Introduction to Continuous Control Systems

EEME E3601

Week 13

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&
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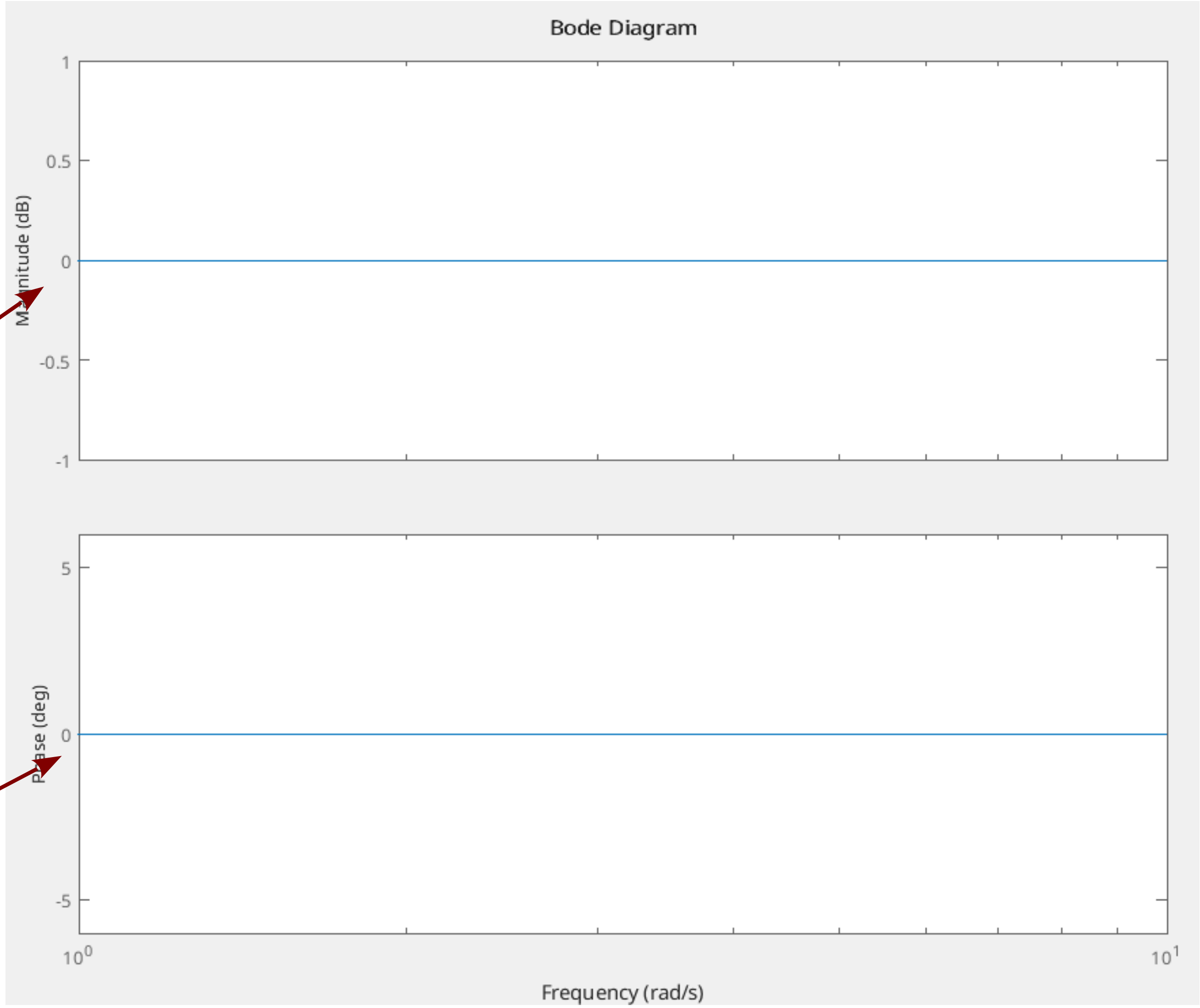
Bode Plot



20 log 1 = 0

0 phase shift for
positive gains

$$G = 1$$

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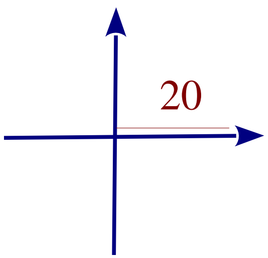
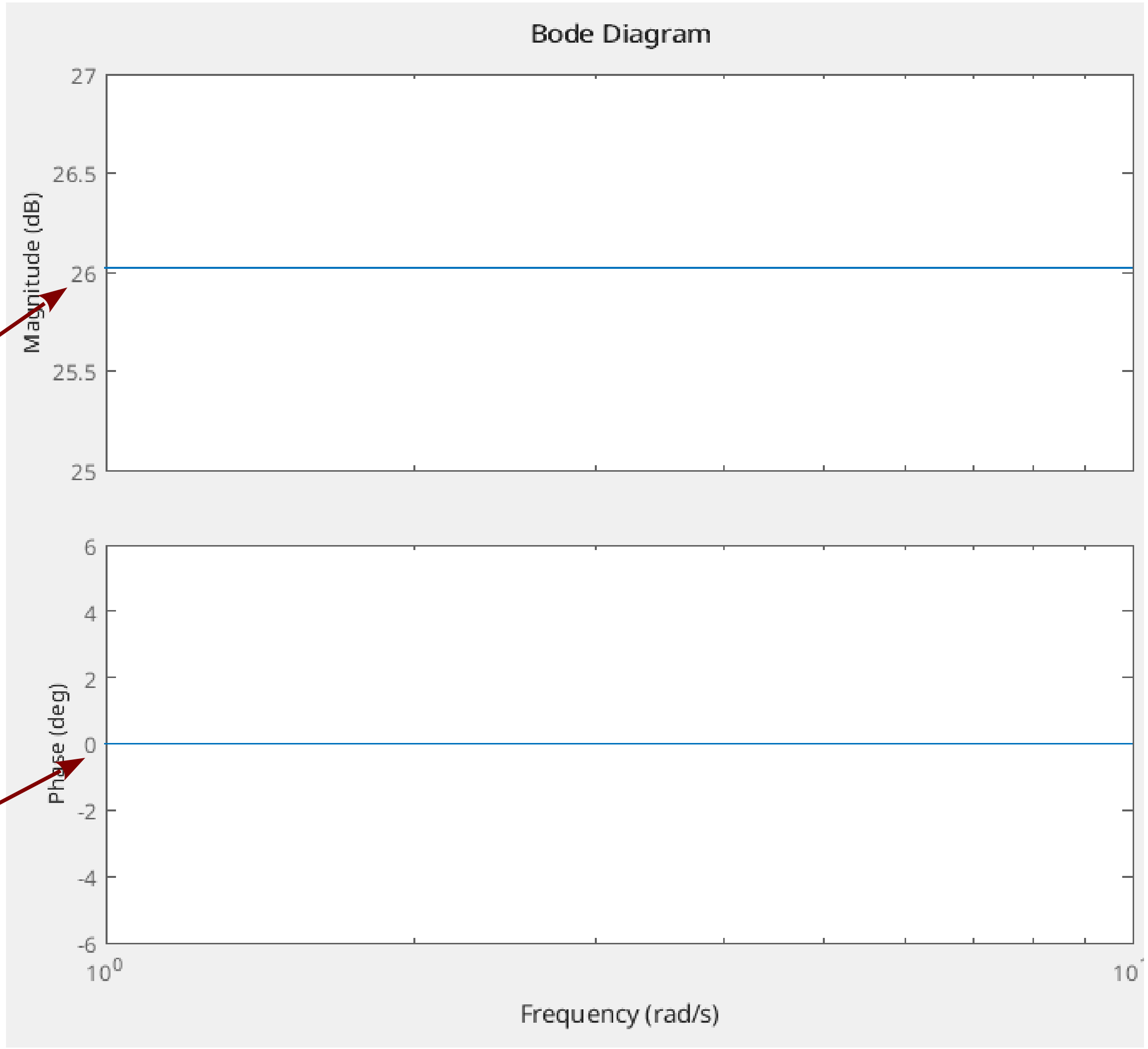
Bode Plot



20 log 20



0 phase shift for positive gains

$$G = 20$$

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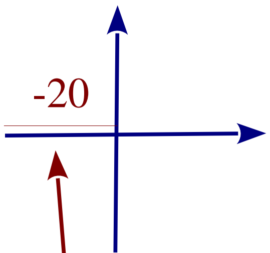
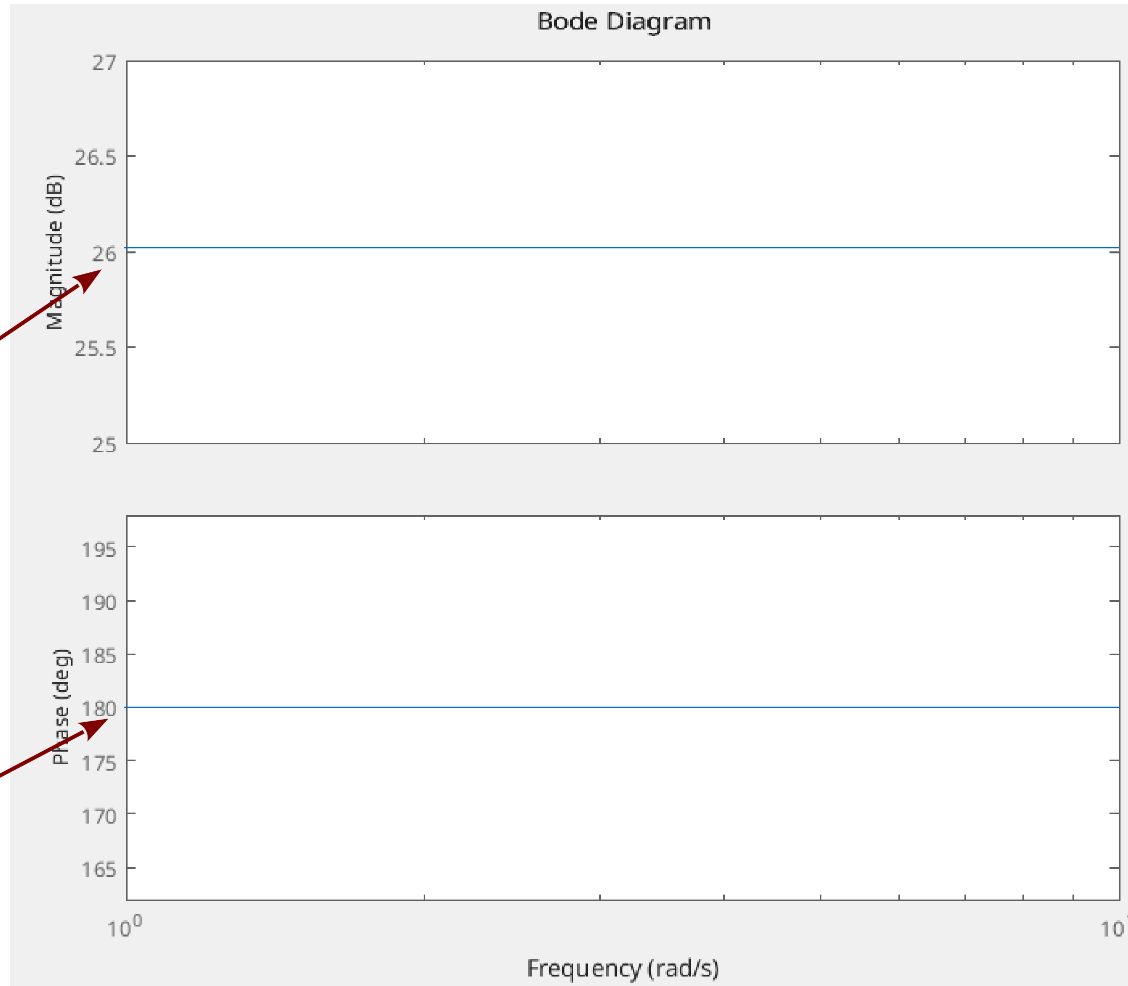
Bode Plot





20 log 20

Opposite (180
degree phase)
For negative gains

$$G = -20$$

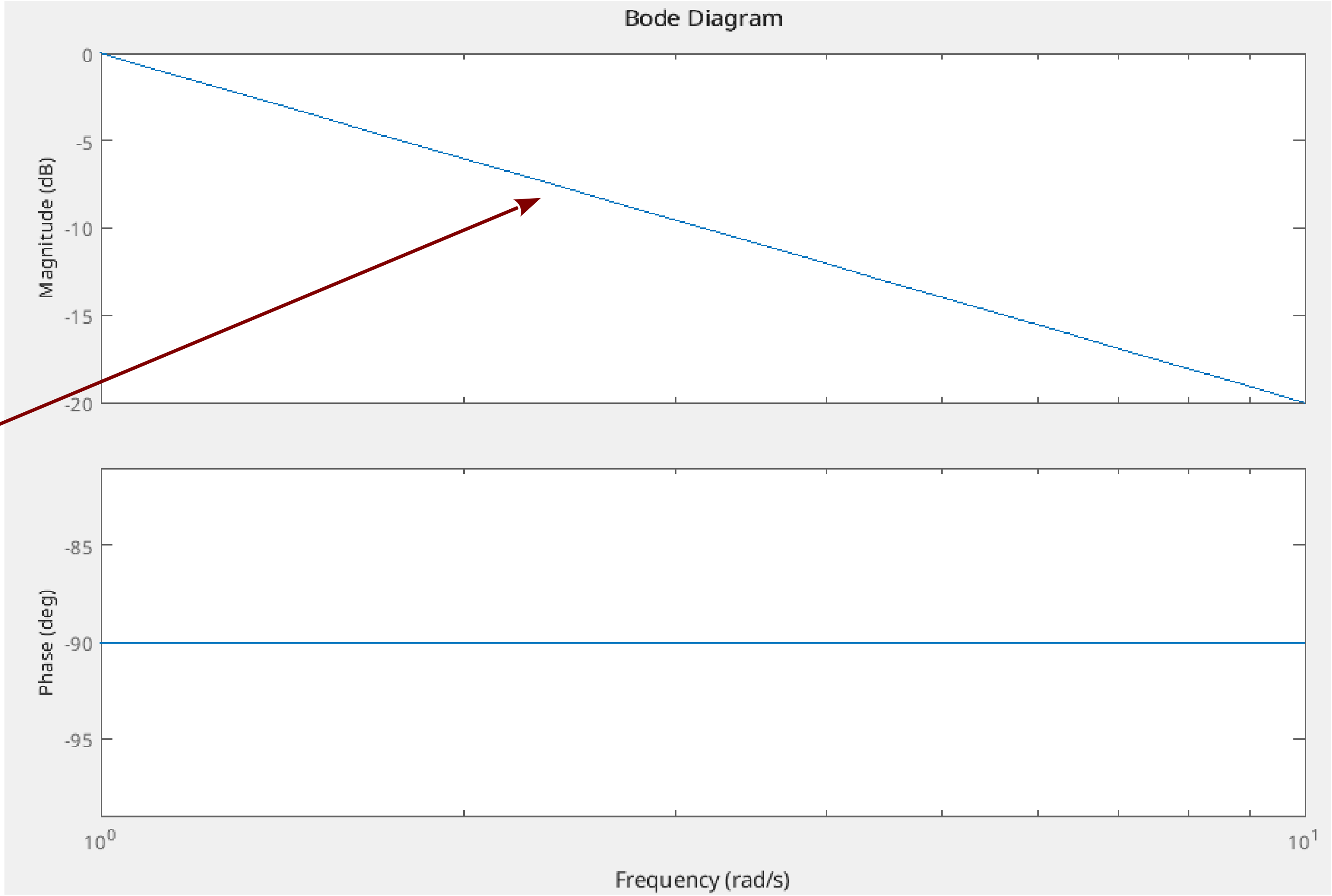
Bode Plot (Integrator)

$$
\begin{aligned}
G(i\omega) &= \frac{1}{i\omega} \\
&= -\frac{1}{\omega} i
\end{aligned}
$$

$$
\begin{aligned}
|G(i\omega)| &= \left| -\frac{1}{\omega} i \right| \\
&= \frac{1}{\omega}
\end{aligned}
$$

$$\angle G(i\omega) = -90^\circ$$

Slope=20db/decade



$$G = \frac{1}{s}$$

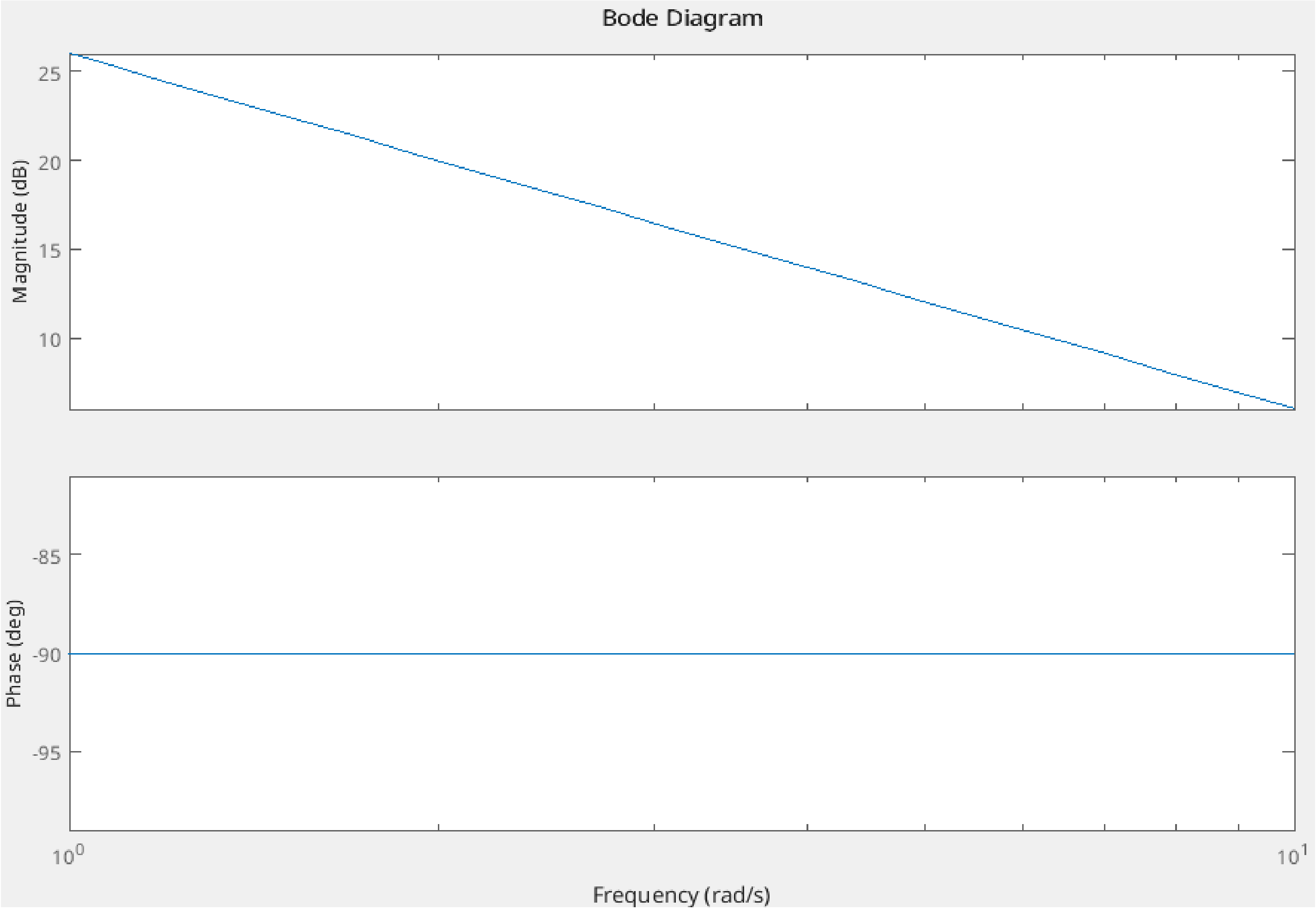
Bode Plot

$$G(s) = \frac{20}{s} = 20\frac{1}{s}$$

$$|G(i\omega)| = |20|\left|-\frac{1}{\omega}i\right|$$

$$20\log(|20|\frac{1}{\omega}) = 20\log(|20|) + 20\log(\frac{1}{\omega})$$

$$\angle G(i\omega) = \angle 20 + \angle -\frac{1}{\omega}i = 0^\circ - 90^\circ = -90^\circ$$



