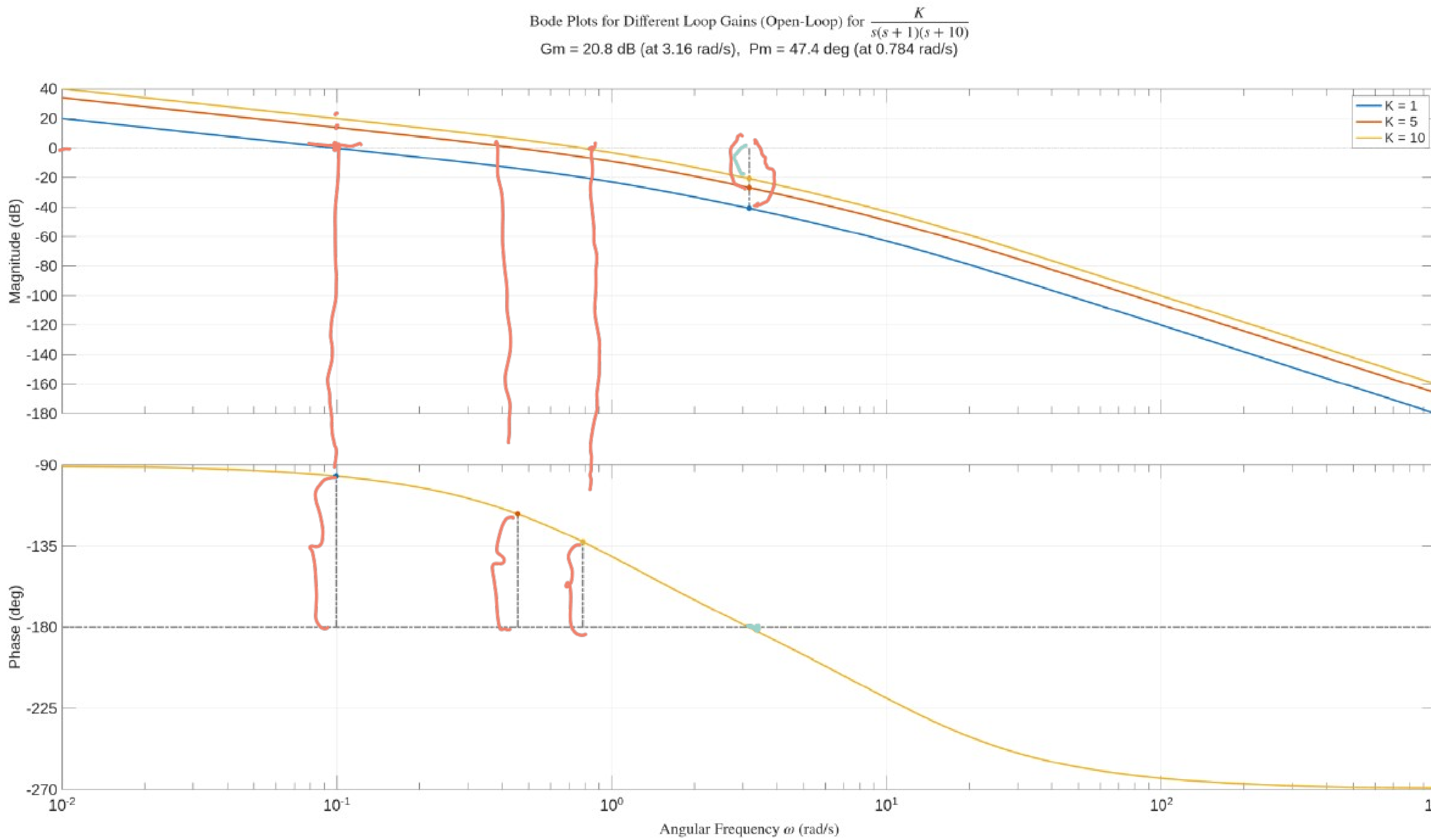


homayoon.beigi@columbia.edu

$$G(s) = \frac{K}{s(s+1)(s+10)}$$

Bode Plot

$u(s) \rightarrow \boxed{K}_{C} \rightarrow \boxed{\frac{1}{s(s+1)(s+10)}}_{G} \rightarrow X(s)$



Nyquist

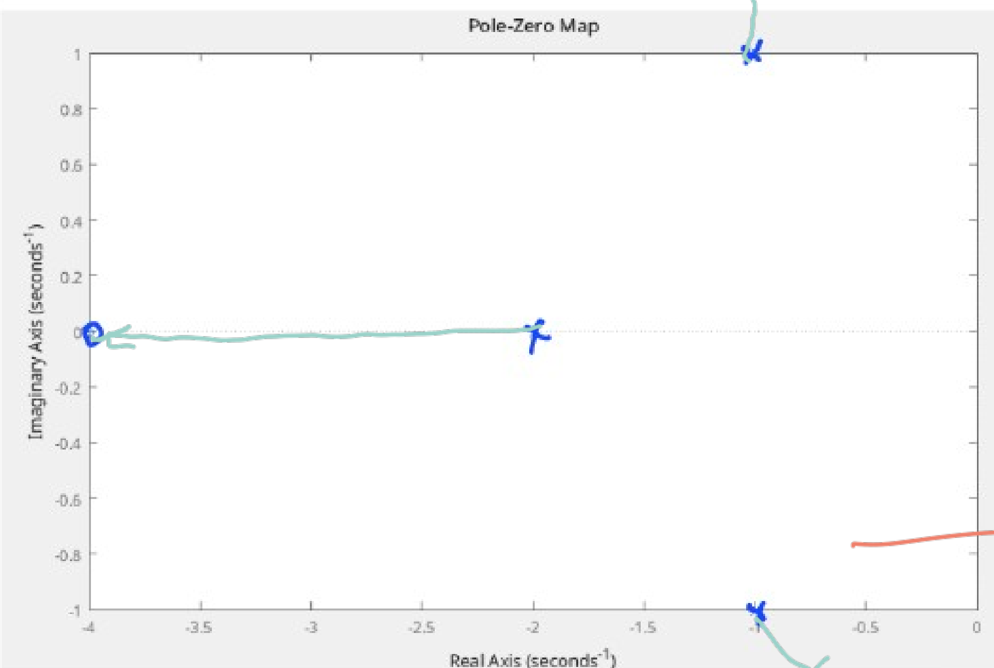
NYQUIST

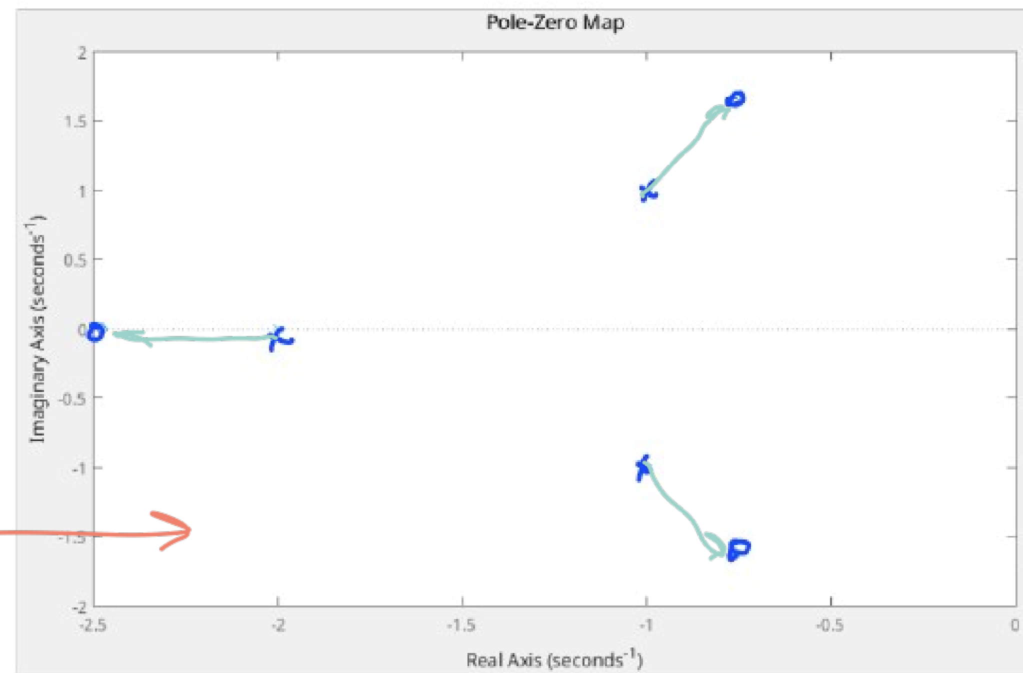
$1 + CGH = 0$

$CGH = \frac{s+4}{(s+2)(s^2+2s+2)}$

$1 + KC'G'H' = 0$

$C'G'H' = -\frac{1}{K}$

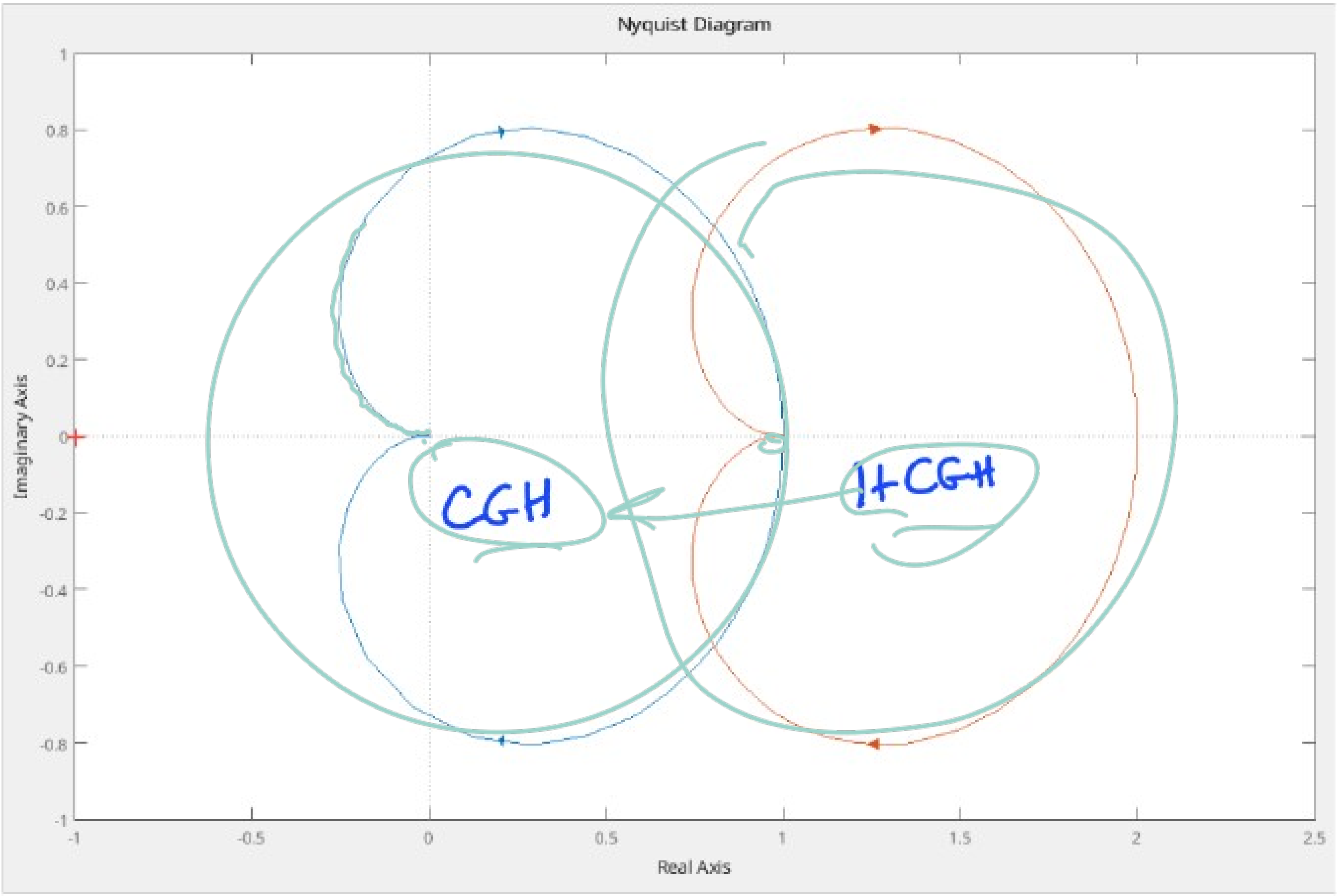




CGH

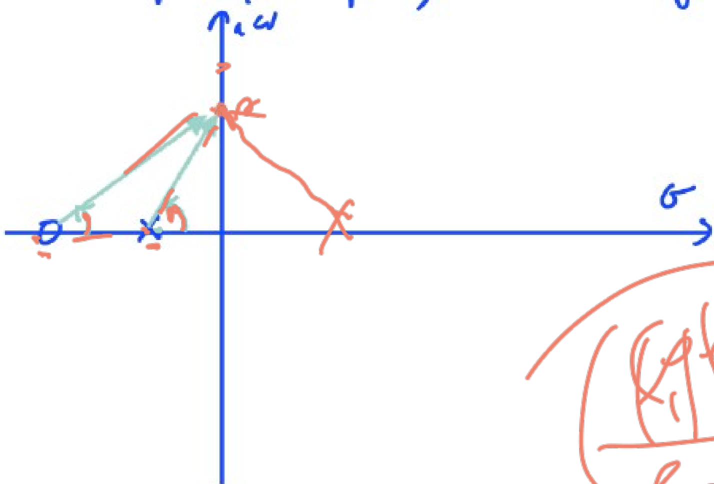
$1 + CGH$

Nyquist

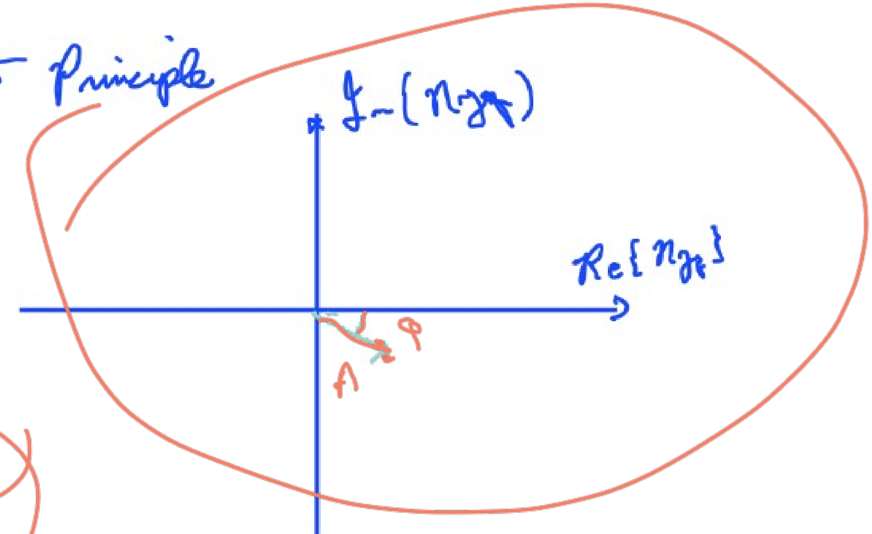

Nyquist Diagram
Imaginary Axis
Real Axis
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.5
0.5
1.5
2
2.5
CGH
1+CGH

Nyquist

Laplace (s-plane)



Cauchy's Argument Principle



$$\frac{\ell_1 e^{i\theta_1}}{\ell_2 e^{i\theta_2}}$$

- select a point
- draw vector from poles and zeros to pt.
- $A = \dfrac{\prod_{j=1}^{m} |s - z_j|}{\prod_{k=1}^{m} |s - p_k|}$

$Ae^{i\varphi}$

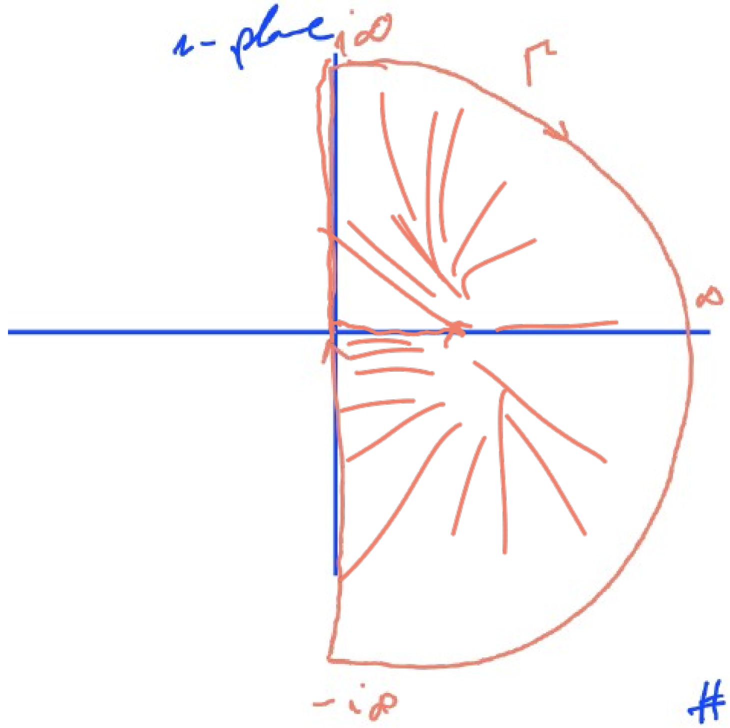
- $\varphi = \sum_{j=1}^{m} \angle(s - z_j) - \sum_{k=1}^{m} \angle(s - p_k)$

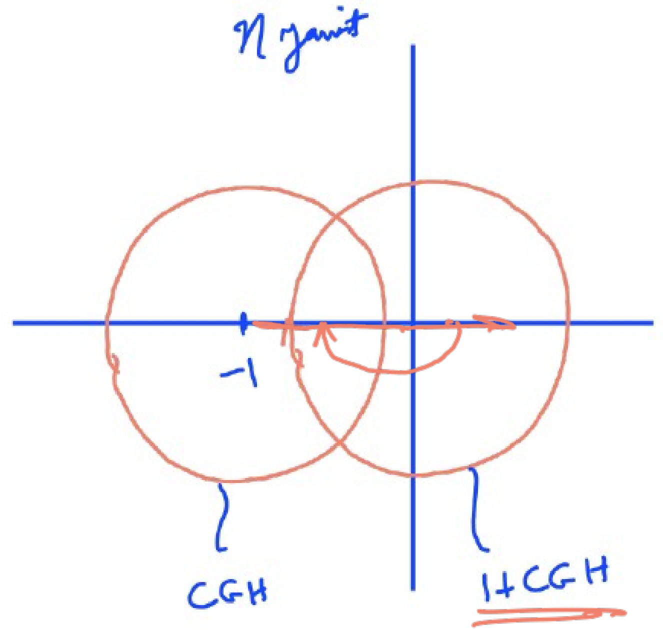
# of times the plot encircles the origin $= Z - P$
$= \#$ of zeros $- \#$ of poles

if contour in s-plane is
CW $\Rightarrow$ ABOVE IS CW
CCW $\Rightarrow$ ABOVE IS CCW

change in direction negates
CW $\rightarrow$ CCW (more poles)