$$G(s) = \frac{20}{s}$$

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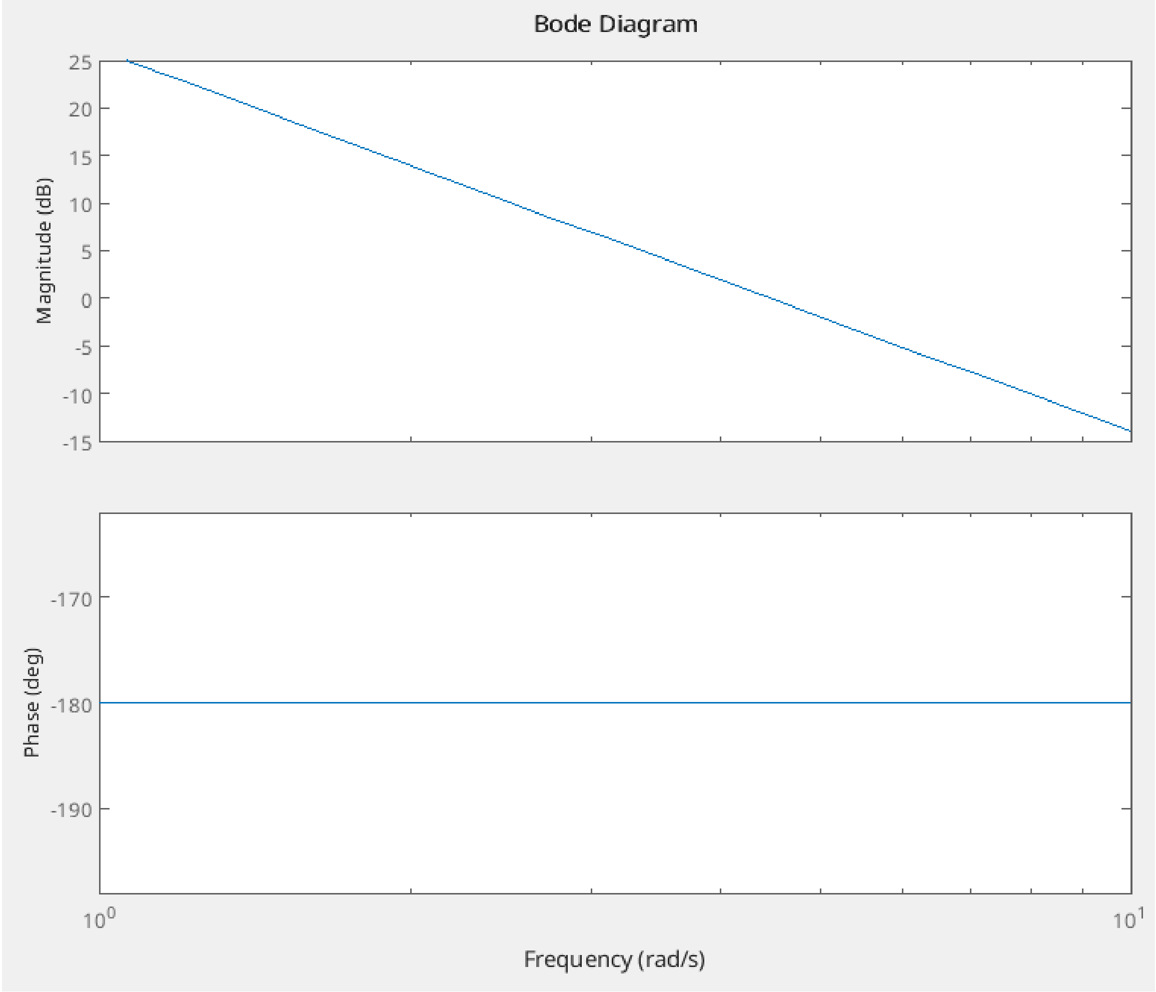
Bode Plot

$$
\begin{aligned}
G(s) &= \frac{20}{s^2} \\
&= 20\frac{1}{s}\frac{1}{s} \\
|G(i\omega)| &= |20|\left|-\frac{1}{\omega}i\right|\left|-\frac{1}{\omega}i\right|
\end{aligned}
$$

$$
\begin{aligned}
&20\log(|20|\frac{1}{\omega}\frac{1}{\omega}) = \\
&20\log(|20|) + 20\log(\frac{1}{\omega}) + 20\log(\frac{1}{\omega}) \\
&= \\
&20\log(|20|) + 40\log(\frac{1}{\omega})
\end{aligned}
$$

$$
\begin{aligned}
\angle G(i\omega) &= \angle 20 + \angle -\frac{1}{\omega}i + \angle -\frac{1}{\omega}i \\
&= 0^\circ - 90^\circ - 90^\circ \\
&= -180^\circ
\end{aligned}
$$



$$
G(s) = \frac{20}{s^2}
$$

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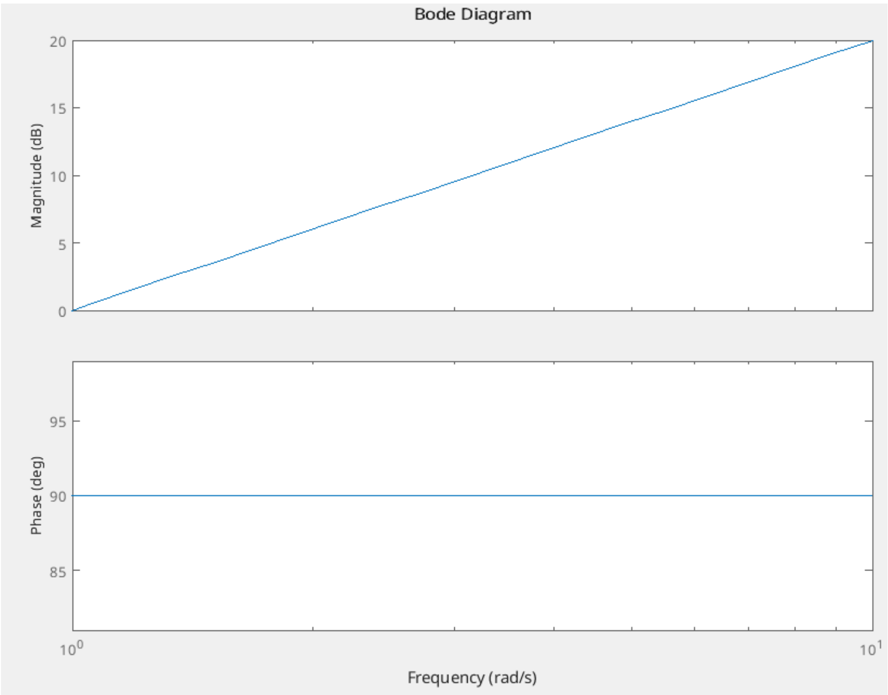
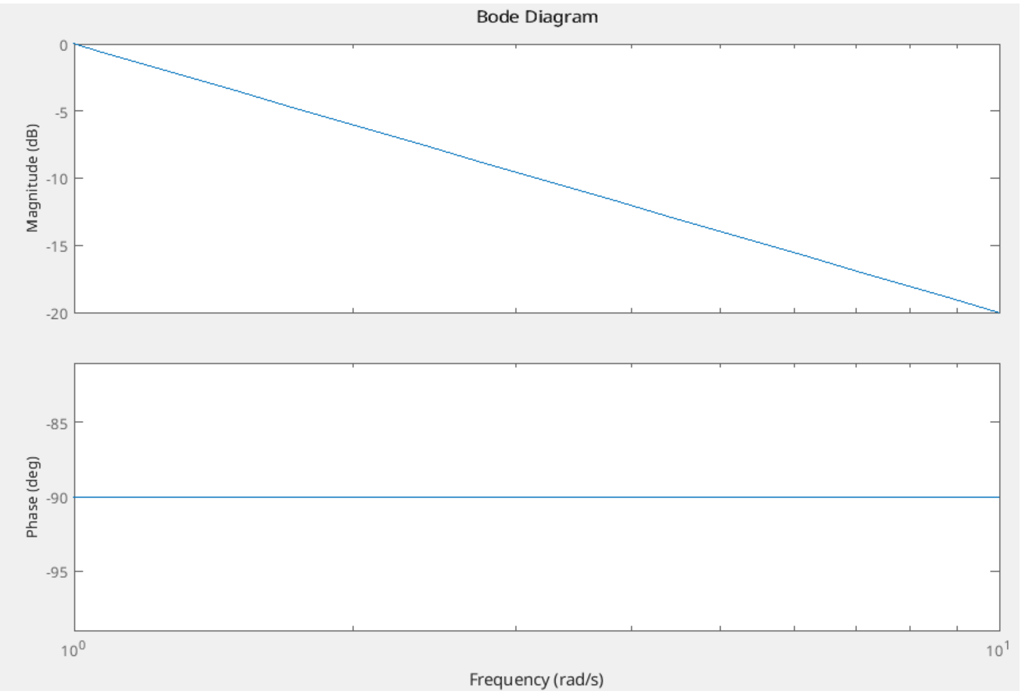
Bode Plot





$$G = s$$

Single Zero

Positive phase
Positive slope

$$G = \frac{1}{s}$$

Single Pole

Negative phase
Negative slope

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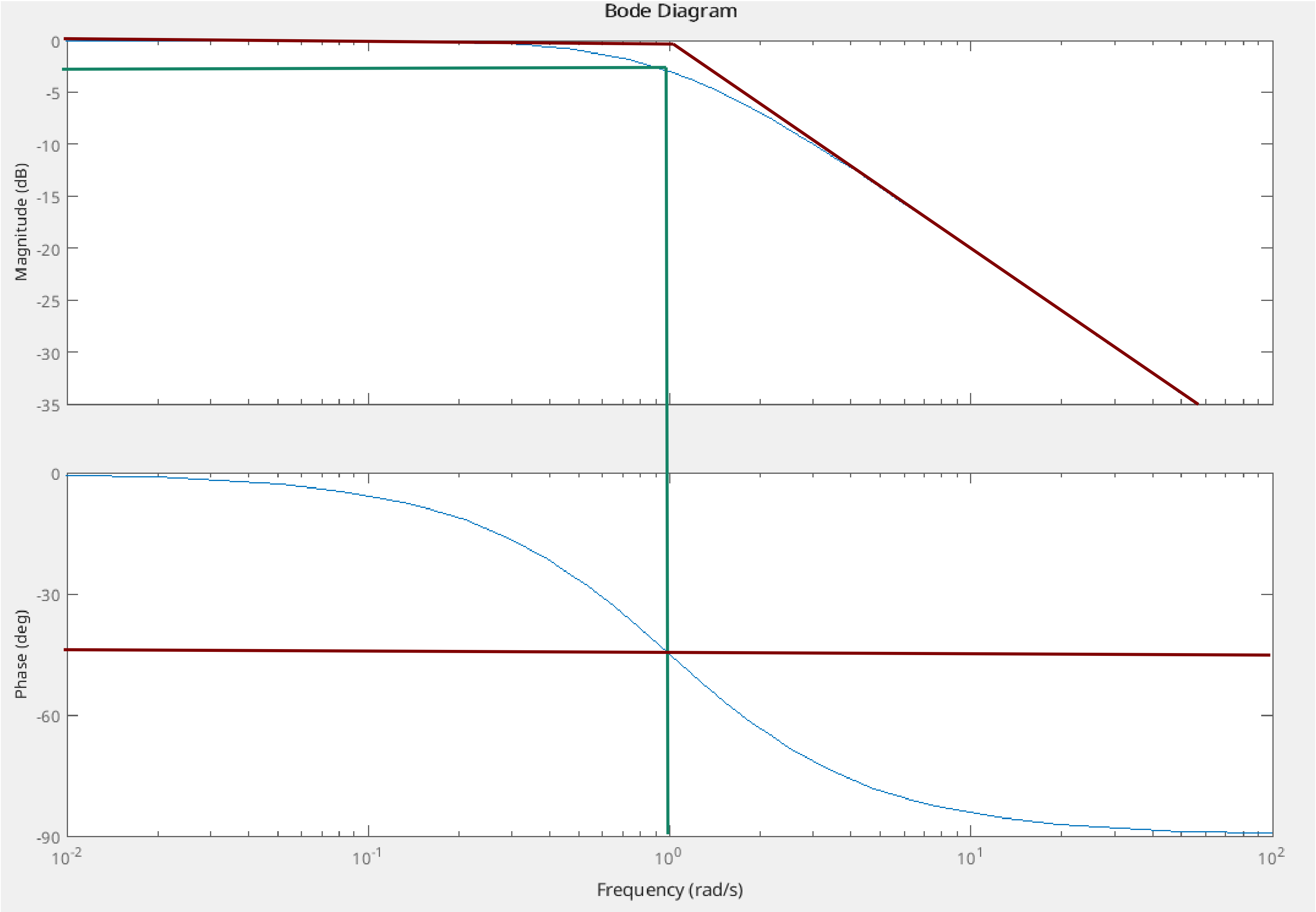
Bode Plot



$$G = \frac{1}{s+1}$$

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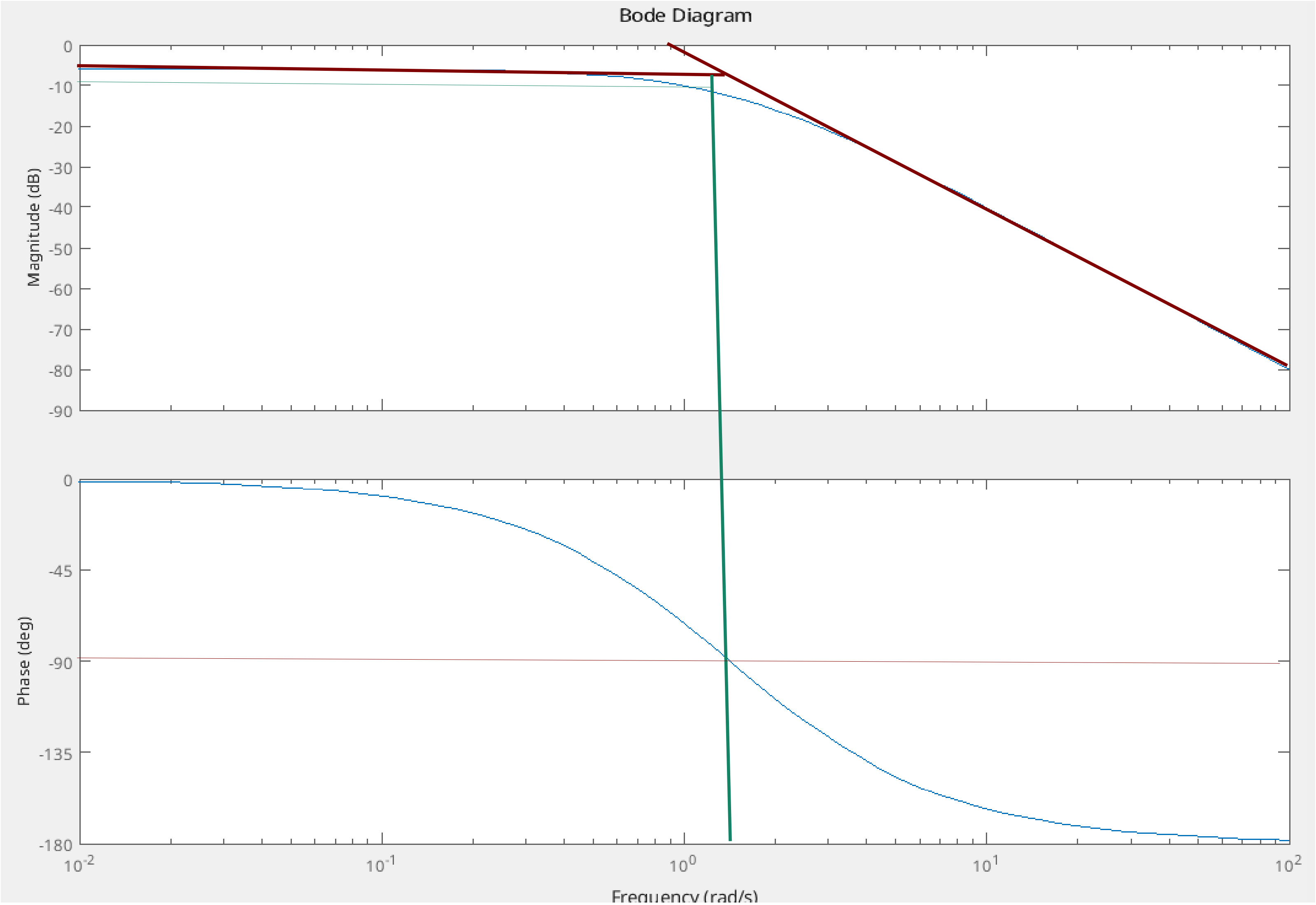
Bode Plot



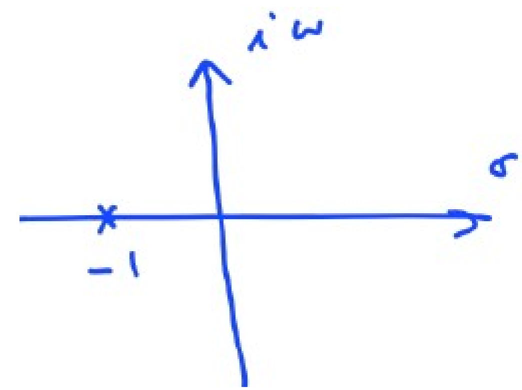
$$G = \frac{1}{s^2 + 3s + 2}$$

Bode Plot

$$\dot{y}(t) = -\lambda y(t) + u(t)$$

$$\tau = \frac{1}{\lambda}$$

time constant



$$(s+\lambda)\,Y(s) = U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s+\lambda}$$

$$= \frac{1}{\lambda + s}$$

$$= \frac{1}{\lambda}\,\frac{1}{1+\frac{s}{\lambda}}$$

$$\boxed{\frac{1}{\lambda}\,\frac{1}{\frac{\lambda+s}{\lambda}} = \frac{1}{\cancel{\lambda}}\,\frac{\cancel{\lambda}}{\lambda+s} = \frac{1}{\lambda+s}}$$

Bode Plot

use the following $G(s) = \dfrac{1}{1 + \frac{s}{\omega_0}}$ ← Break frequency

set $s = i\omega$

$$G(i\omega) = \frac{1}{1 + i\frac{\omega}{\omega_0}} \cdot \frac{1 - i\frac{\omega}{\omega_0}}{1 - i\frac{\omega}{\omega_0}}$$

$$= \frac{1 - i\frac{\omega}{\omega_0}}{1 + \left(\frac{\omega}{\omega_0}\right)^2}$$

gain $|G(i\omega)| = \sqrt{Re\{G(i\omega)\}^2 + Im\{G(i\omega)\}^2}$

$$= \frac{1}{1 + \left(\frac{\omega}{\omega_0}\right)^2} \sqrt{\left[1 + \left(\frac{\omega}{\omega_0}\right)^2\right]}$$

$$= \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}}$$

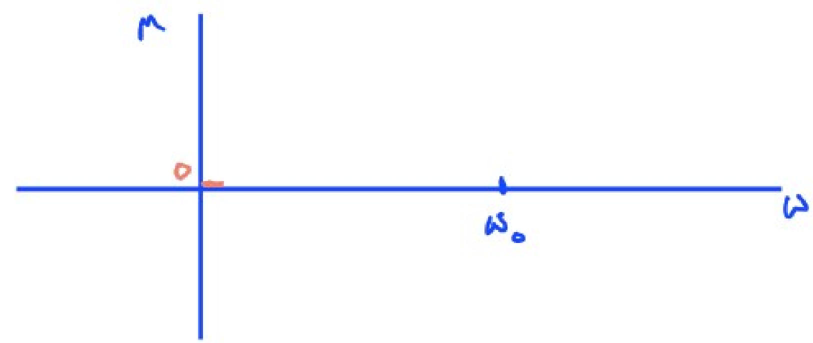
Bode Plot

$$|G(i\omega)| = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_0}\right)^2}}$$

gain $m = 20\log|G(i\omega)| = -20\log\left(\sqrt{1+\left(\frac{\omega}{\omega_0}\right)^2}\right)$

phase $\varphi = \tan^{-1}\left[\frac{\mathcal{Im}\{G(i\omega)\}}{\mathcal{Re}\{G(i\omega)\}}\right] = \tan^{-1}\left[-\frac{\omega}{\omega_0}\right]$

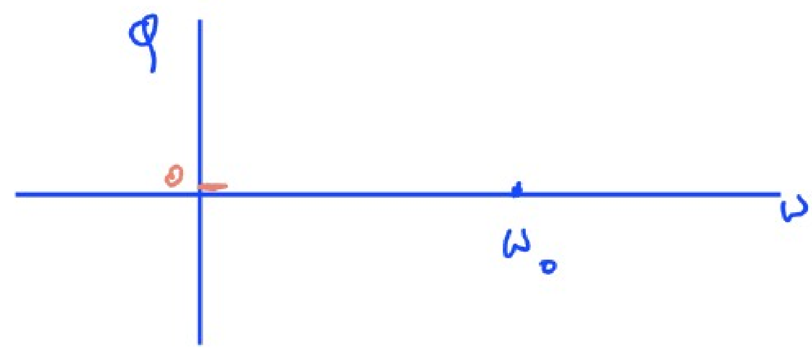
mind the quadrant



if $\omega \ll \omega_0$

$m \to 0$

$\varphi \to 0$



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Bode Plot

if

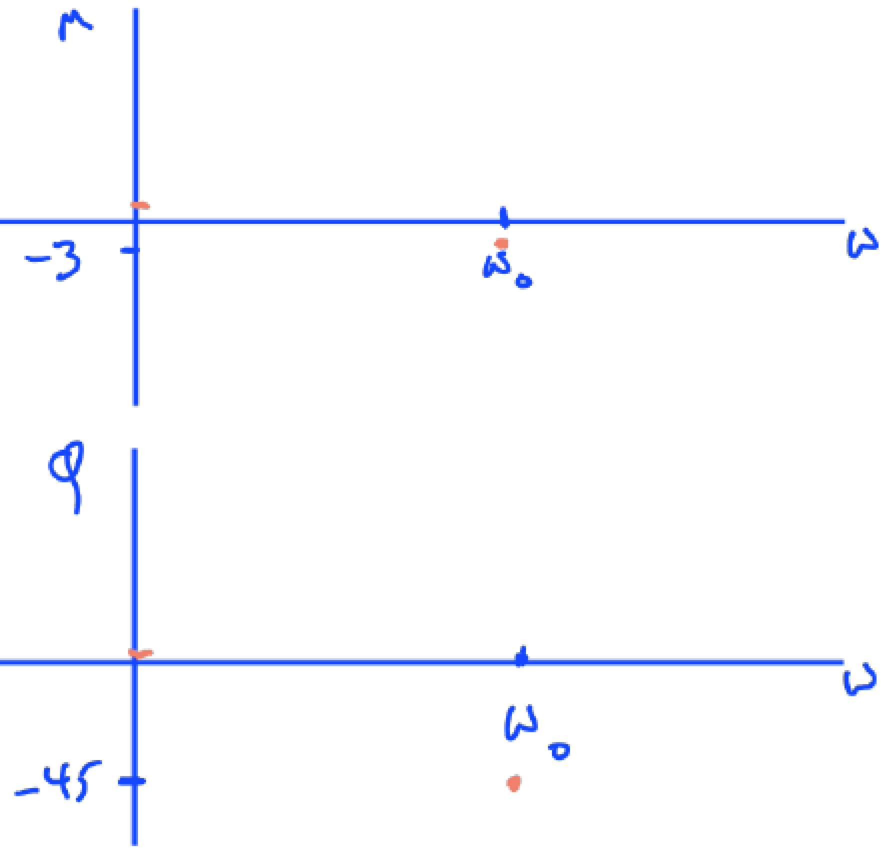
$\omega = \omega_0$

$M = -20\log\sqrt{2}$

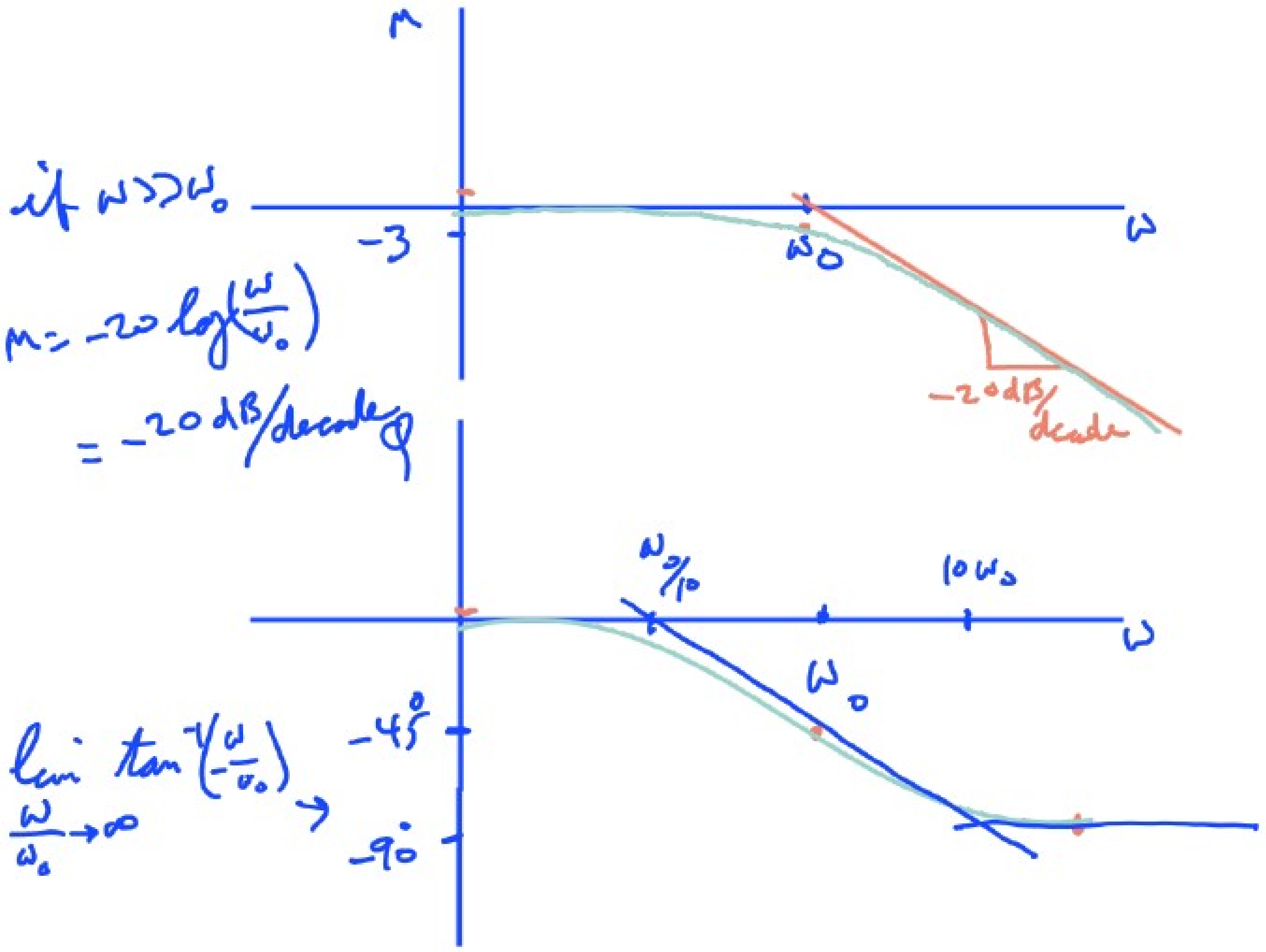
$= -3dB$

$\varphi = \tan^{-1}(-1)$

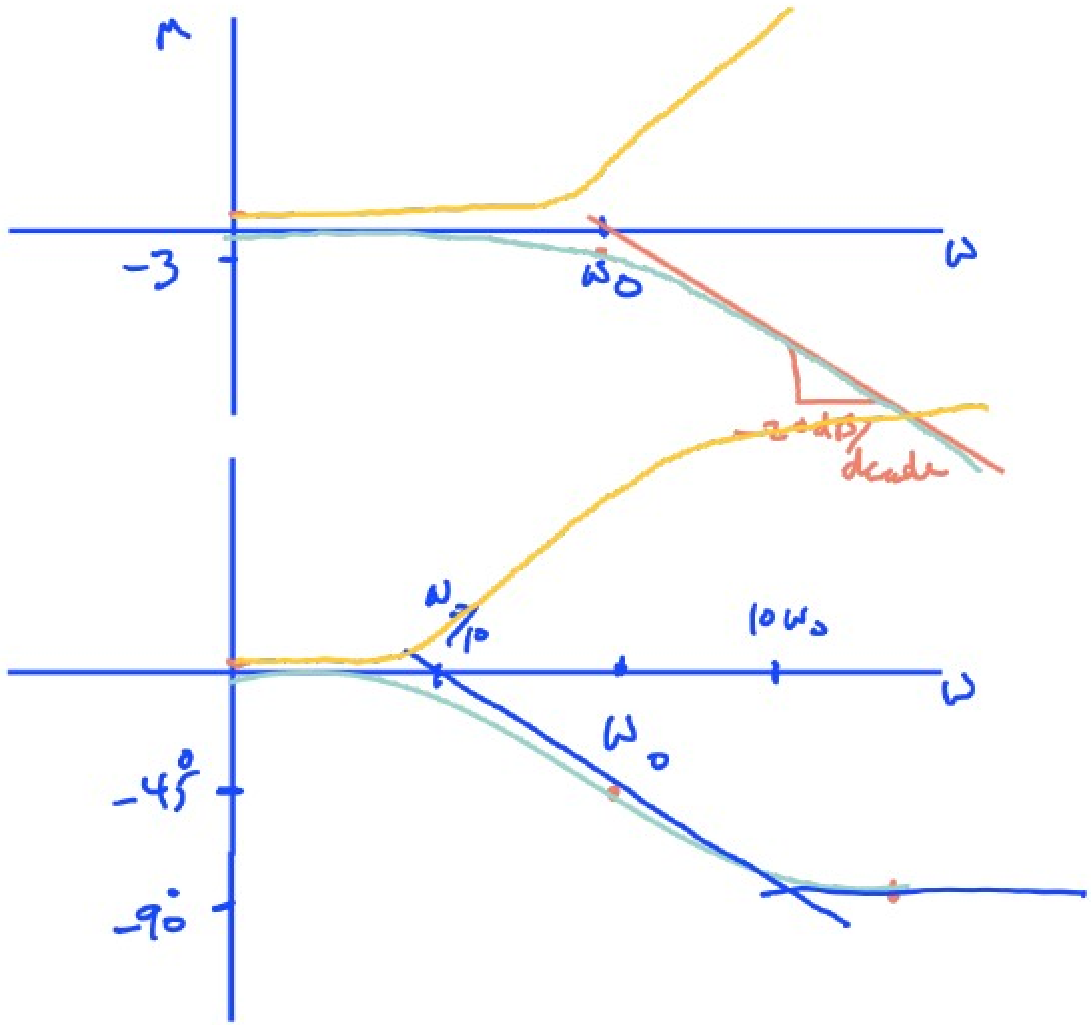
$= -45^\circ$



Bode Plot


M
if ω>>ω₀
-3
ω₀
ω
M = -20 log(ω/ω₀)
= -20 dB/decade
-20 dB/decade
φ
ω₀/10
10ω₀
ω
ω₀
-45°
-90°
lim tan⁻¹(-ω/ω₀) →
ω/ω₀→∞

Bode Plot



for a simple zero

$$G(s) = 1 + \frac{s}{\omega_0}$$

$$G(j\omega) = 1 + j\frac{\omega}{\omega_0}$$

$$M = 20\log\left(\sqrt{1+\left(\frac{\omega}{\omega_0}\right)^2}\right)$$

$$\varphi = \tan^{-1}\left(\frac{\omega}{\omega_0}\right)$$

Bode Plot

complex conjugate poles

recall the form

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

$$= \frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\frac{s}{\omega_0} + 1}$$

$$G(i\omega) = \frac{1}{\left(\frac{i\omega}{\omega_0}\right)^2 + 2\zeta\frac{\omega}{\omega_0} i + 1}$$

$$= \frac{1}{-\left(\frac{\omega}{\omega_0}\right)^2 + 2\zeta\frac{\omega}{\omega_0} + 1} = \frac{1}{1 - \left(\frac{\omega}{\omega_0}\right)^2 + 2\zeta\frac{\omega}{\omega_0} i}$$

$$ms^2 + cs + k = 0$$

$$\frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

$c^2 < 4km \Rightarrow$ complex conjugate

$$\zeta \triangleq \sqrt{\frac{c^2}{4km}} = \frac{c}{2\sqrt{km}}$$

damping ratio

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Bode Plot

$$G(i\omega) = \frac{1}{1-\left(\frac{\omega}{\omega_0}\right)^2 + 2\zeta\frac{\omega}{\omega_0}i} \cdot \frac{1-\left(\frac{\omega}{\omega_0}\right)^2 - 2\zeta\frac{\omega}{\omega_0}i}{1-\left(\frac{\omega}{\omega_0}\right)^2 - 2\zeta\frac{\omega}{\omega_0}i}$$

$$= \frac{1-\left(\frac{\omega}{\omega_0}\right)^2 - 2\zeta\frac{\omega}{\omega_0}i}{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_0}\right]^2}$$

$$|G(i\omega)| = \frac{1}{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_0}\right]^2}\left[\sqrt{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_0}\right)^2}\right]$$

$$= \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2}}$$

Bode Plot

$$|G(i\omega)| = \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2+\left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2}}$$

$$M = 20\log|G(i\omega)| = -20\log\sqrt{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2+\left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2}$$

$$\varphi = \tan^{-1}\left[\frac{-2\zeta\frac{\omega}{\omega_0}}{1-\left(\frac{\omega}{\omega_0}\right)^2}\right]$$

$$\text{if } \omega \ll \omega_0 \quad \left.\begin{array}{l} \mathrm{Re}\{G(i\omega)\} \to 1 \\ \mathrm{Im}\{G(i\omega)\} \to 0 \end{array}\right\} \Longrightarrow |G(i\omega)| \to 1$$

$$\Rightarrow M \to 0$$

$$\varphi = \tan^{-1}\left(\frac{\mathrm{Im}(G(i\omega))}{\mathrm{Re}(G(i\omega))}\right) \to 0^\circ$$

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Bode Plot

if $\omega \gg \omega_0$

$$\mathrm{Re}\{G(i\omega)\} \to \frac{-\left(\frac{\omega}{\omega_0}\right)^2}{\left(\frac{\omega}{\omega_0}\right)^4 + \epsilon} = -\left(\frac{\omega}{\omega_0}\right)^{-2}$$

$$\mathrm{Im}\{G(i\omega)\} \to \frac{-2\zeta\frac{\omega}{\omega_0}}{\left(\frac{\omega}{\omega_0}\right)^4} = -2\zeta\left(\frac{\omega}{\omega_0}\right)^{-3}$$

$$|G(i\omega)| \to \sqrt{\left[-\left(\frac{\omega}{\omega_0}\right)^{-2}\right]^2 + \underbrace{\left[-2\zeta\left(\frac{\omega}{\omega_0}\right)^{-3}\right]^2}_{\to\ \epsilon}}$$

$$\approx \sqrt{\left(\left(\frac{\omega}{\omega_0}\right)^{-2}\right)^2}$$

$$= \left(\frac{\omega}{\omega_0}\right)^{-2}$$

$$M = 20\log|G(i\omega)| = 20\log\left(\frac{\omega}{\omega_0}\right)^{-2} = \underbrace{-40\log\left(\frac{\omega}{\omega_0}\right)}_{-40\,\mathrm{dB/decade}}$$