Nyquist





# of times −1 is encircled

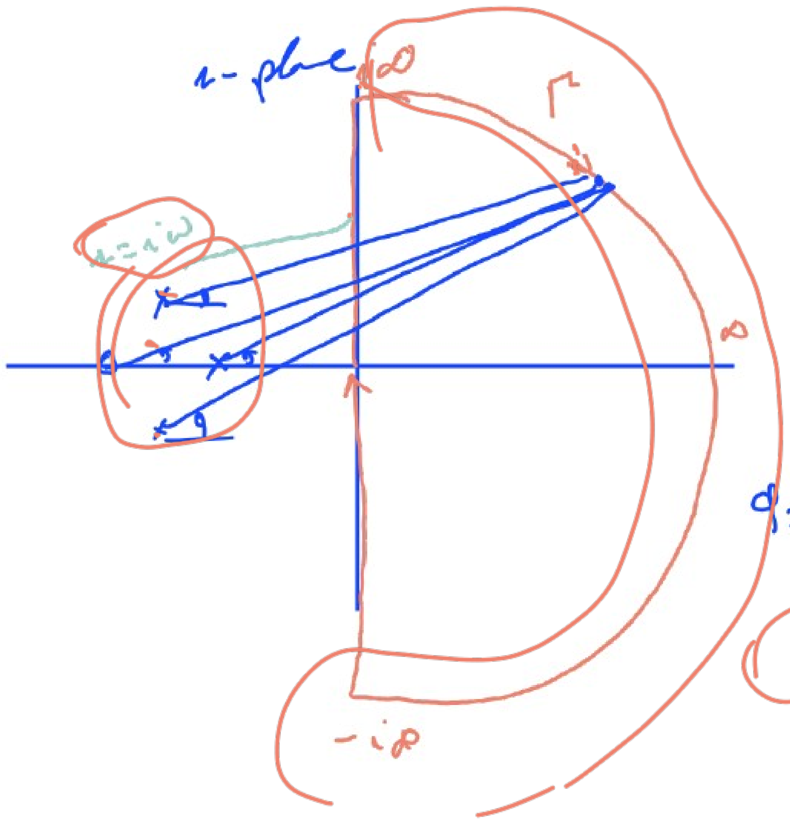
# of poles in the RHP of 1+CGH =
# of poles in the RHP of CGH