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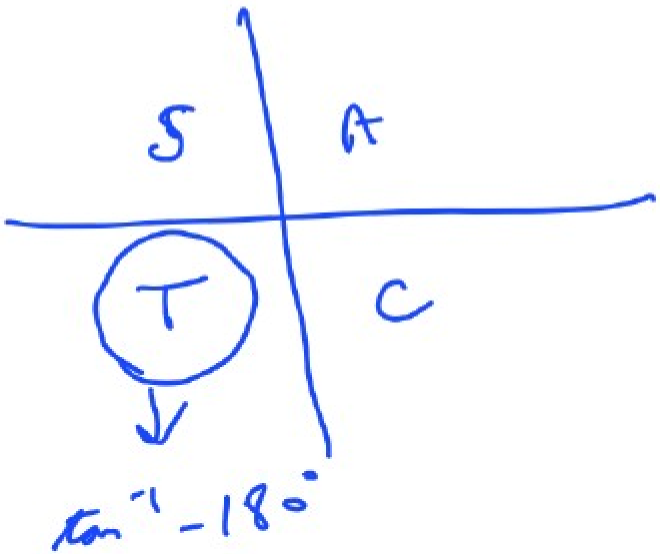
Bode Plot

$$\phi = \tan^{-1}\left[\frac{-2\zeta\left(\frac{\omega}{\omega_0}\right)^{-3}}{-\left(\frac{\omega}{\omega_0}\right)^{-2}}\right]$$

$$= \tan^{-1}\left(2\zeta\left(\frac{\omega}{\omega_0}\right)^{-1}\right) - 180^\circ$$

$$= 0 - 180^\circ$$

$$= -180^\circ$$



Bode Plot

If $w = w_0$

$$Re\{G(i\omega)\} = \frac{\cancel{1-\left(\frac{\omega}{\omega_0}\right)^2} \to 0}{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left[2\zeta\frac{\omega}{\omega_0}\right]^2} \approx 0$$

$$Im\{G(i\omega)\} = \frac{-2\zeta\frac{\omega}{\omega_0} \to -2\zeta}{\cancel{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2} \to 0 + \cancel{\left[2\zeta\frac{\omega}{\omega_0}\right]^2} \; (2\zeta)^2} \approx -\frac{1}{2\zeta}$$

$$|G(i\omega)| = \sqrt{0^2 + \left(-\frac{1}{2\zeta}\right)^2}$$

$$= \frac{1}{2\zeta}$$

$$M = 20\log|G(i\omega)| = 20\log\frac{1}{2\zeta}$$

$$= -20\log(2\zeta)$$

$$\phi = \tan^{-1}(\infty) = -90$$

3rd quadrant

if $\zeta = 0 \Rightarrow M \to \infty$

if $\zeta = 0.5 \Rightarrow M = 0$

if $\zeta > 0.5 \Rightarrow M \downarrow$ (negative)

if $\zeta > 0.5 \Rightarrow$ max. at ω_0 $= -20\log(2\zeta)$

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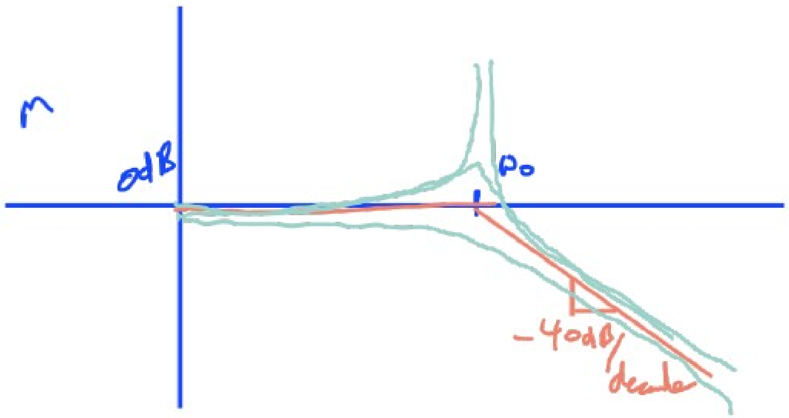
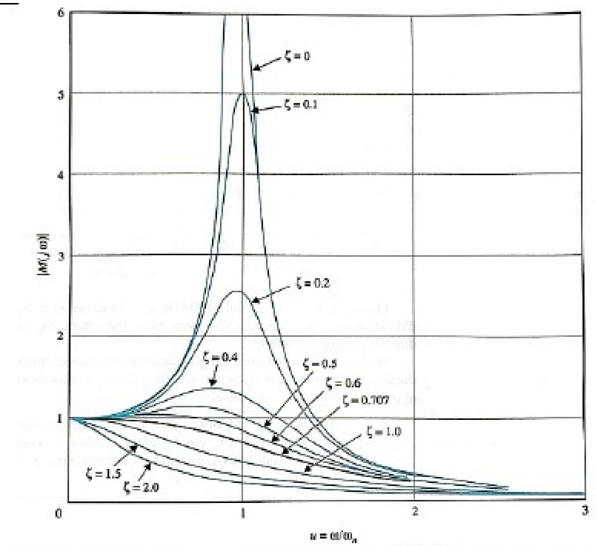
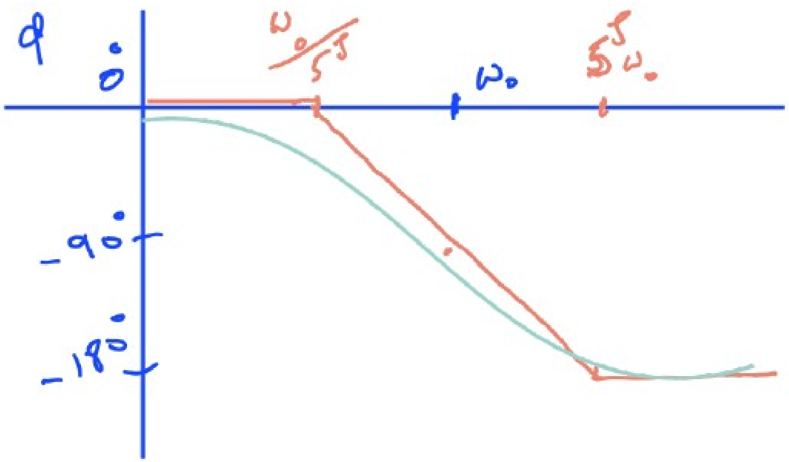
Bode Plot







- trouble spot: $\underbrace{M = 0 \ \& \ \phi = -180}$

$\downarrow$

$CGH = -1$

should not happen simultaneously $\left(\begin{array}{l} |\tilde{G}| = 1 \quad 20 \log|\tilde{G}| = 0 \\ \angle \tilde{G} = -180° \end{array} \right.$

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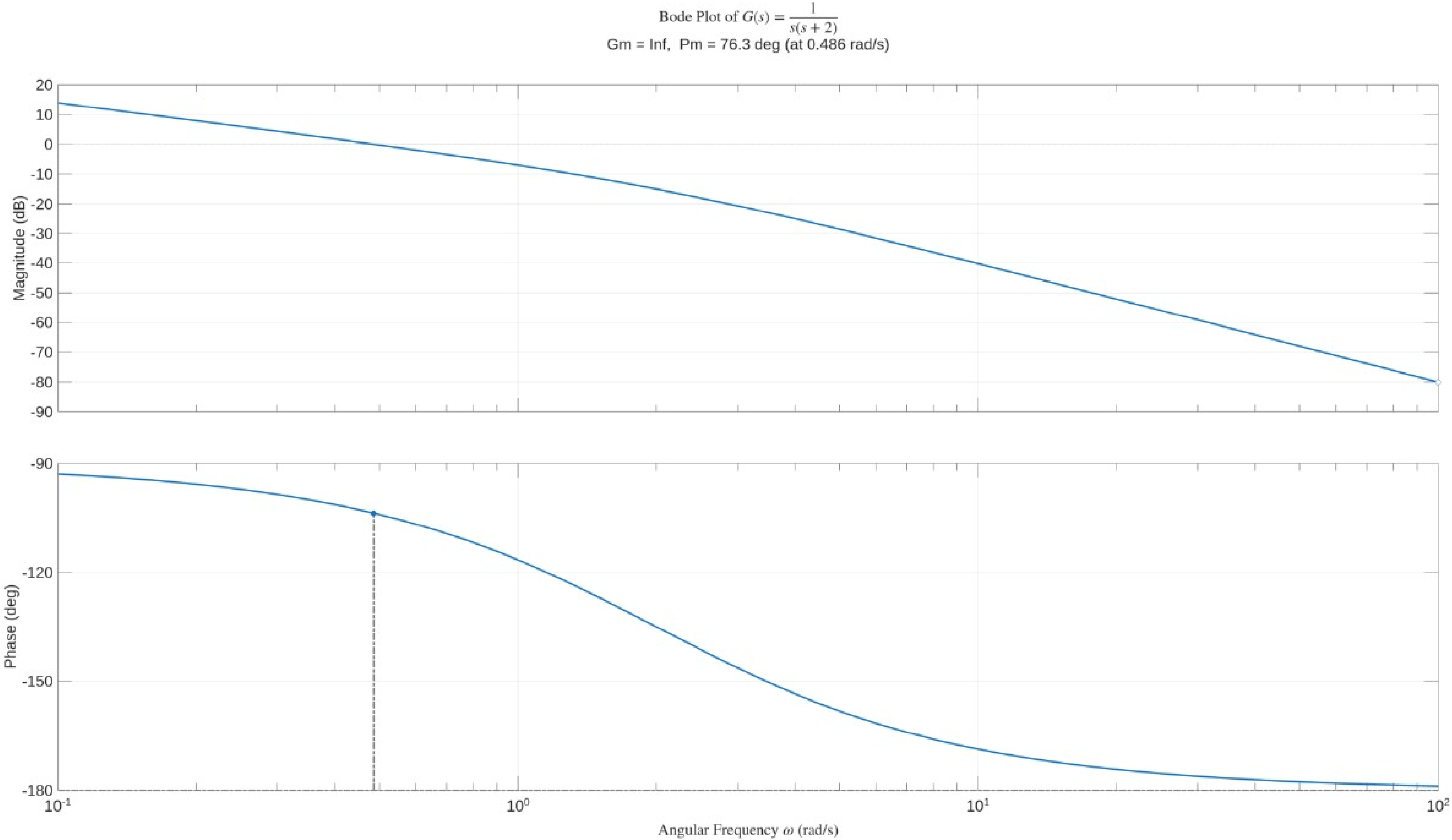
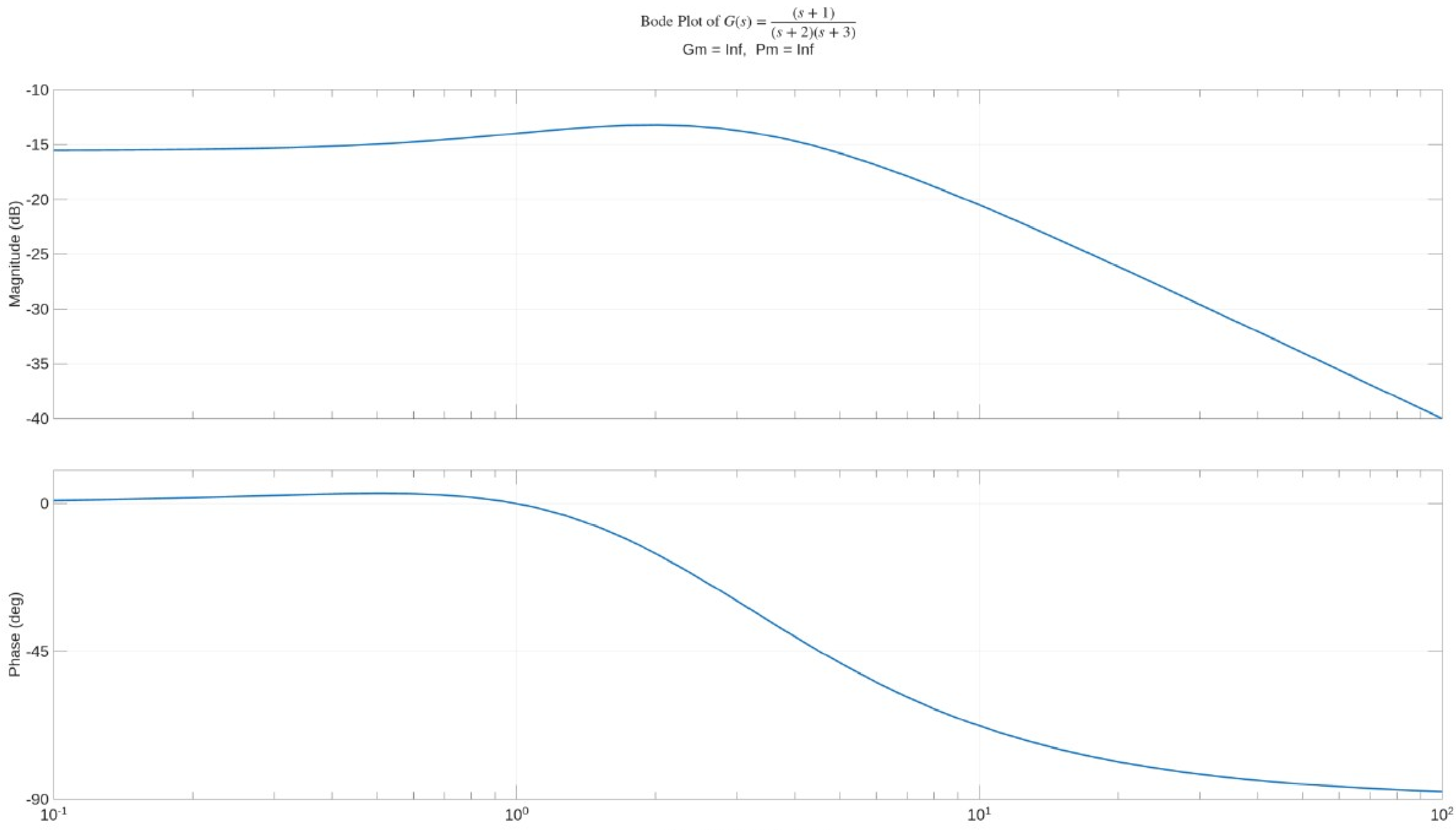
Bode Plot

$$G(s) = \frac{1}{s(s+2)}$$



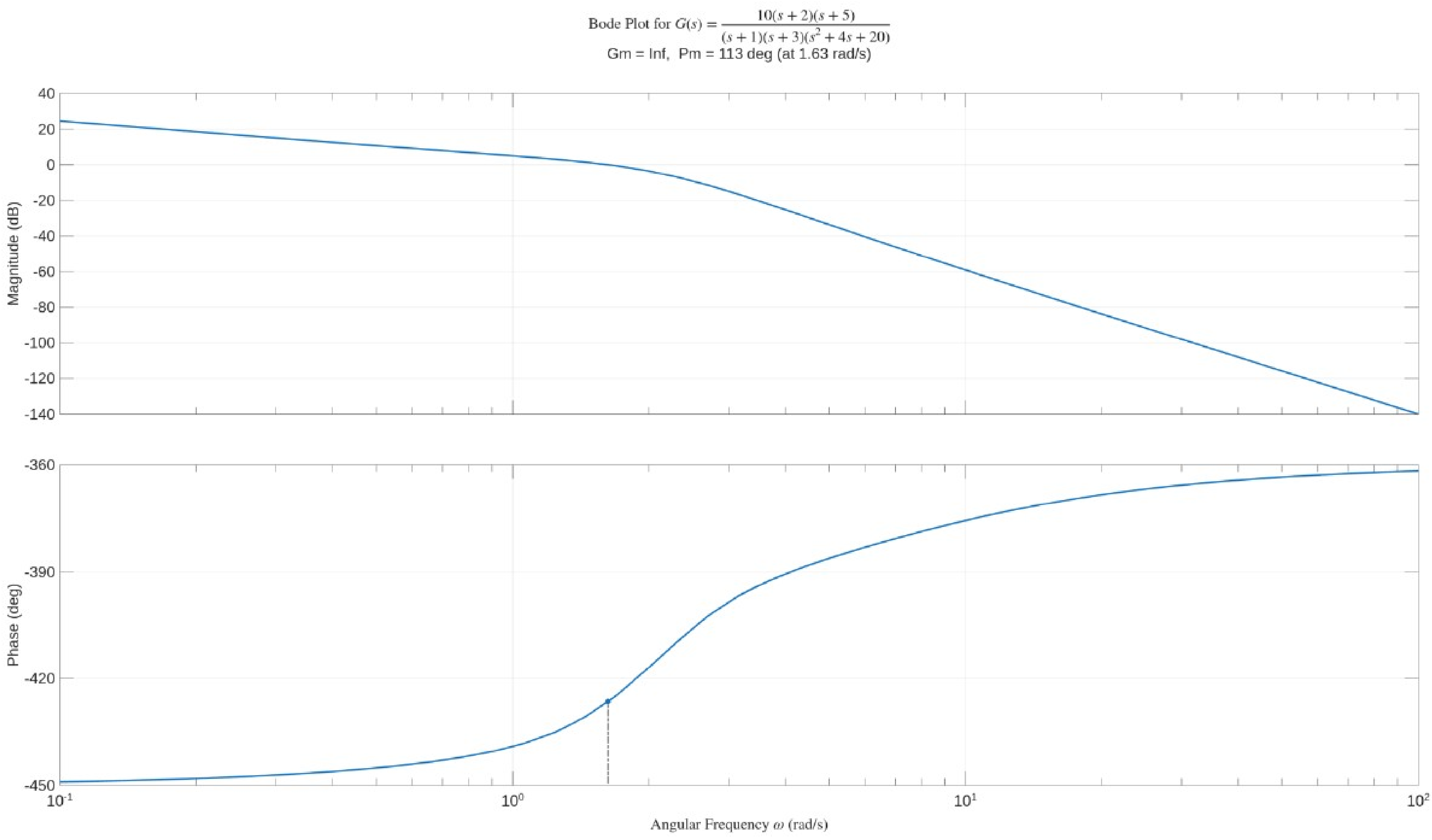
Bode Plot

$$G(s) = \frac{(s+1)}{(s+2)(s+3)}$$



$$G(s) = \frac{10(s+2)(s+5)}{(s+3)(s+1)(s^2+4s+20)}$$

Bode Plot



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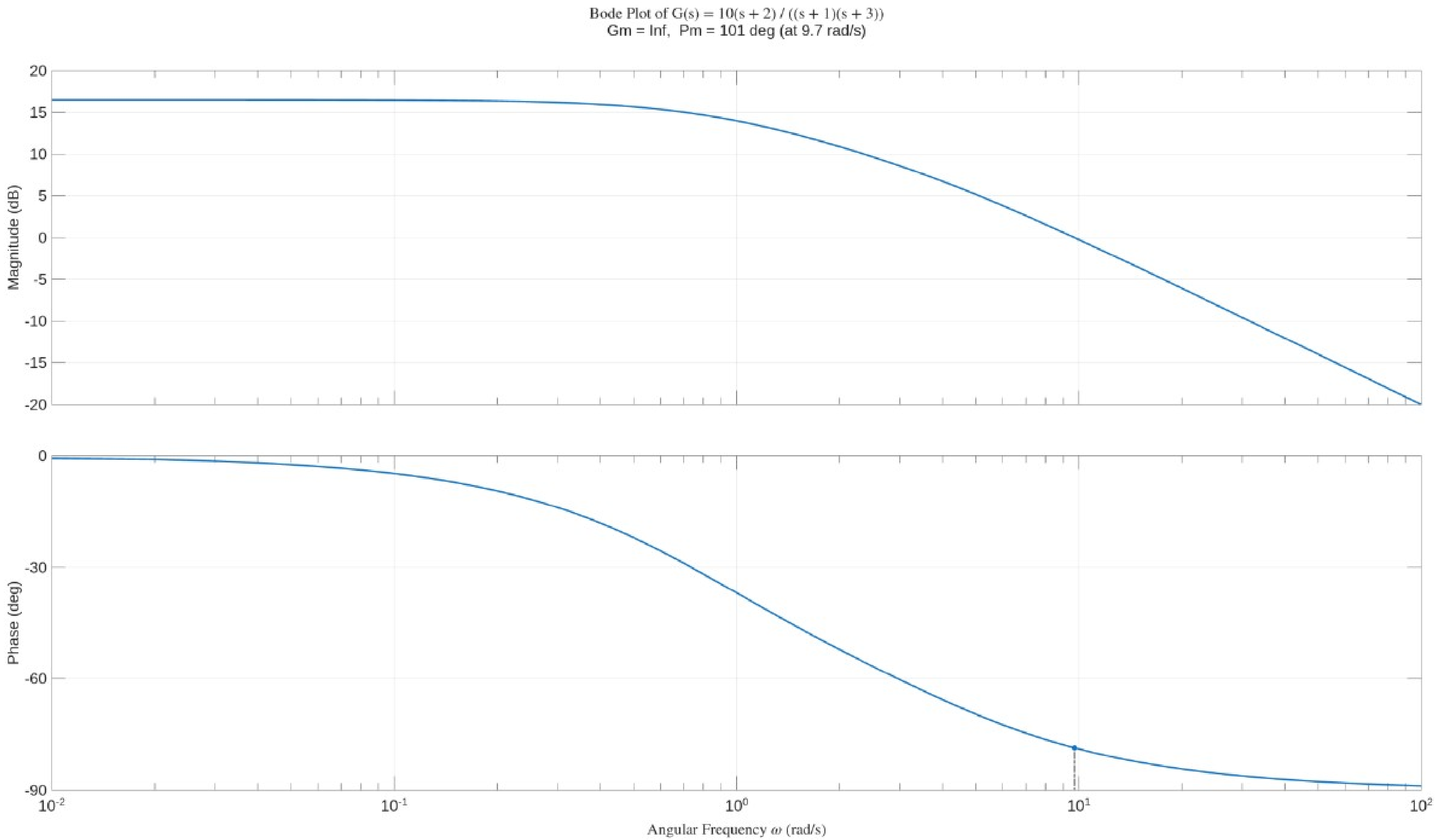
$$G(s) = \frac{10(s+2)}{(s+3)(s+1)}$$

Bode Plot



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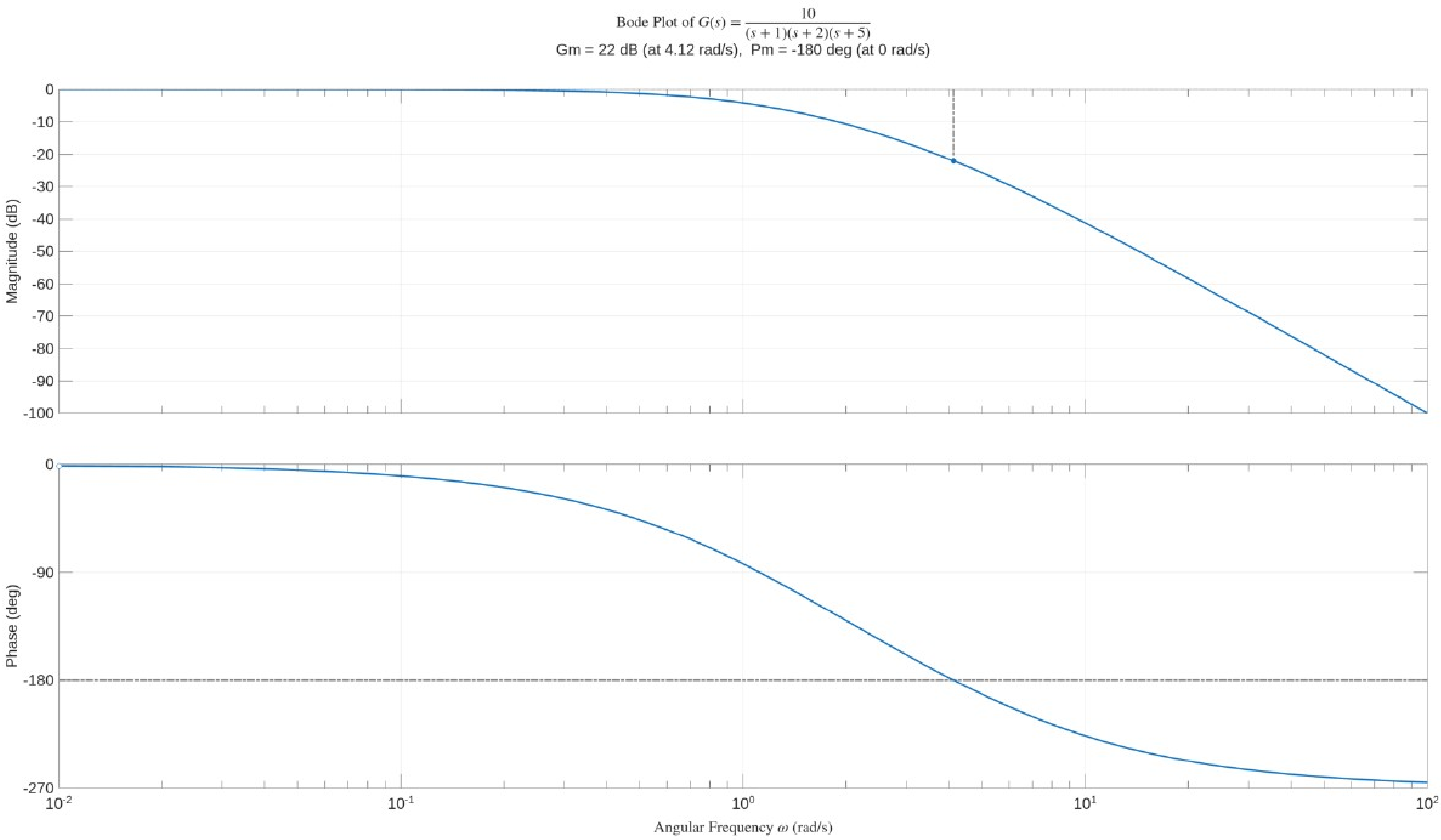
Bode Plot

$$G(s) = \frac{10}{(s+1)(s+2)(s+3)}$$



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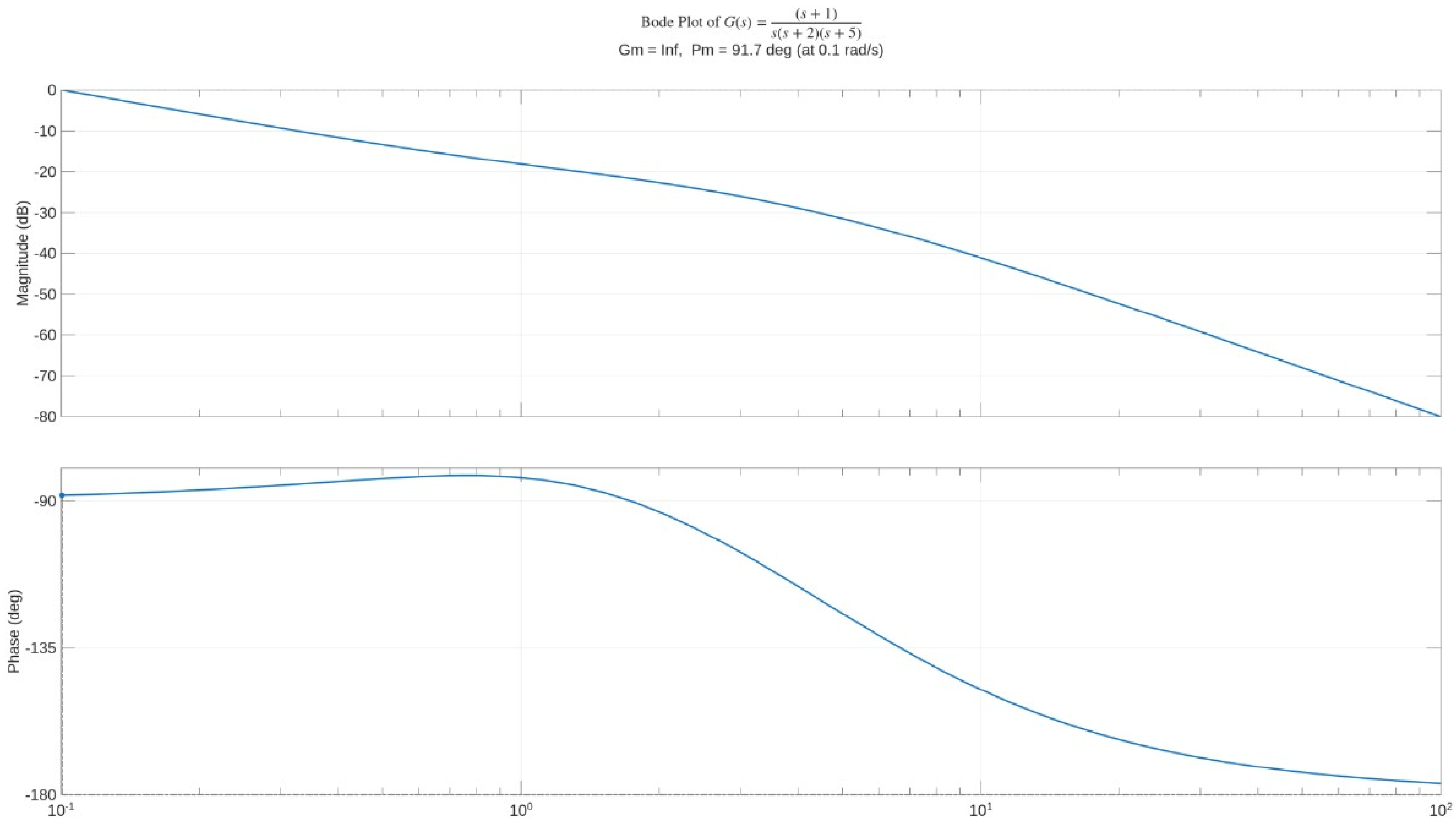
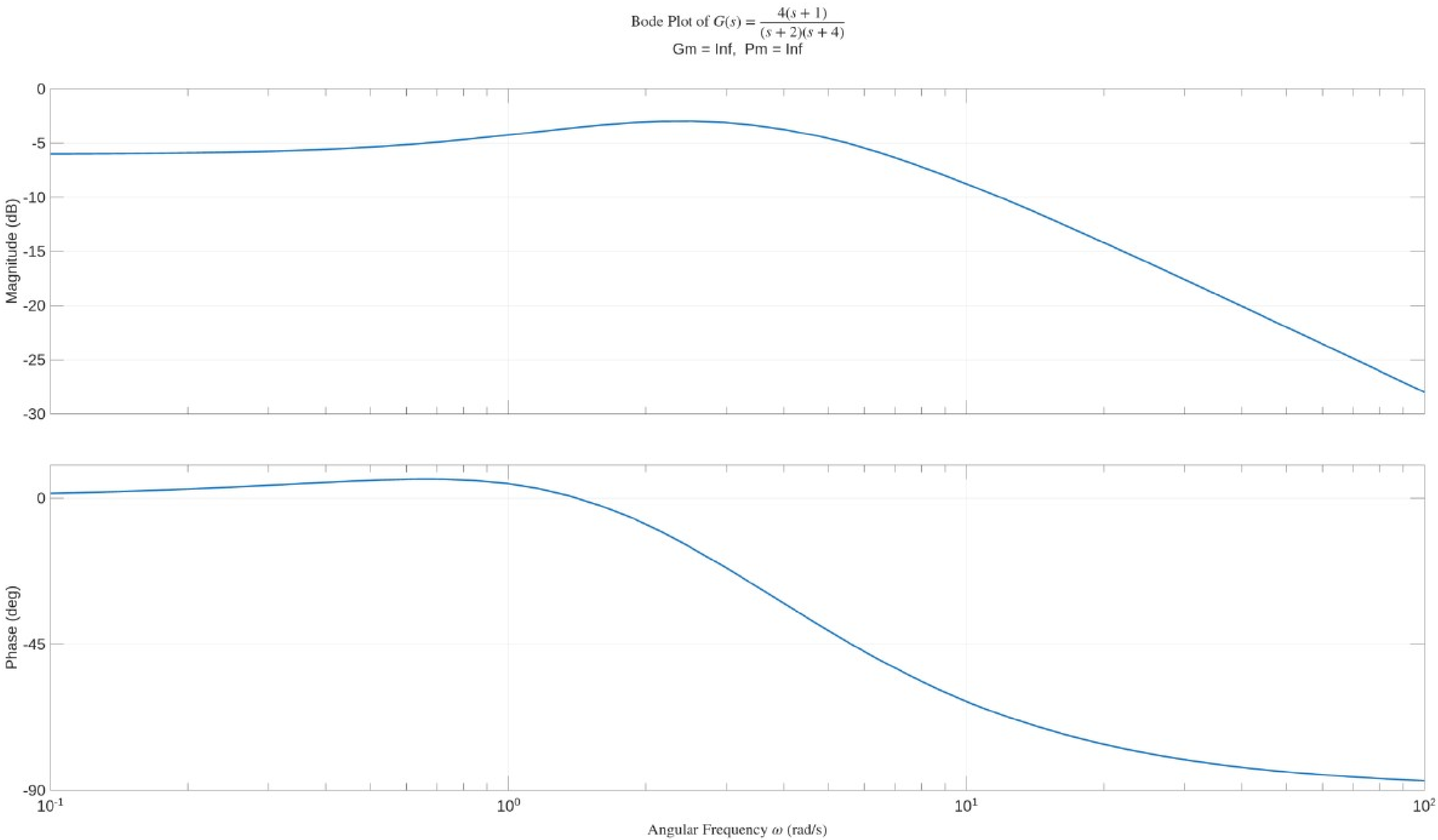
Bode Plot



Bode Plot


Bode Plot of $G(s) = \frac{4(s+1)}{(s+2)(s+4)}$
Gm = Inf, Pm = Inf
Magnitude (dB)
0
-5
-10
-15
-20
-25
-30
Phase (deg)
0
-45
-90
10^{-1}
10^{0}
10^{1}
10^{2}
Angular Frequency ω (rad/s)

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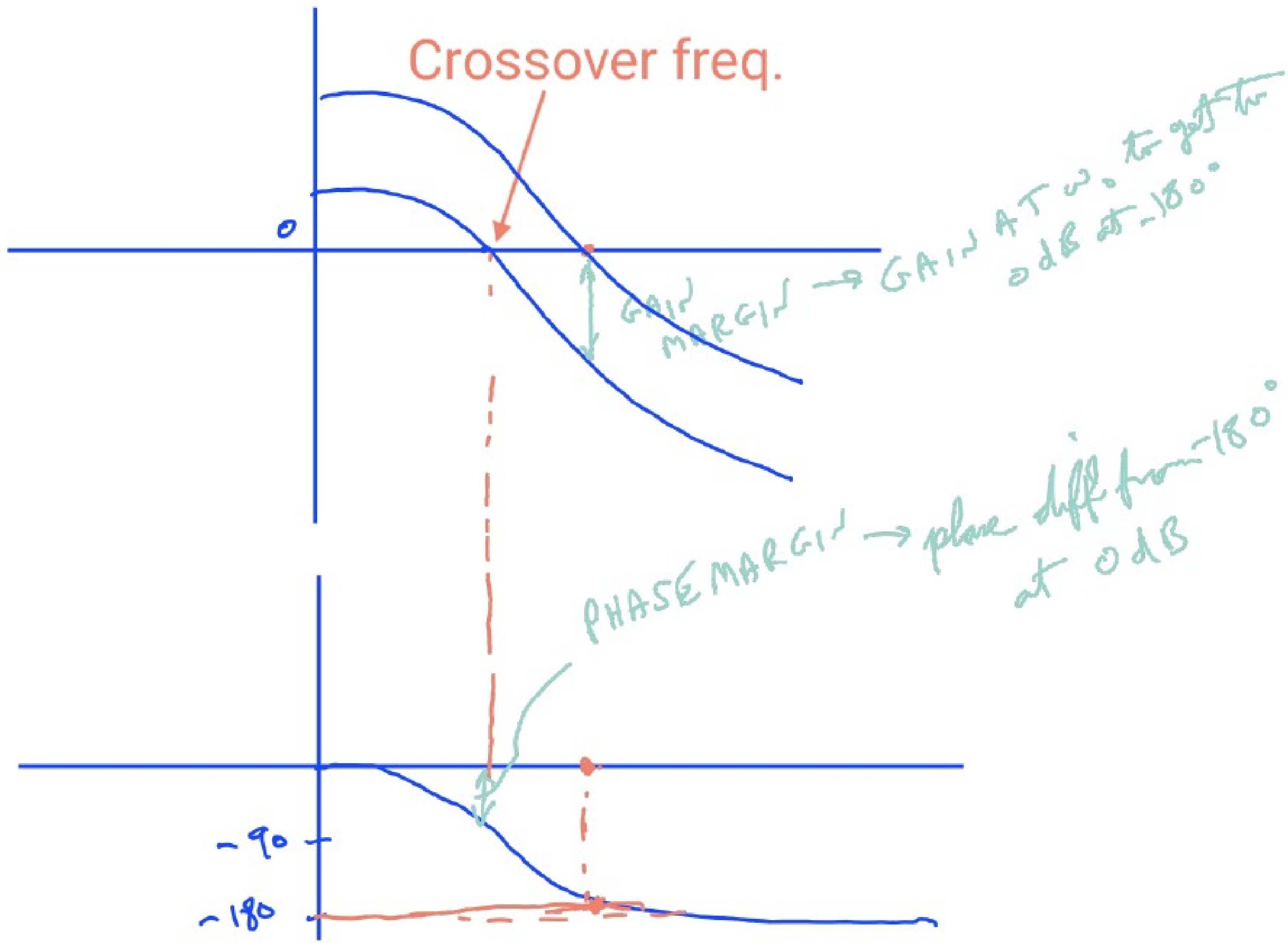
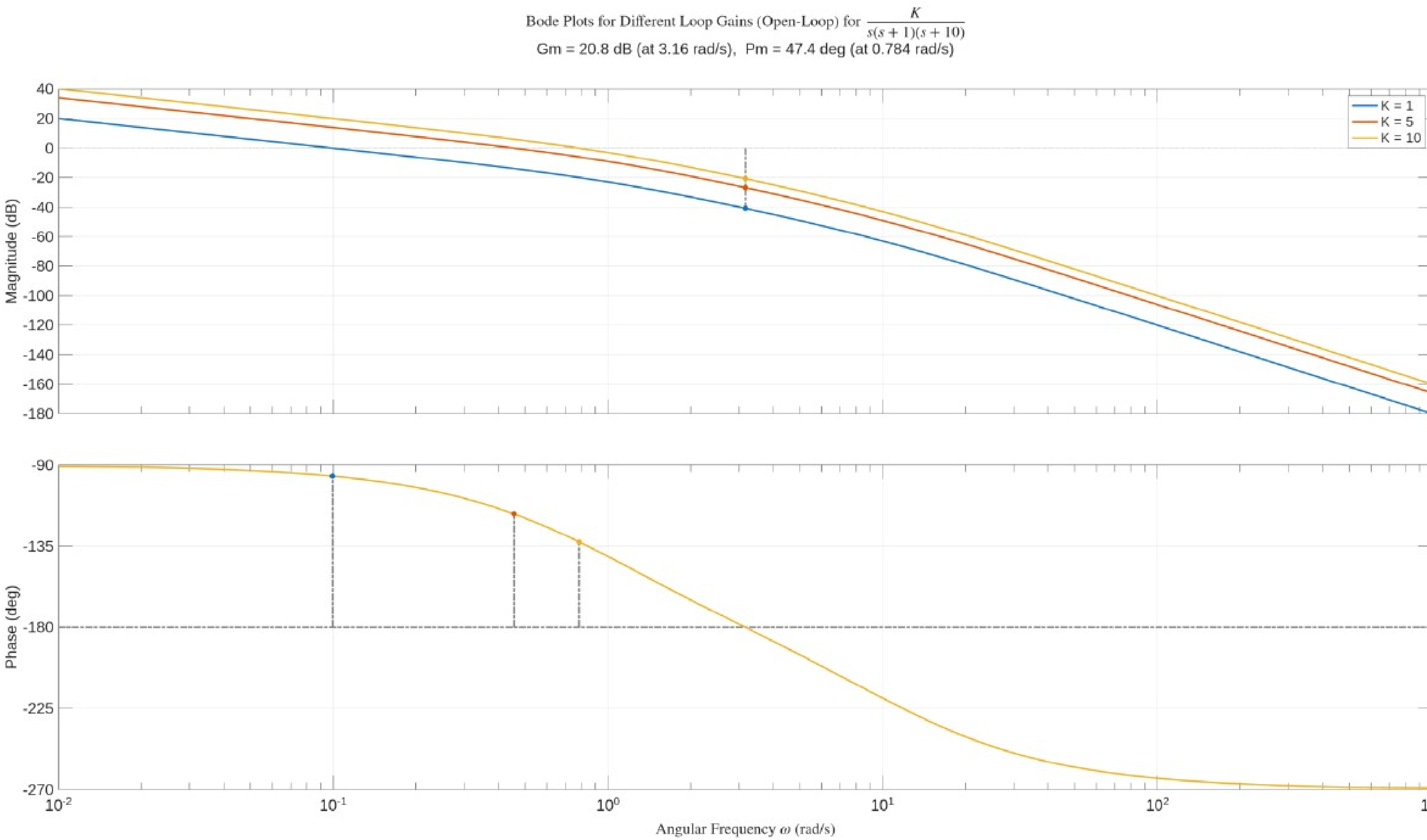
Bode Margins