# of zeros in the RHP → $Z = N + P$

# of O of −1   # of poles of CGH in the RHP

homayoon.beigi@columbia.edu

Nyquist



strictly proper # of z < # of p

proper # of z = # of p

not proper # of z > # of p

$$\varphi = \sum \theta_z - \sum \theta_p$$

$$A = \frac{\prod_{j}^{m} |s-z_j|}{\prod_{k}^{n} |s-p_k|} = 0 \text{ for } m<n$$

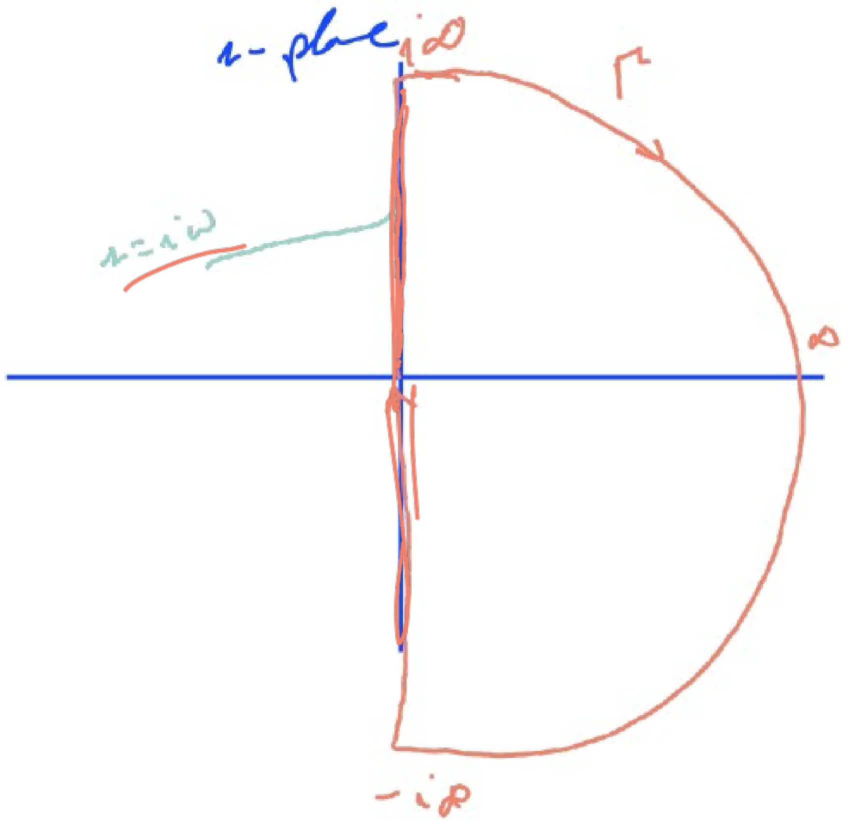
$$= M < \infty \text{ for } m=n$$