$$G(s) = \frac{K}{s(s+1)(s+10)}$$

Bode Plot



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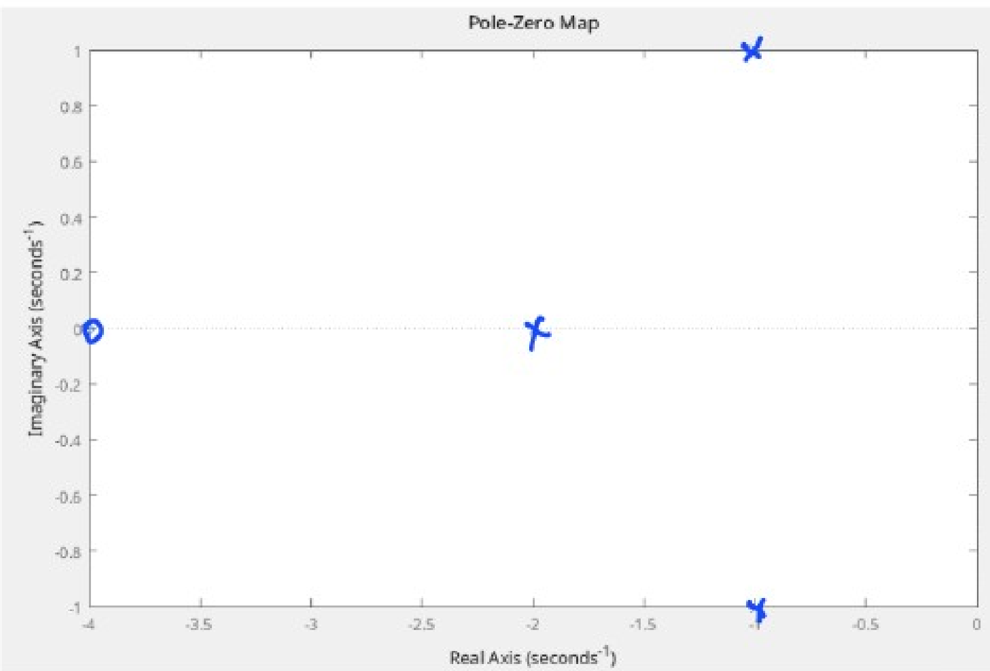
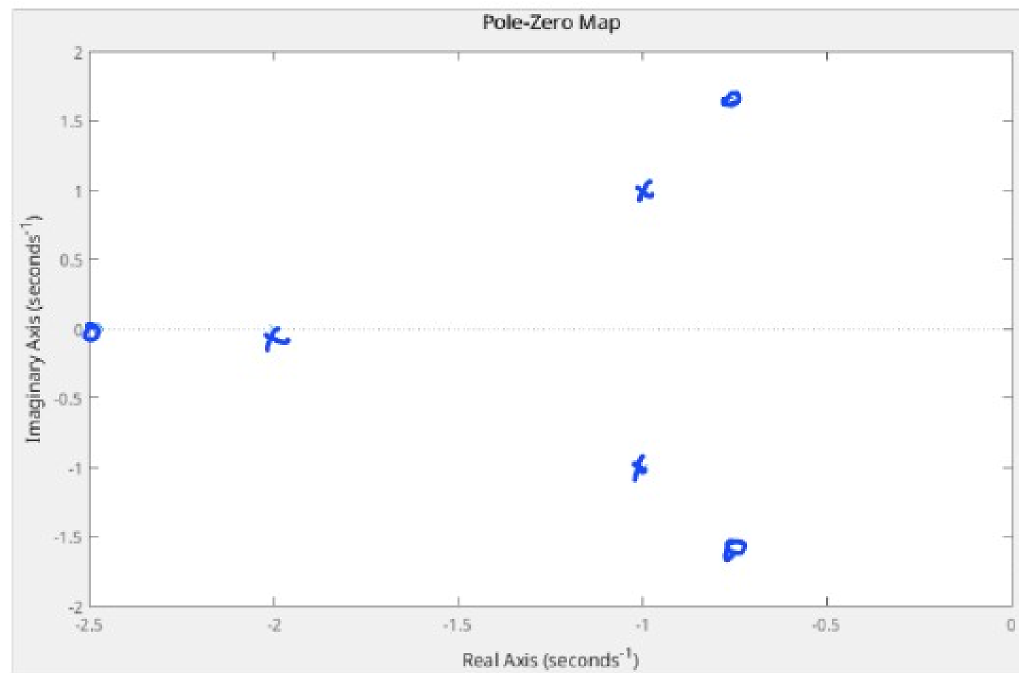
Nyquist

NYQUIST

$1 + CGH = 0$

$$CGH = \frac{s+4}{(s+2)(s^2+2s+2)}$$

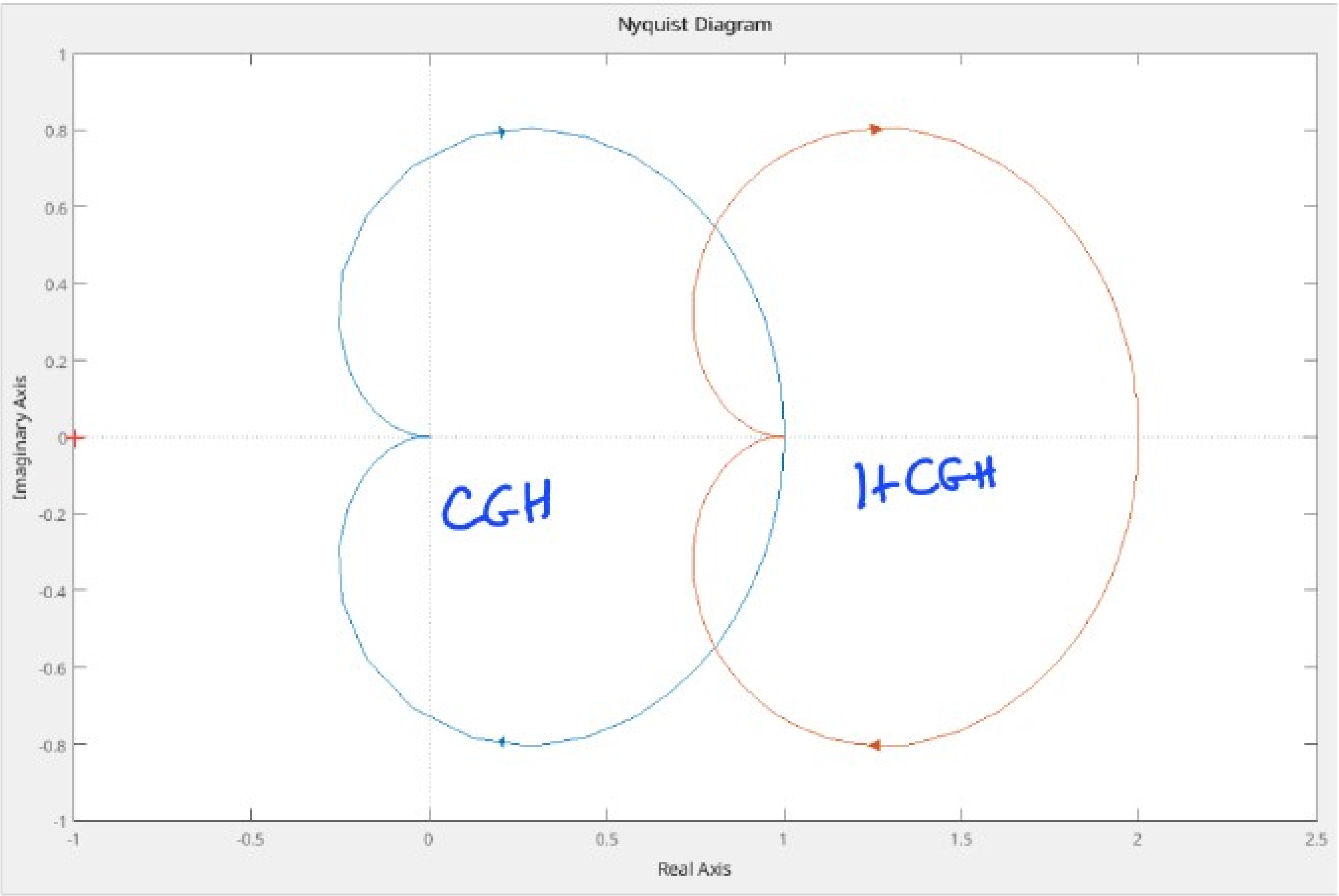




CGH

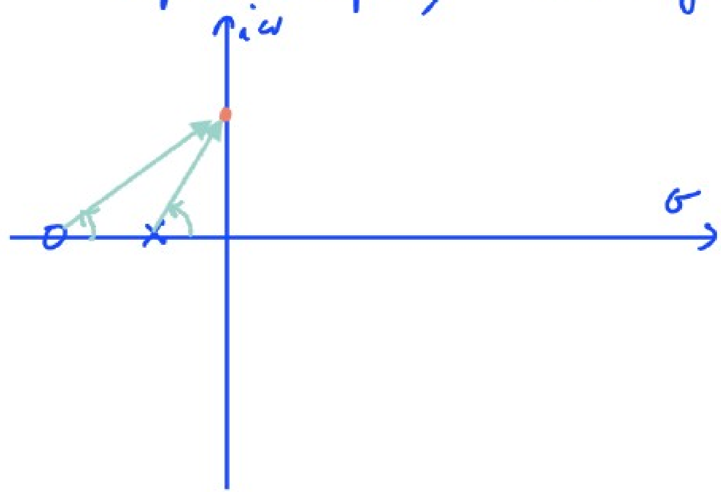
1 + CGH

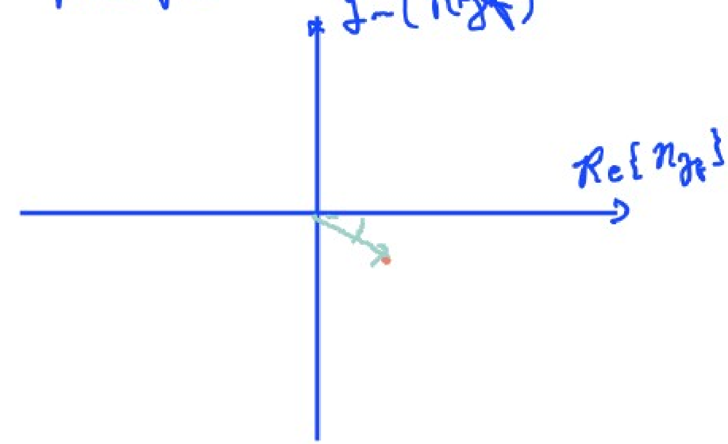
Nyquist


Nyquist Diagram
Imaginary Axis
Real Axis
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1
1.5
2
2.5
CGH
1+CGH

Nyquist

Laplace (s-plane)   Cauchy's Argument Principle





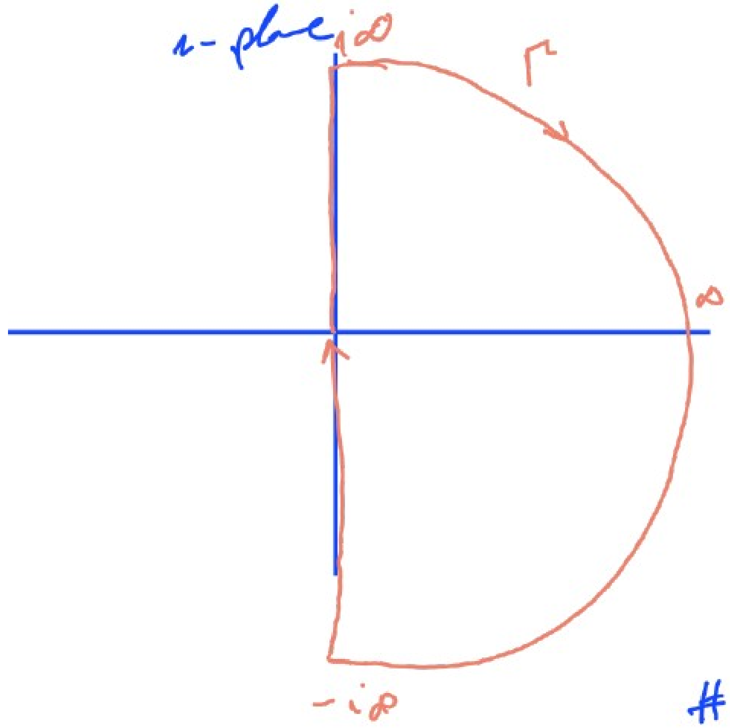
- select a point
- draw vector from poles and zeros to pt.
- $A = \dfrac{\prod_{j=1}^{m} |s - z_j|}{\prod_{k=1}^{m} |s - p_k|}$
- $\varphi = \sum_{j=1}^{m} \angle(s - z_j) - \sum_{k=1}^{m} \angle(s - p_k)$

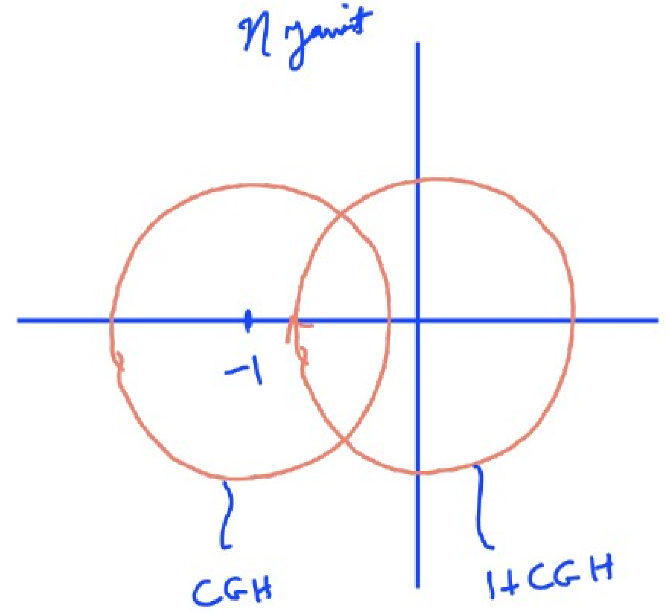
# of times the plot encircles the origin $= Z - P$
$=$ # of zeros $-$ # of poles

if contour in s-plane is
CW ⇒ ABOVE IS CW
CCW ⇒ ABOVE IS CCW

change in direction negates
CW → CCW (more poles)

Nyquist





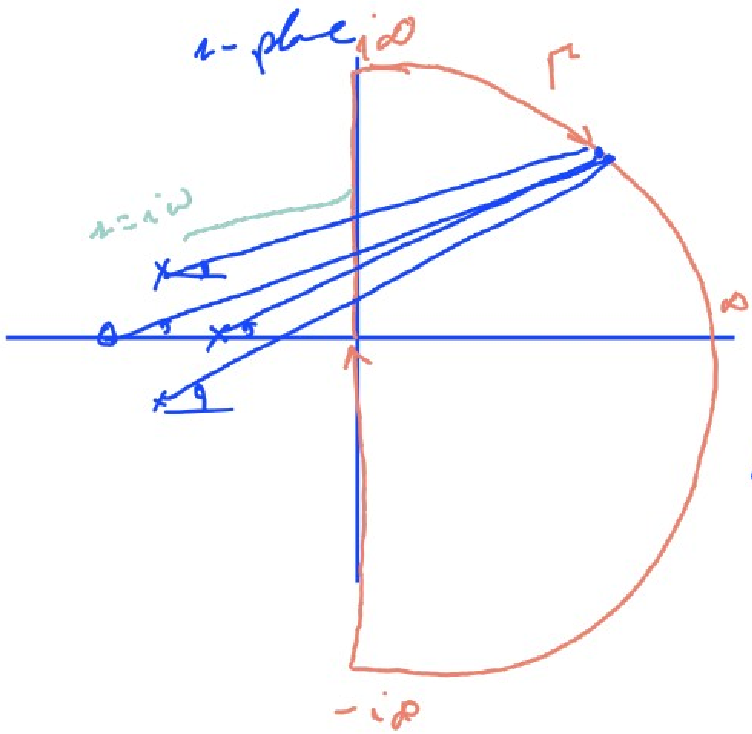
# of times -1 is encircled

# of poles in the RHP of 1+CGH =
# of poles in the RHP of CGH

# of zeros with the RHP → $Z = N + P$

of ⟳ of -1 # of poles of CGH in the RHP

Nyquist



strictly proper # of z < # of p

proper # of z = # of p

not proper # of z > # of p

$$\varphi = \sum \theta_z - \sum \theta_p$$

$$A = \frac{\prod_j^m |s - z_j|}{\prod_k^n |s - p_k|} = 0 \text{ for } m < n$$

$$= M < \infty \text{ for } m = n$$

φ does not matter since $A = 0$ (at the origin) for $m < n$

$\varphi = 0$ for $m = n$ since $\sum \theta_z - \sum \theta_p = 0$ at ∞

$A = M < \infty$ for $m = n$

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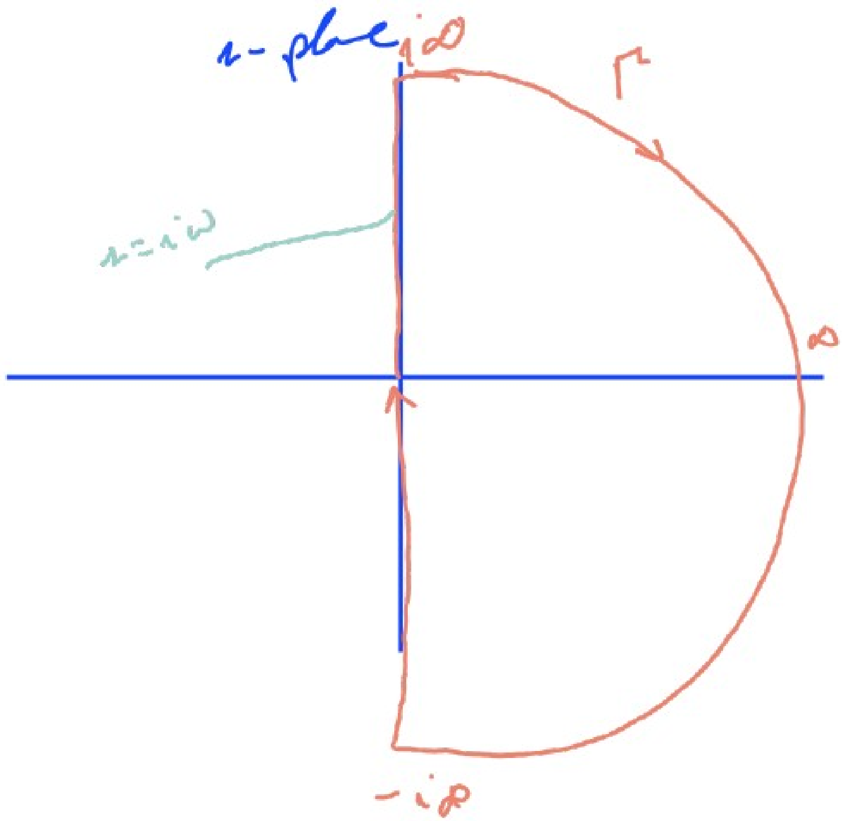
Nyquist



strictly proper # of z < # of p

proper # of z = # of p

not proper # of z > # of p

consider $i\omega$ for $0<\omega<\infty$

$\omega = 0$

$\omega = \infty$

ω ✗ imaginary axis → set $\sigma = 0$

ω ✗ real axis → set $\omega = 0$

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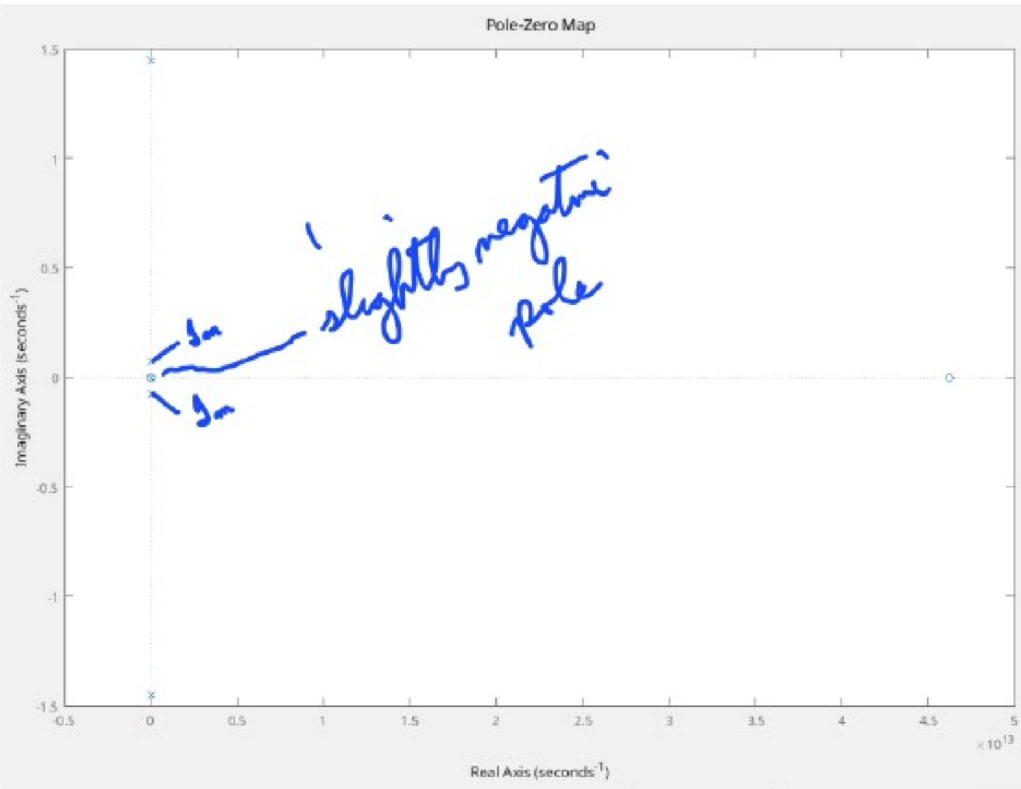
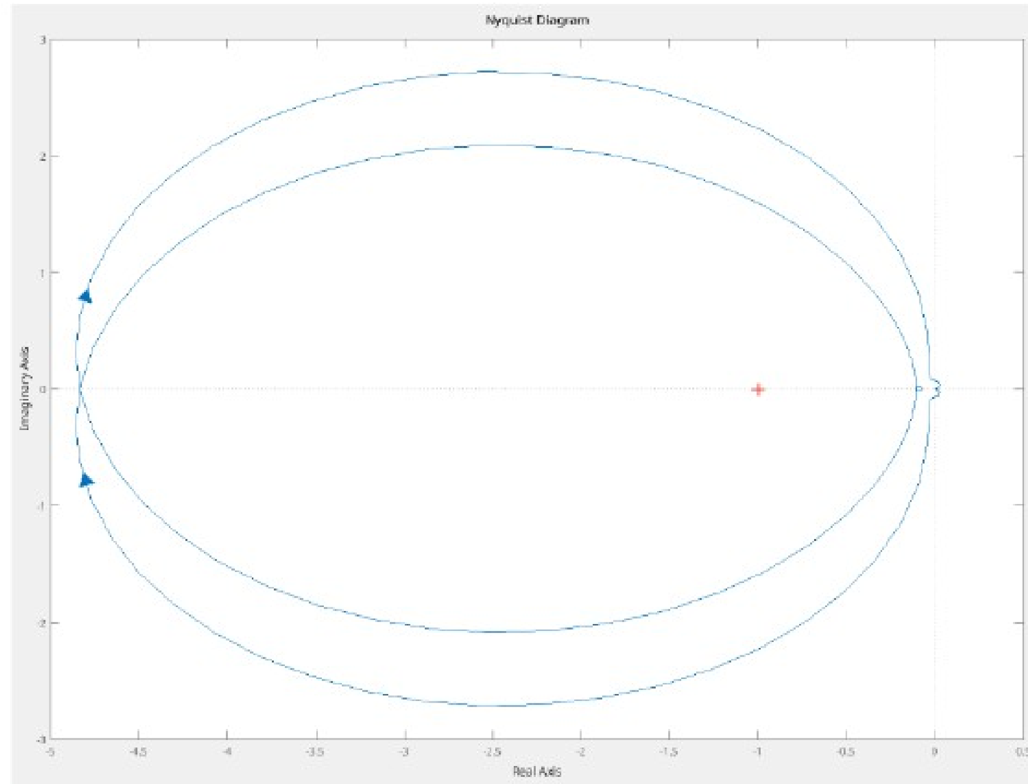
F-16 Aircraft – Nyquist Plot



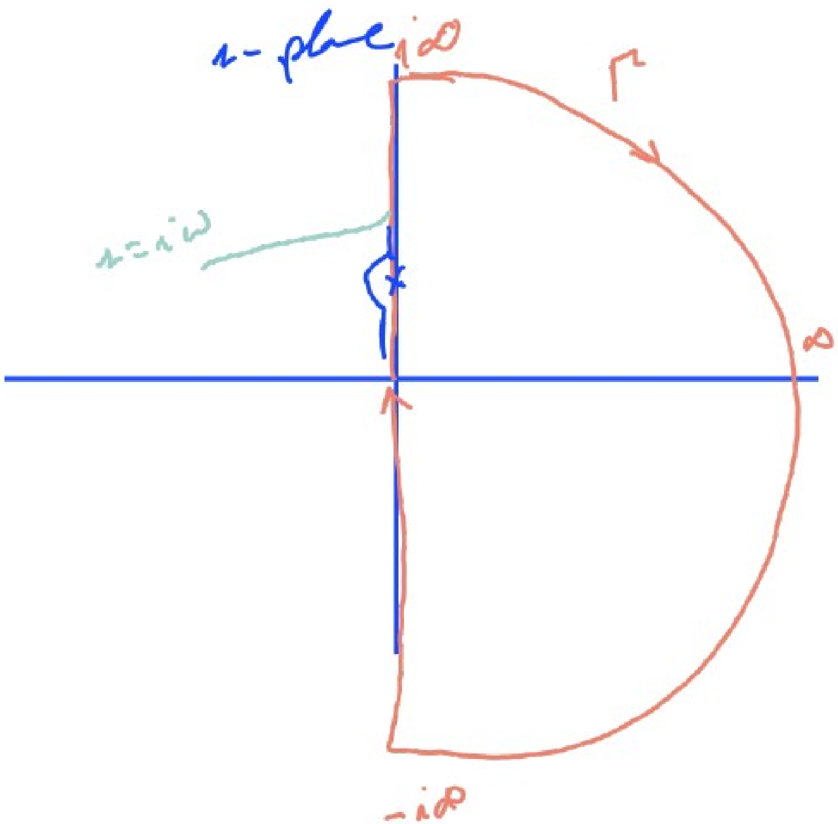


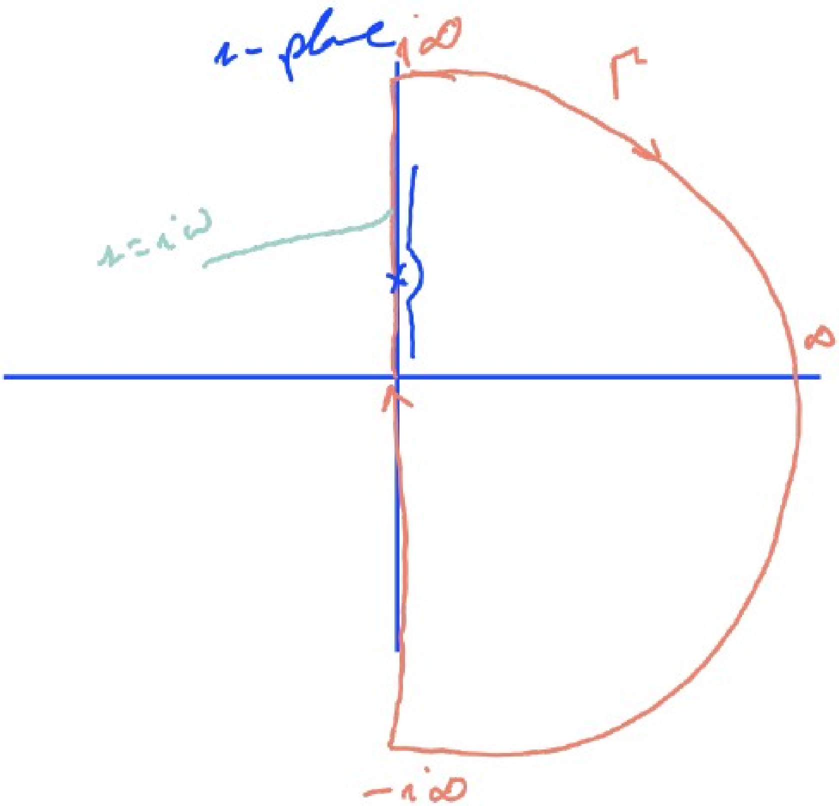
Intentionally made unstable
to be maneuverable

2 encirclements
$2 > 1 \Rightarrow$ unstable

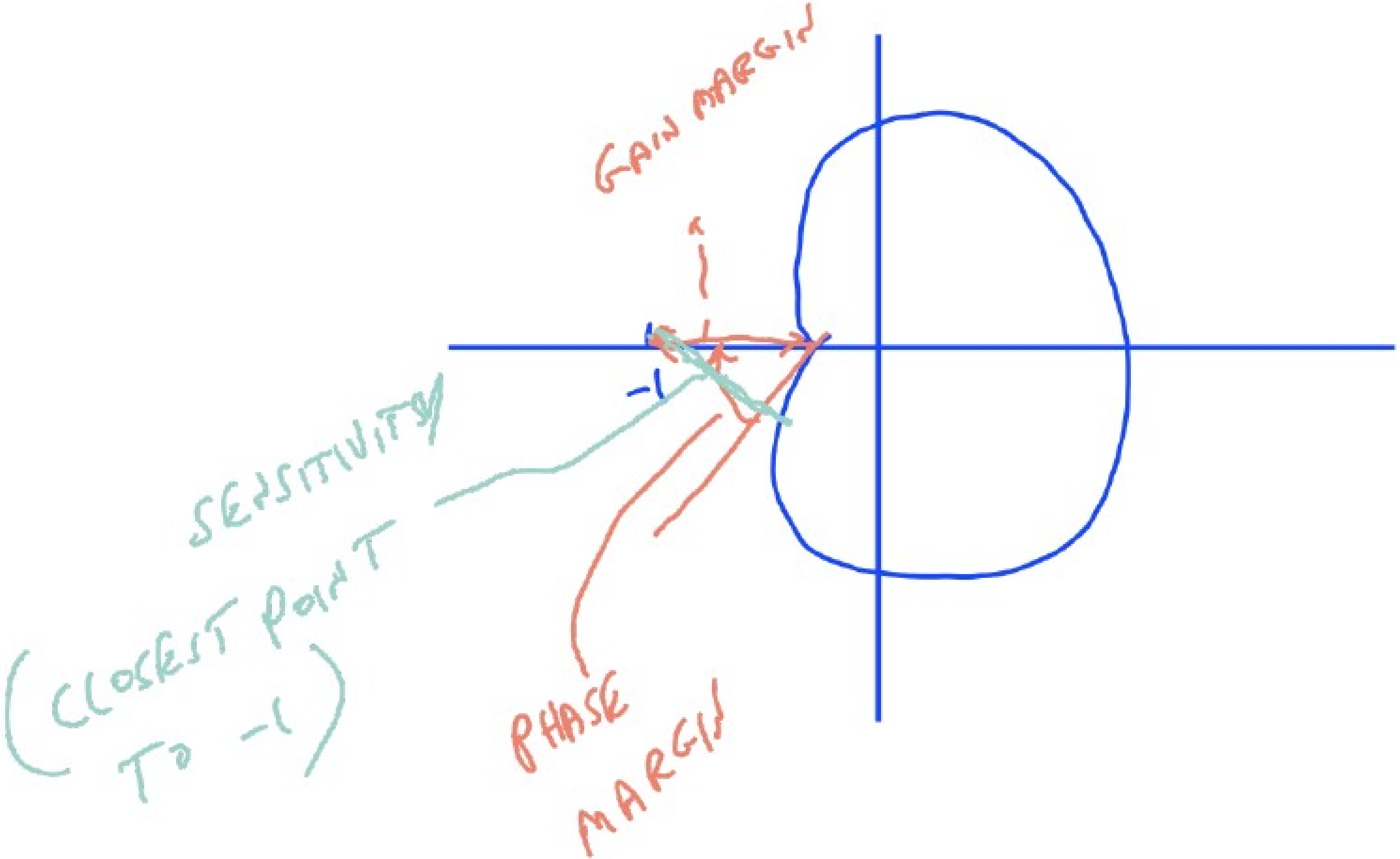
Nyquist

poles or zeros on Imaginary axis





Nyquist Margins



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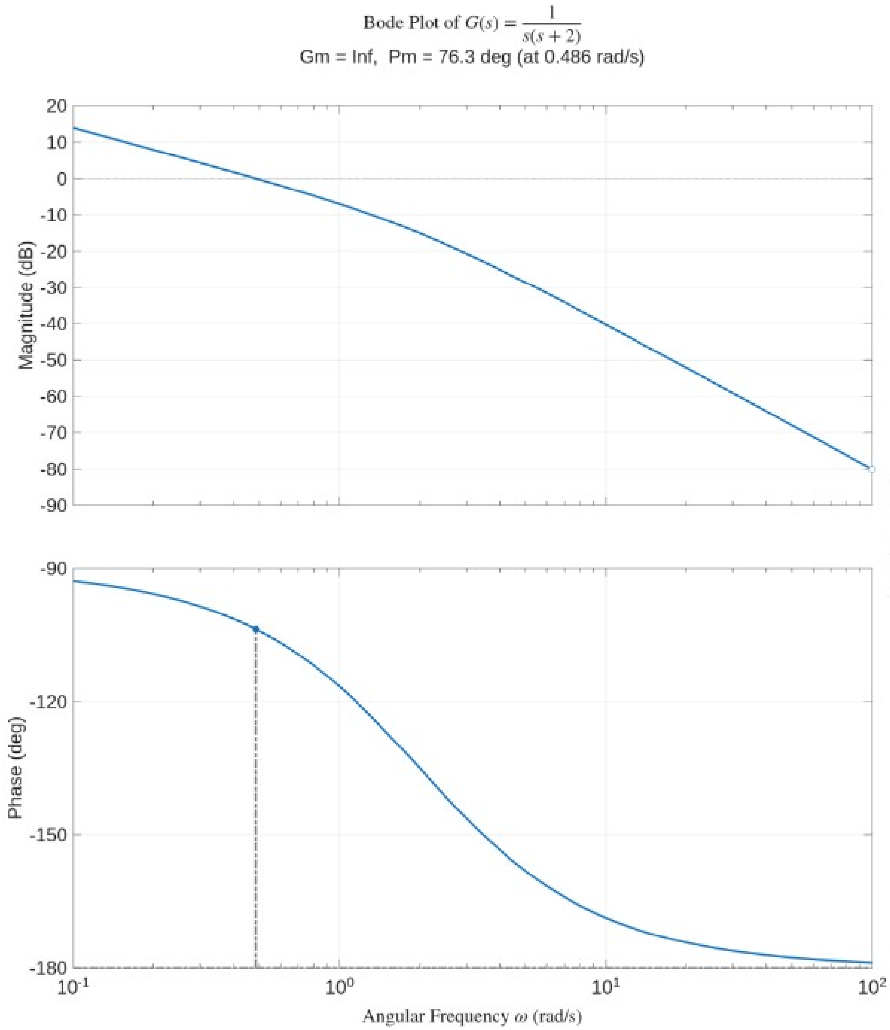
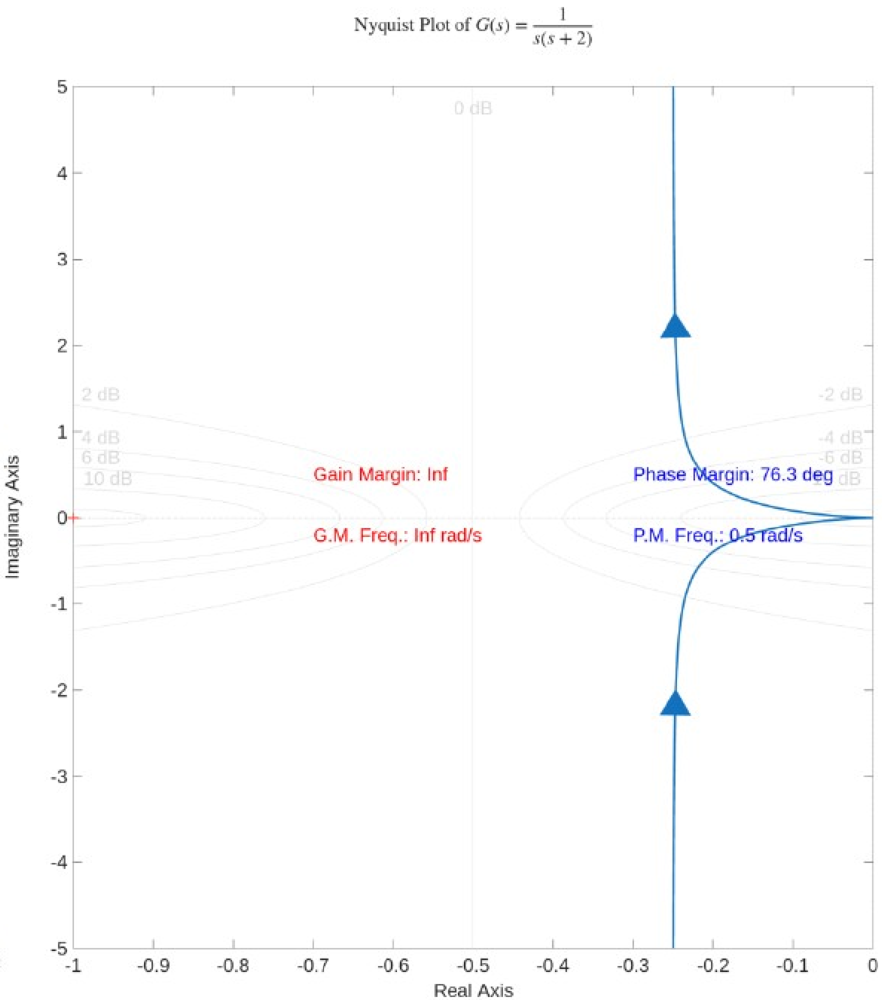
$$G(s) = \frac{1}{s(s+2)}$$