φ does not matter since $A=0$ (at the origin) for $m<n$

$\varphi = 0$ for $m=n$ since $\sum\theta_z - \sum\theta_p = 0$ at ∞

$A = M < \infty$ for $m=n$

Nyquist



strictly proper $\# \text{ of } z < \# \text{ of } p$

proper $\# \text{ of } z = \# \text{ of } p$

not proper $\# \text{ of } z > \# \text{ of } p$

consider $i\omega$ for $0 < \omega < \infty$

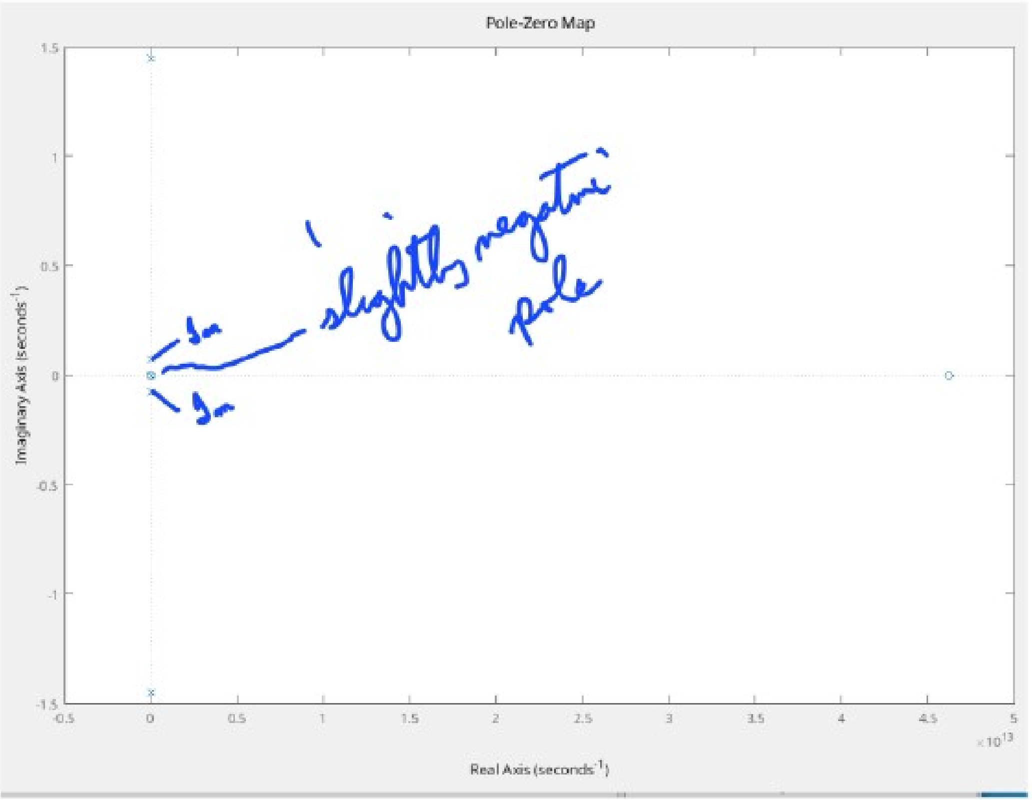
$\omega = 0$

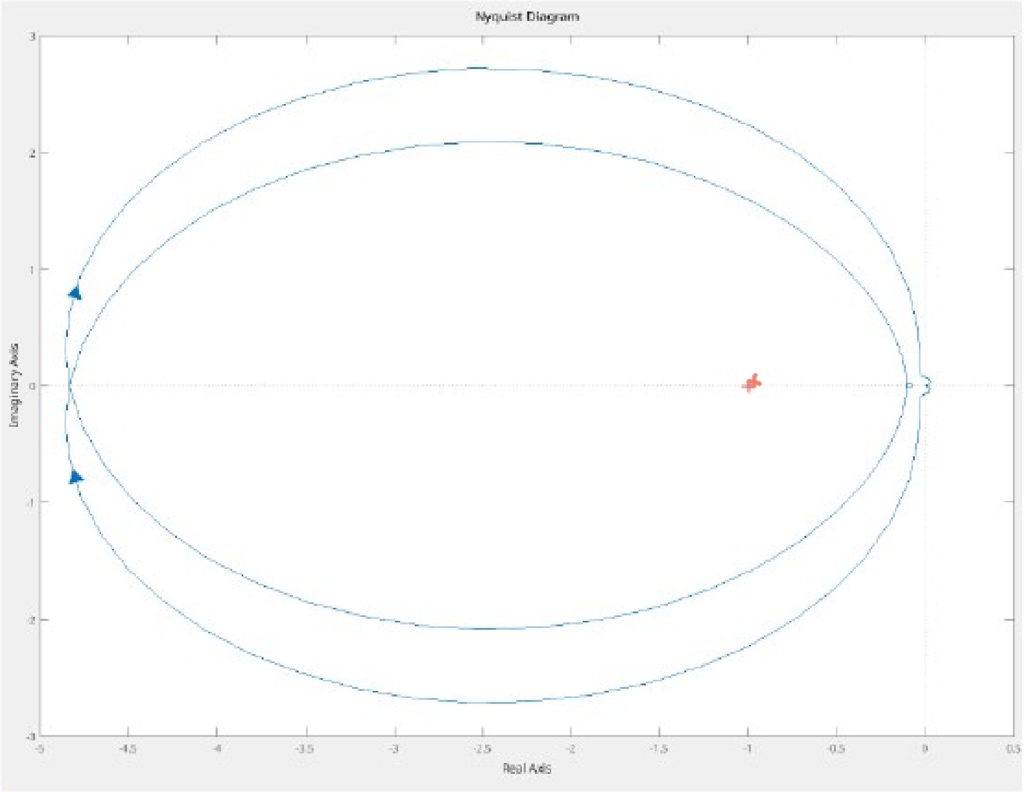
$\omega = \infty$

ω X imaginary axis $\rightarrow$ set $\sigma = 0$

ω X real axis $\rightarrow$ set $\omega = 0$

F-16 Aircraft – Nyquist Plot



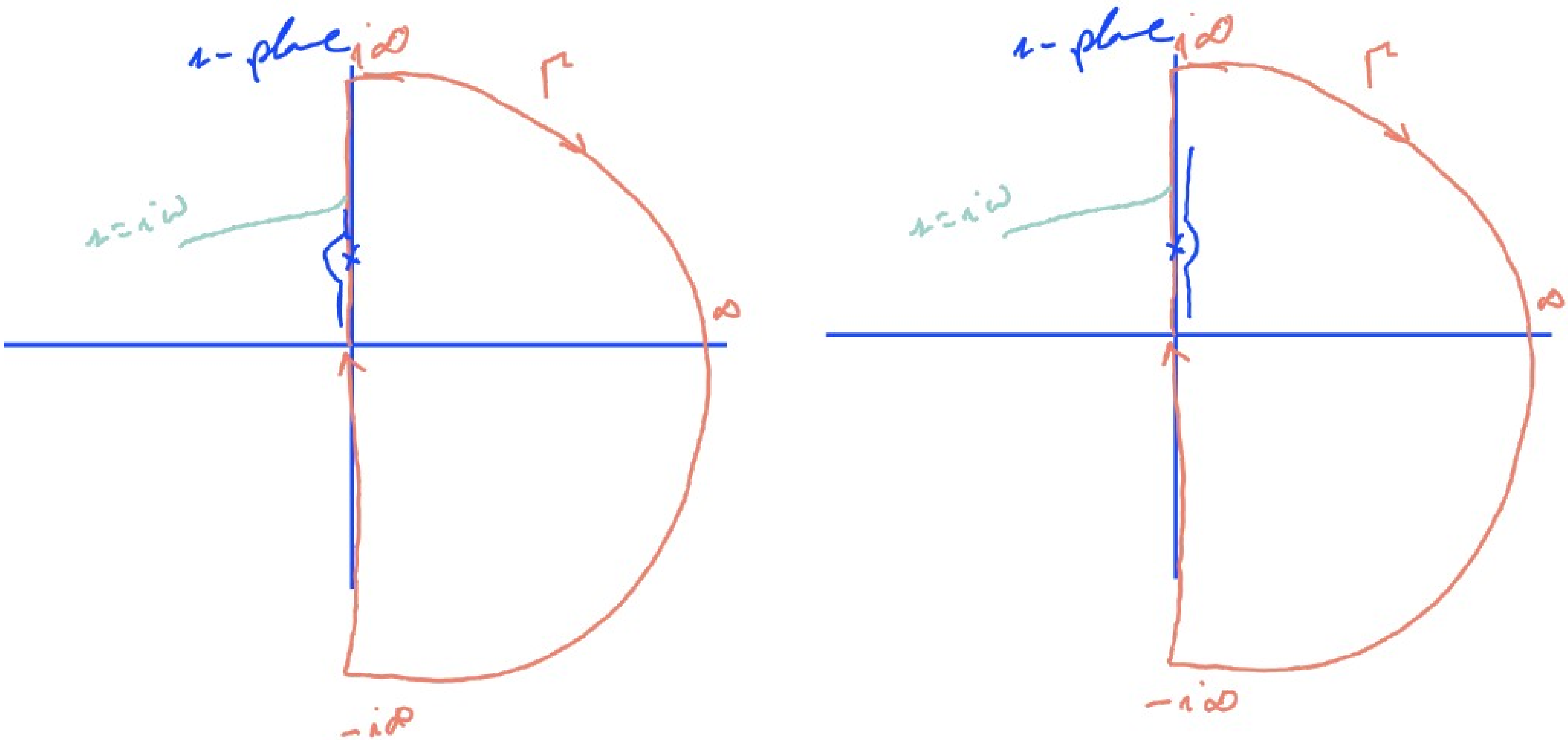


2 encirclements

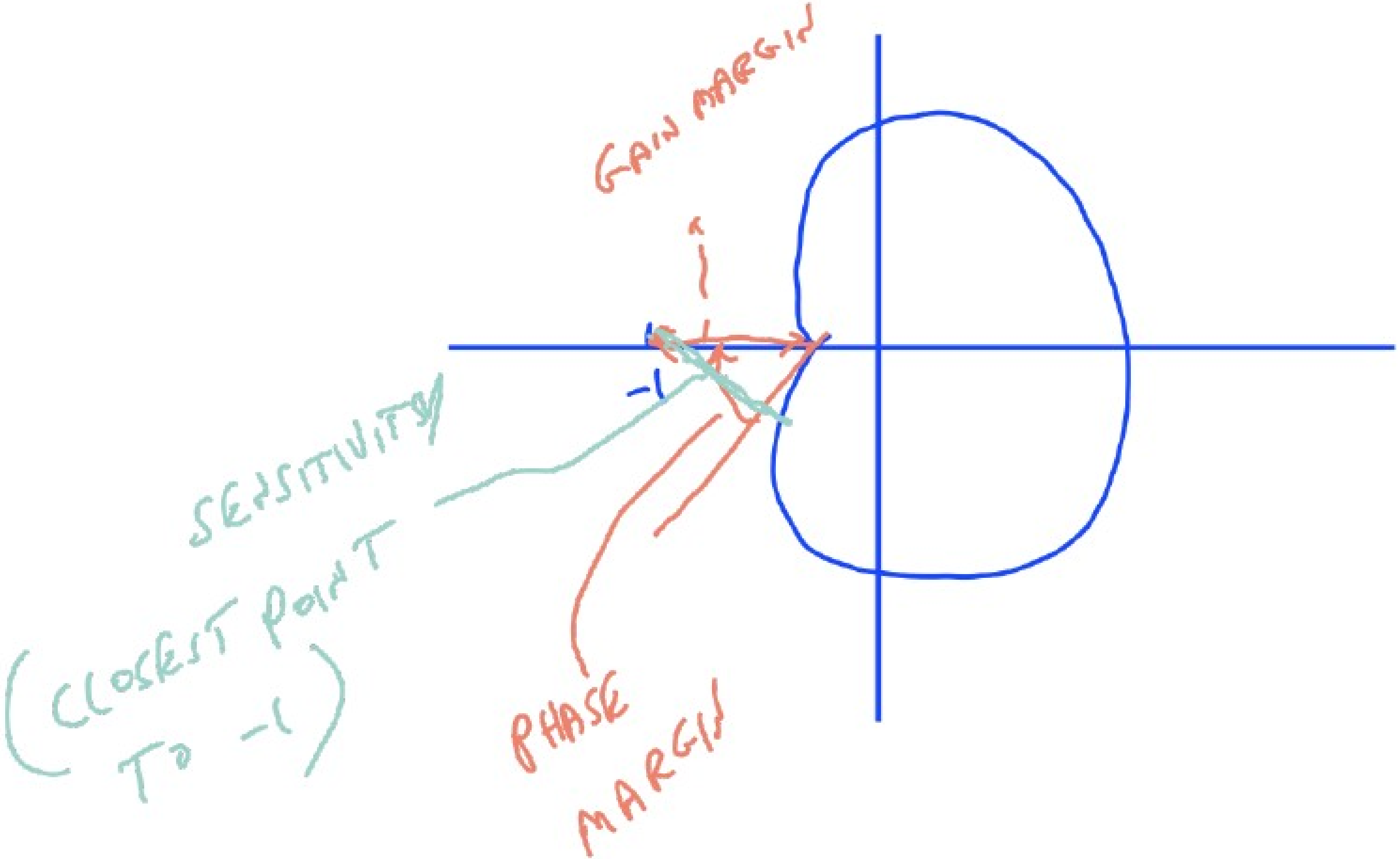
$2 > 1 \Rightarrow$ unstable

Nyquist

poles or zeros on Imaginary axis

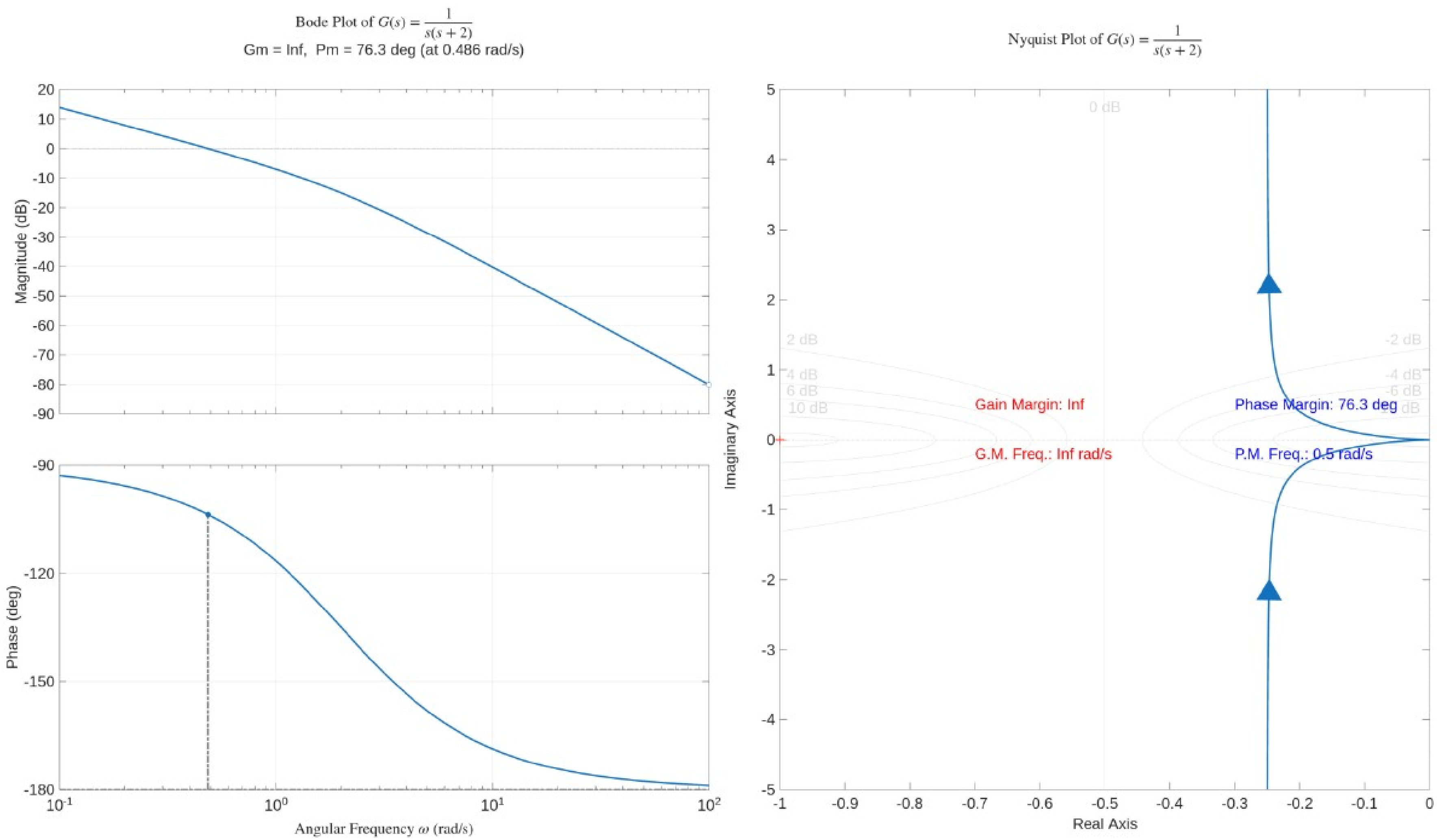


Nyquist Margins



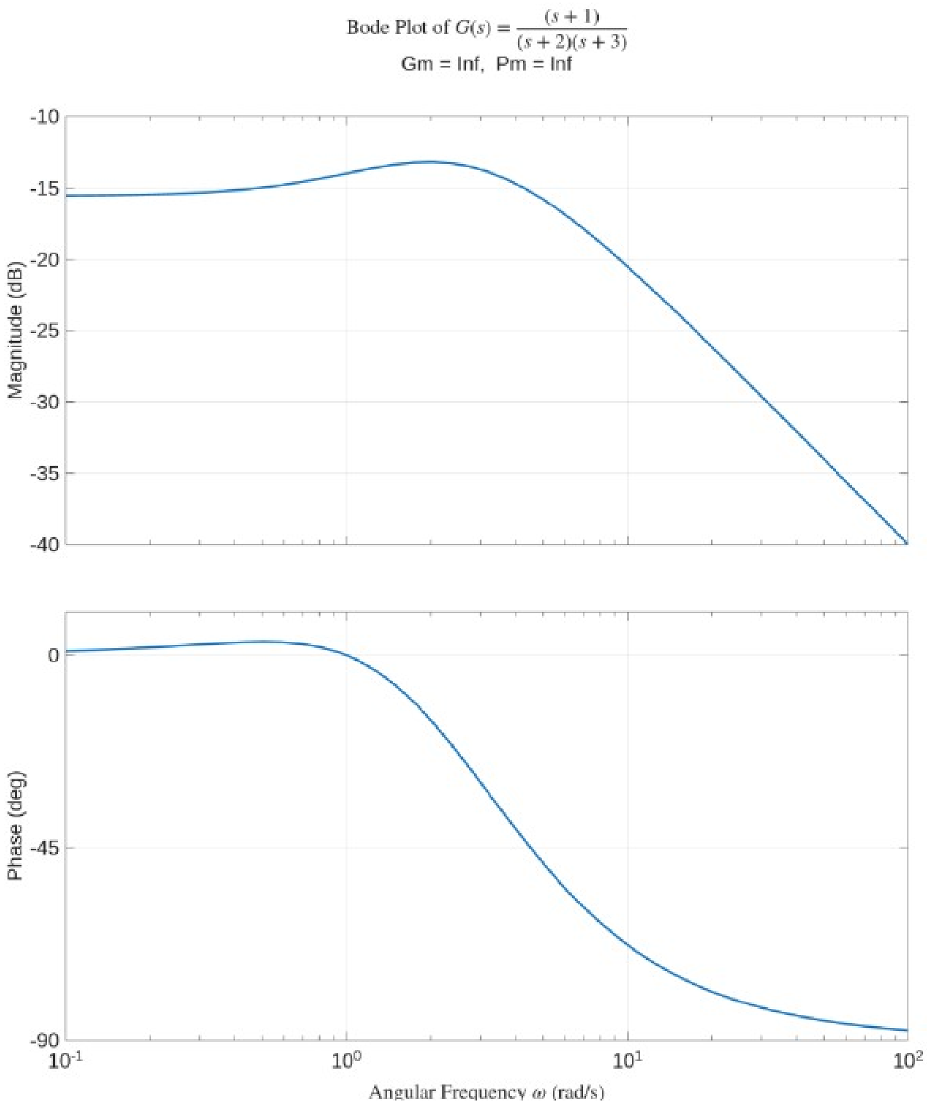
$$G(s) = \frac{1}{s(s+2)}$$

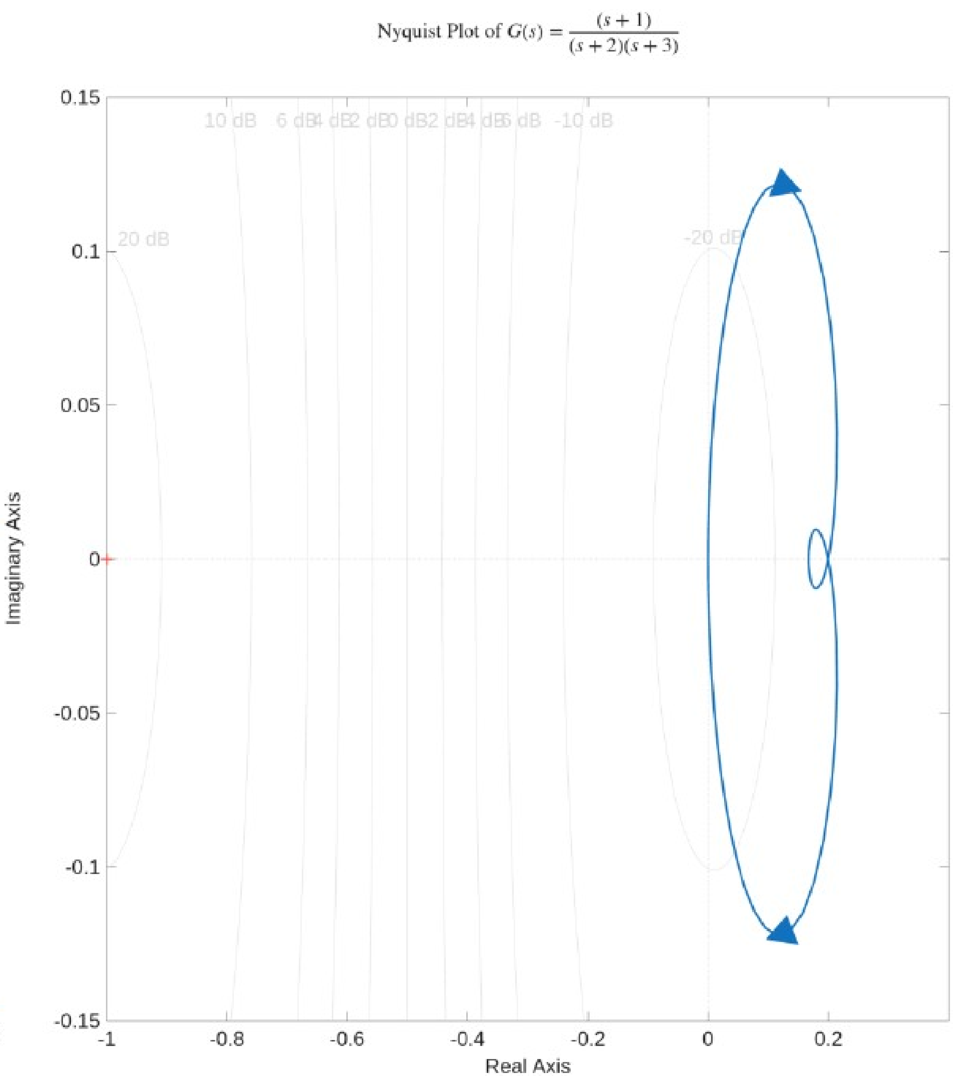
Nyquist Plot



Nyquist Plot

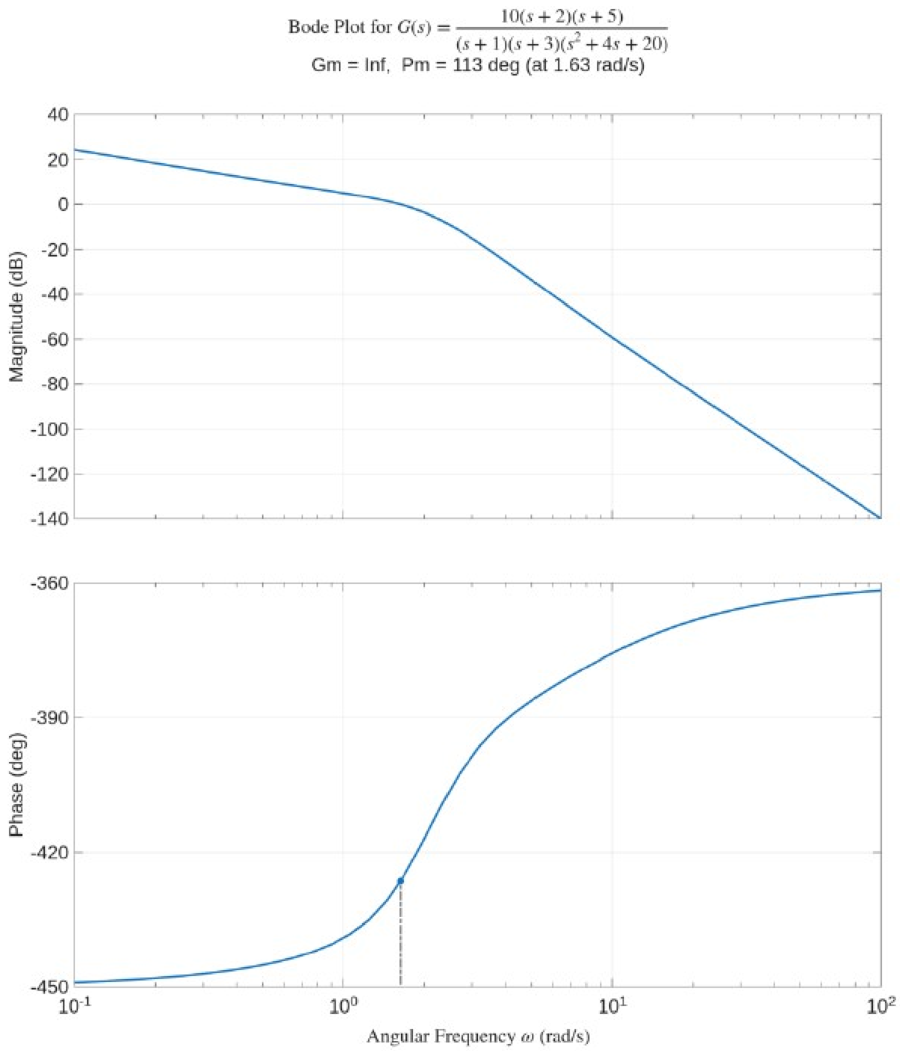
$$G(s) = \frac{(s+1)}{(s+2)(s+3)}$$

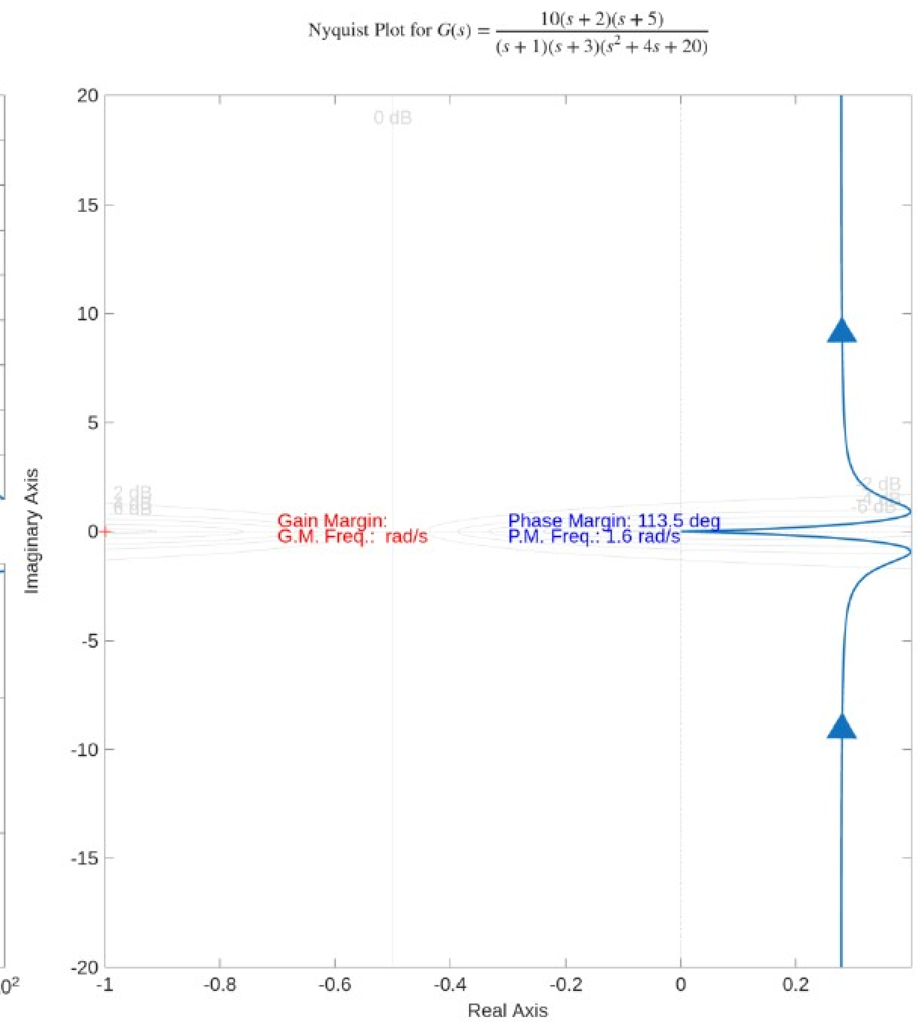




$$G(s) = \frac{10(s+2)(s+5)}{(s+3)(s+1)(s^2+4s+20)}$$

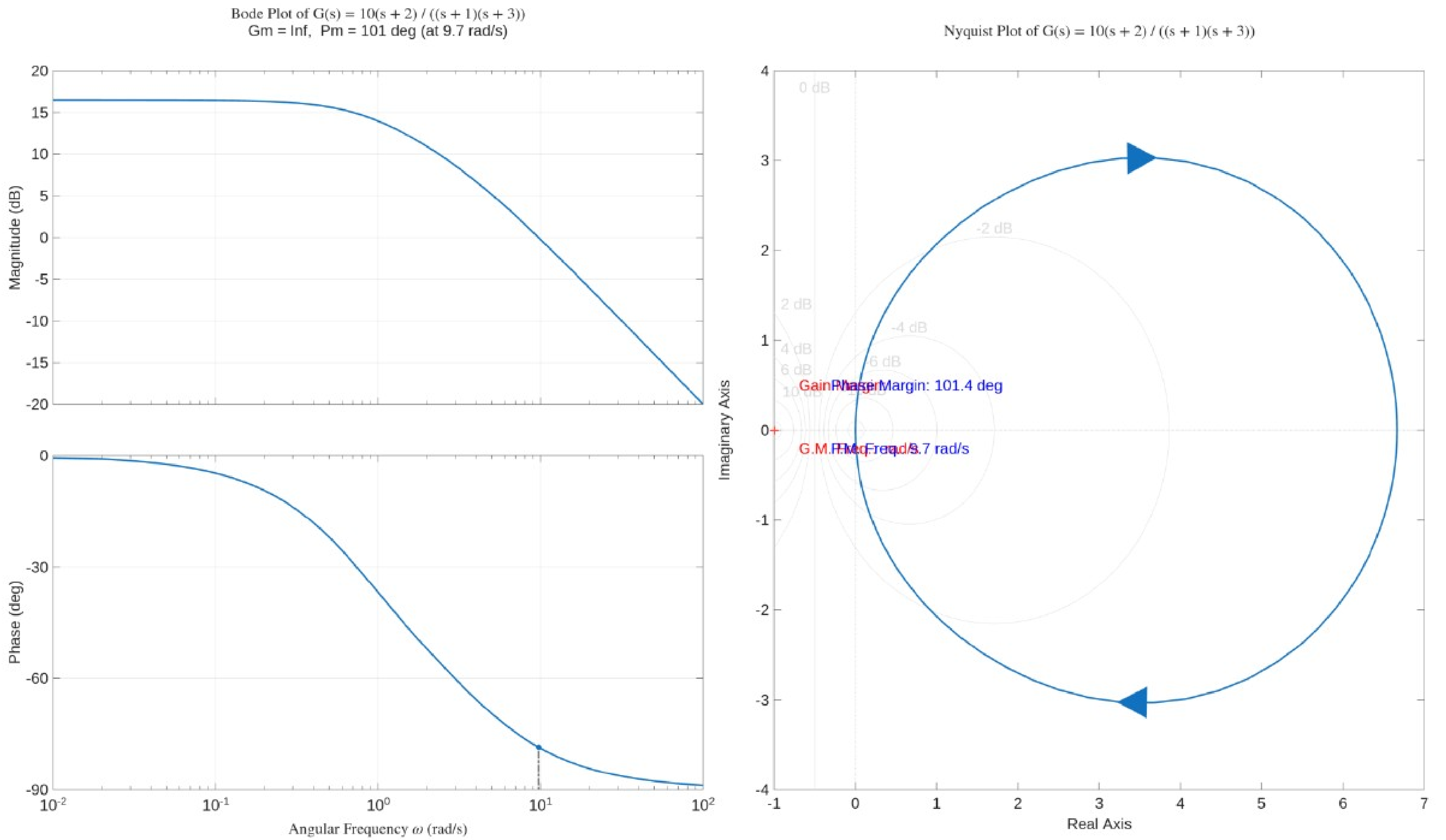
Nyquist Plot





$$G(s) = \frac{10(s+2)}{(s+3)(s+1)}$$

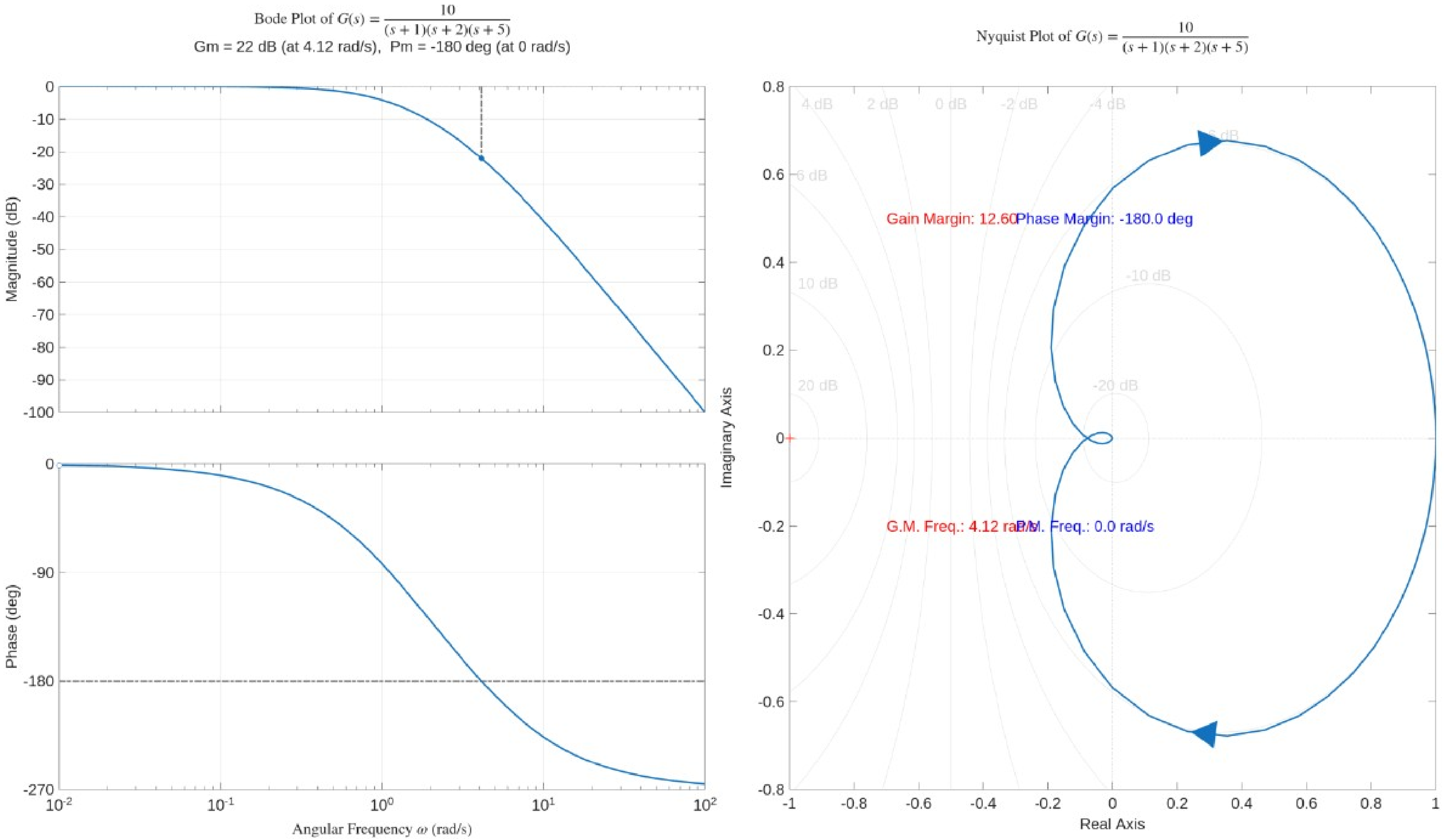
Nyquist Plot



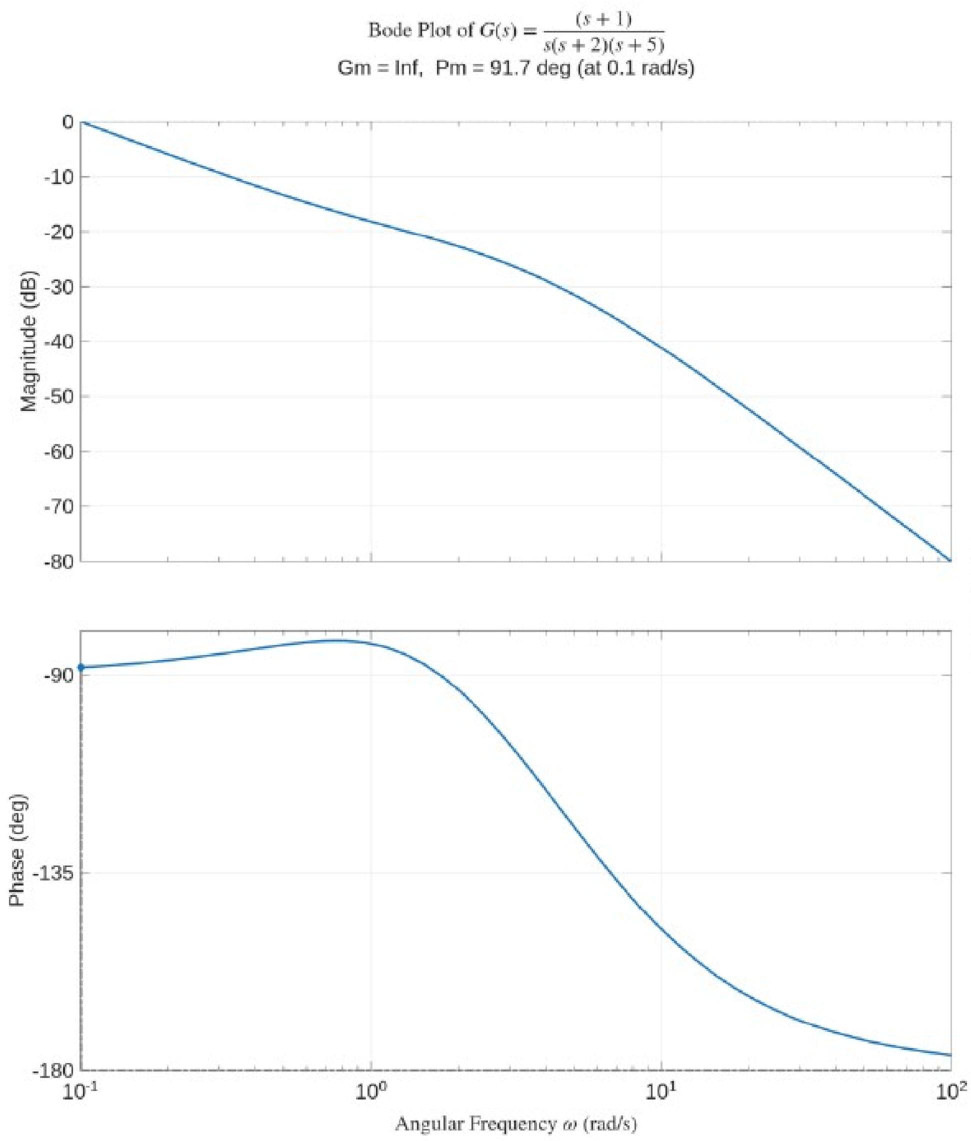
homayoon.beigi@columbia.edu

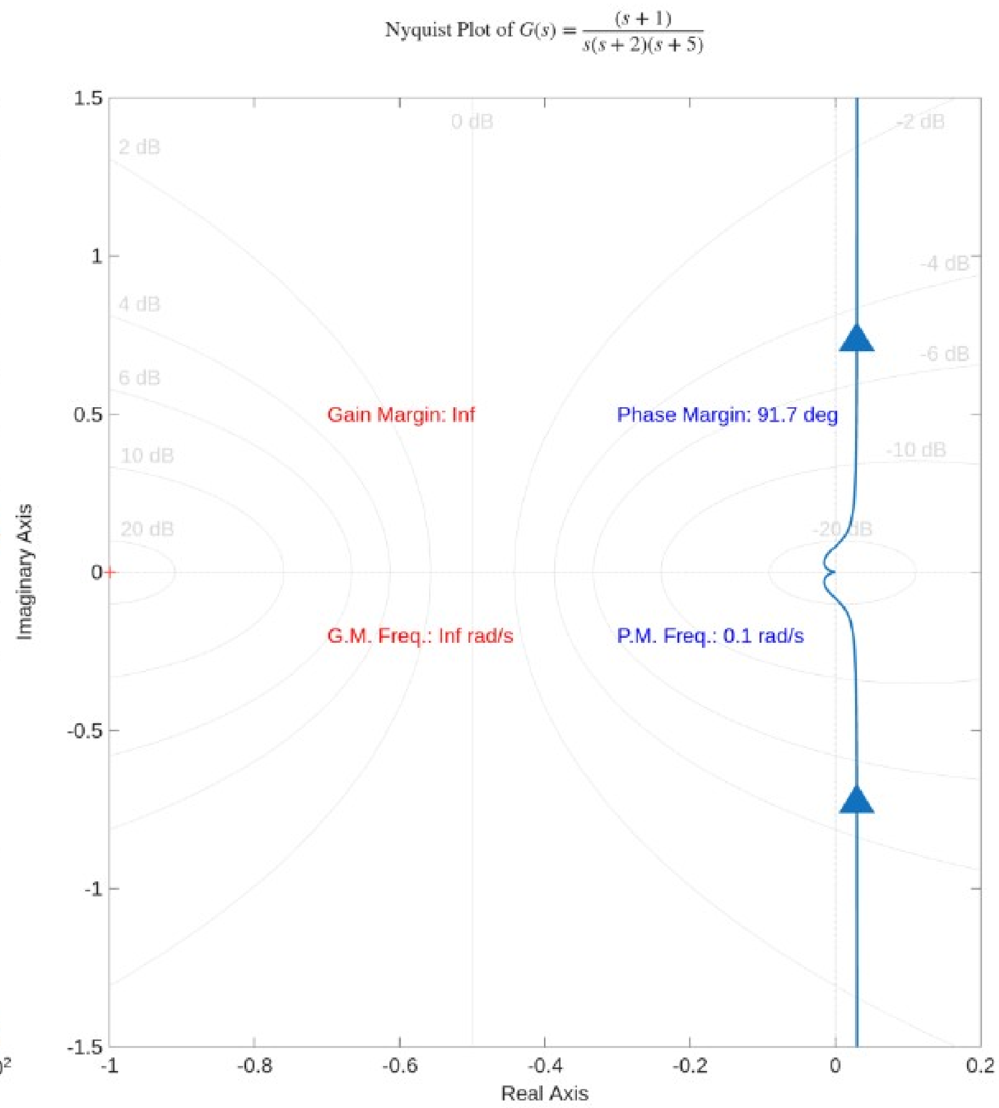
$$G(s) = \frac{10}{(s+1)(s+2)(s+3)}$$

Nyquist Plot

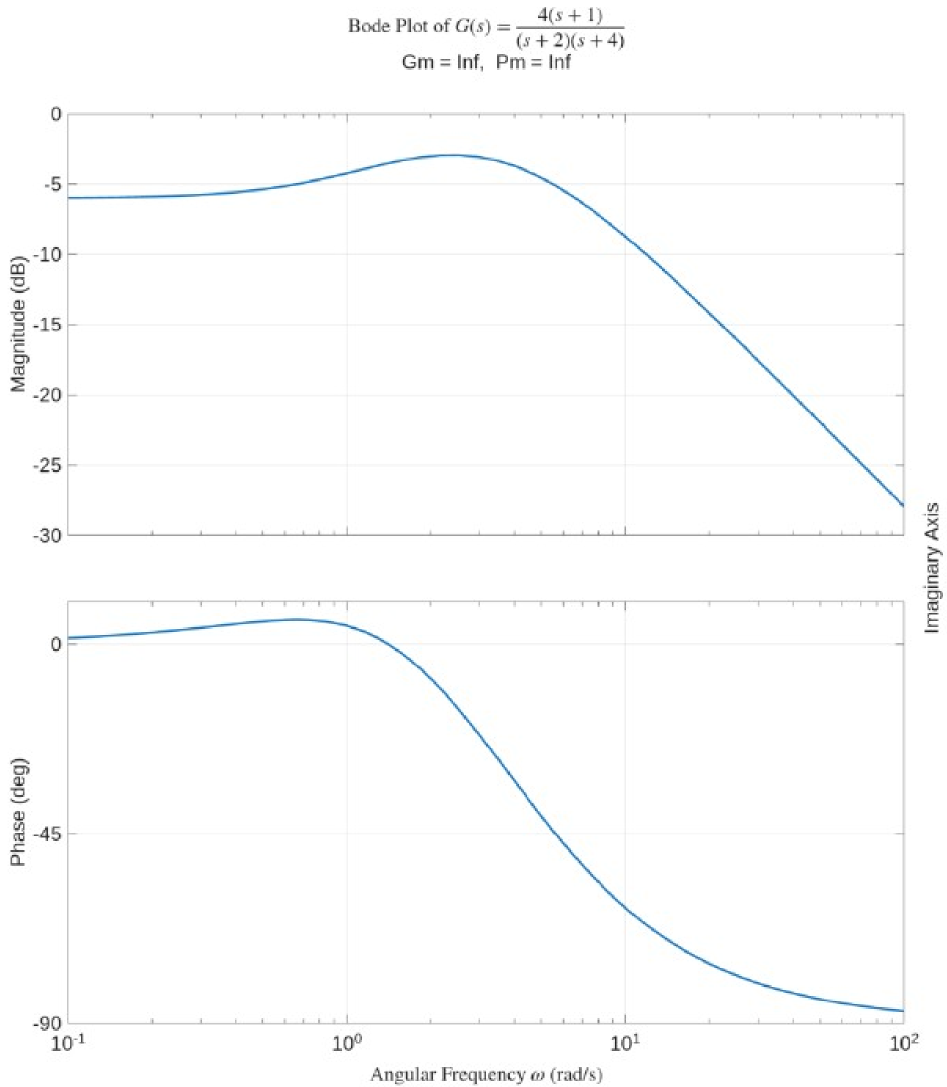


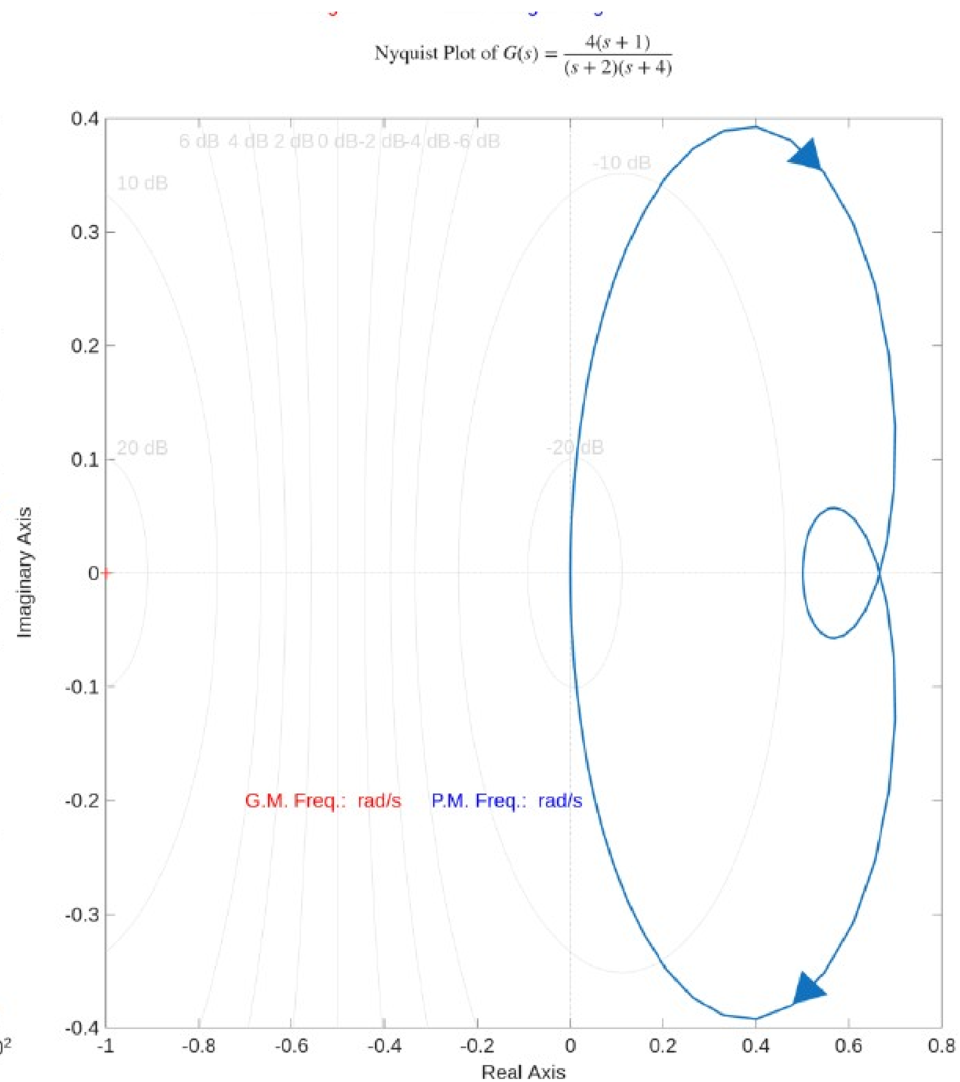
Nyquist Plot





Nyquist Plot





$$G(s) = \frac{K}{s(s+1)(s+10)}$$

Nyquist Plot