Nyquist Plot





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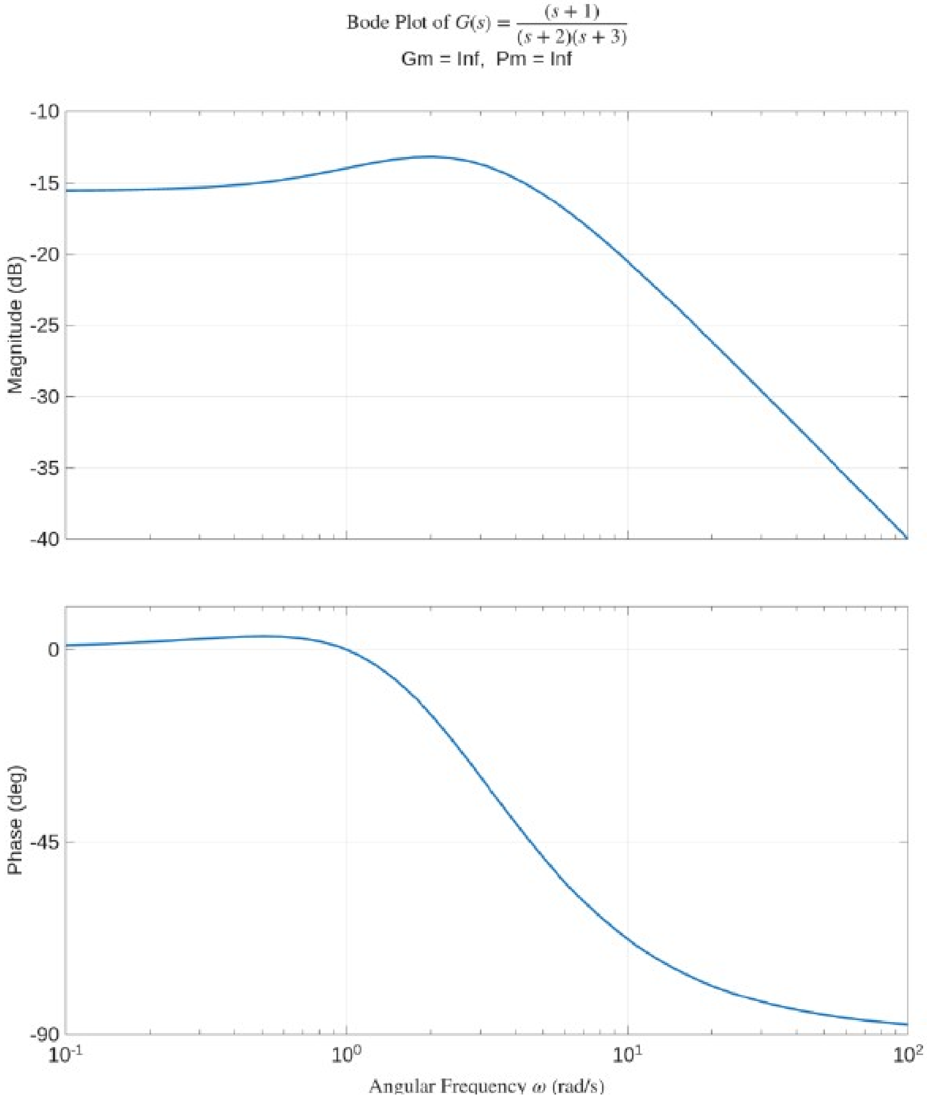
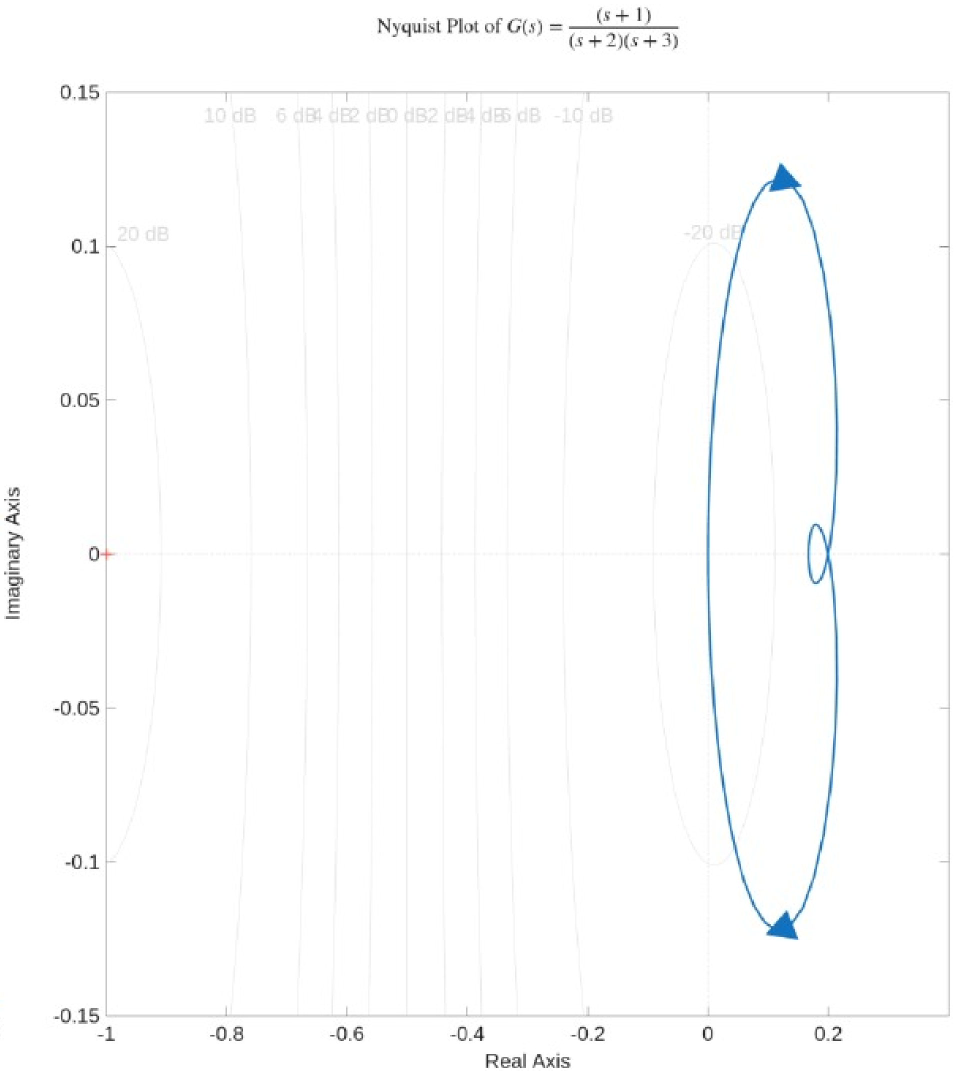
$$G(s) = \frac{(s+1)}{(s+2)(s+3)}$$

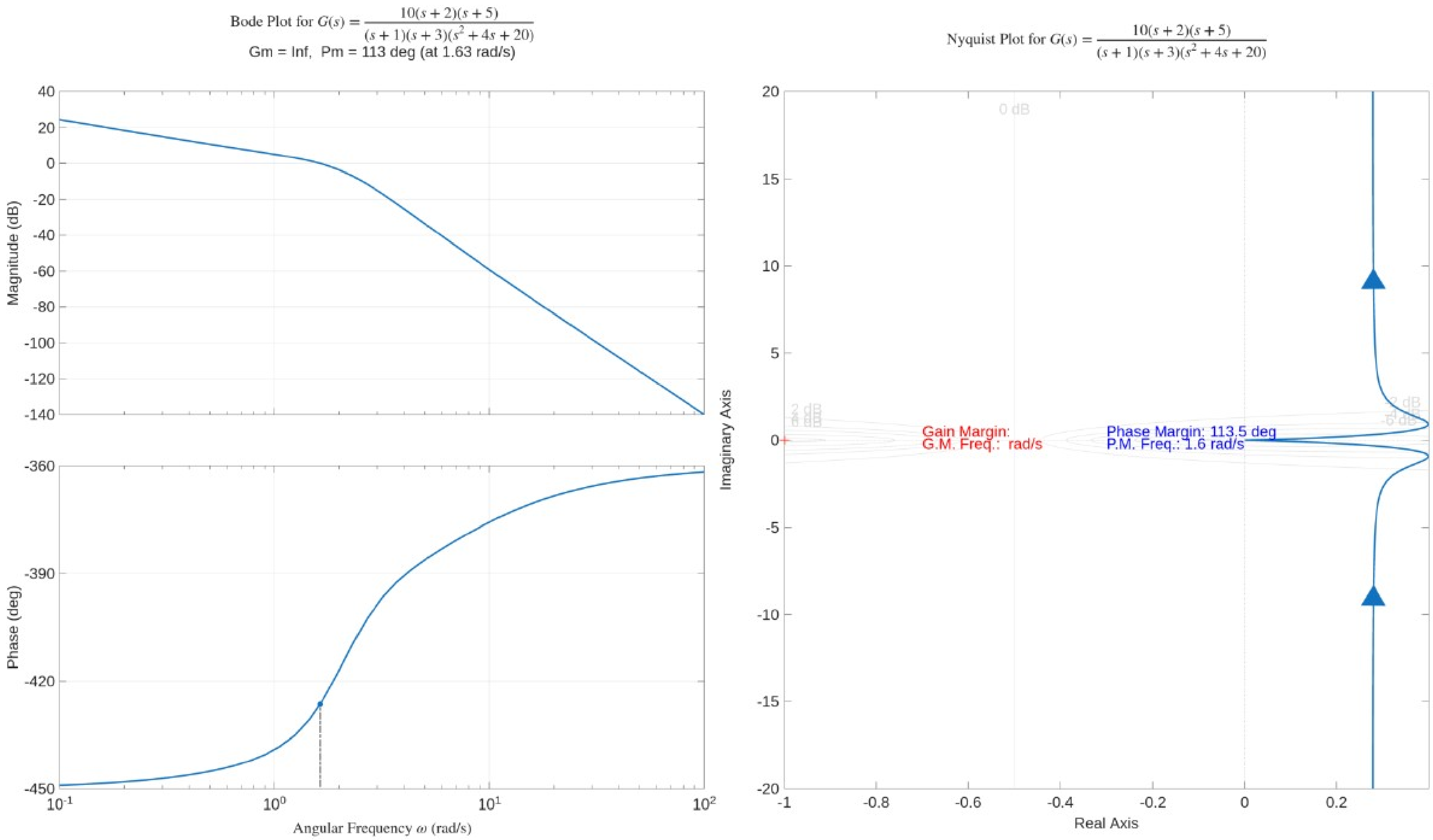
Nyquist Plot





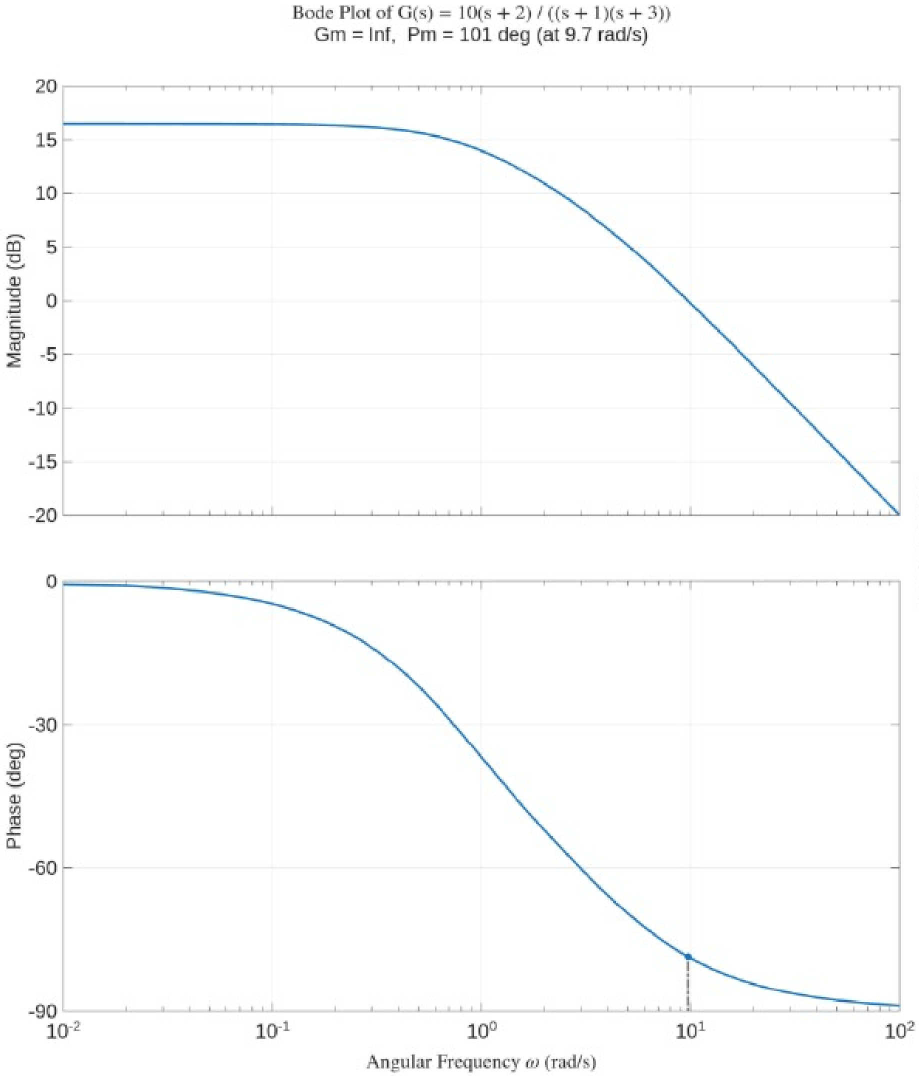
$$G(s) = \frac{10(s+2)(s+5)}{(s+3)(s+1)(s^2+4s+20)}$$

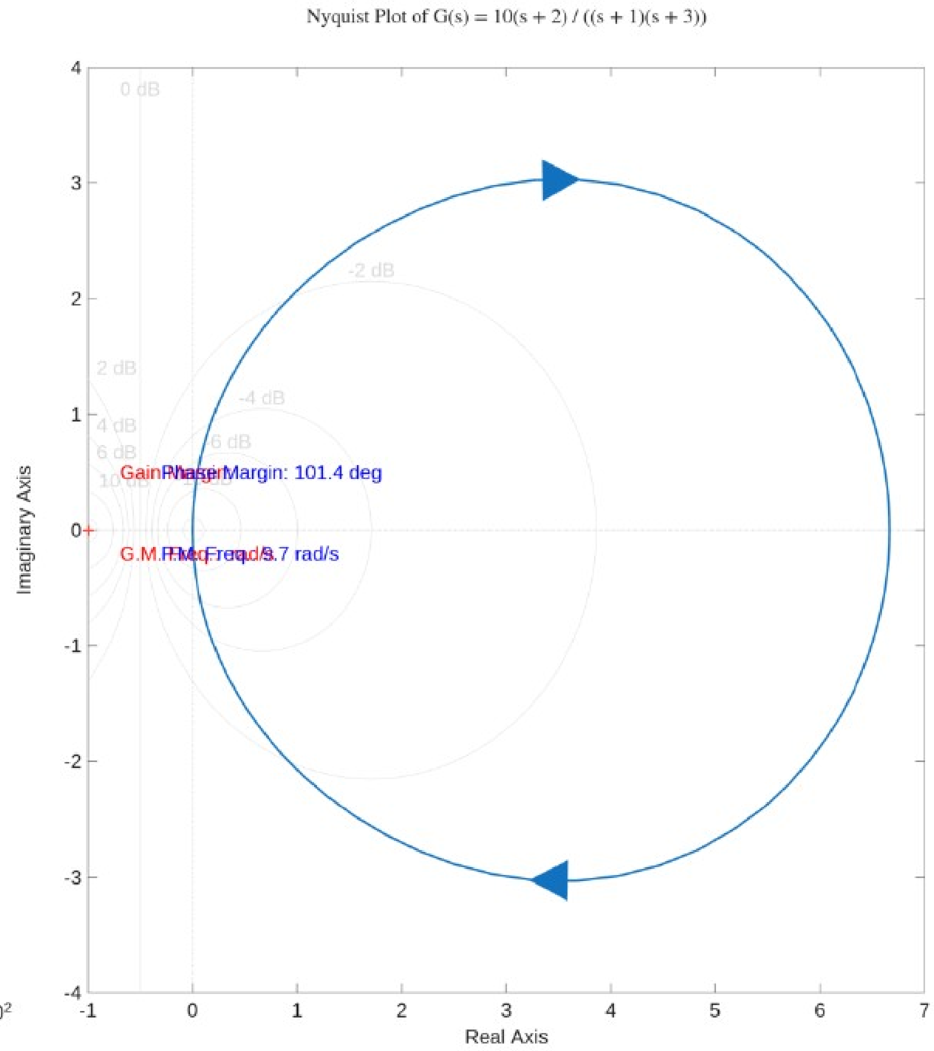
Nyquist Plot



$$G(s) = \frac{10(s+2)}{(s+3)(s+1)}$$

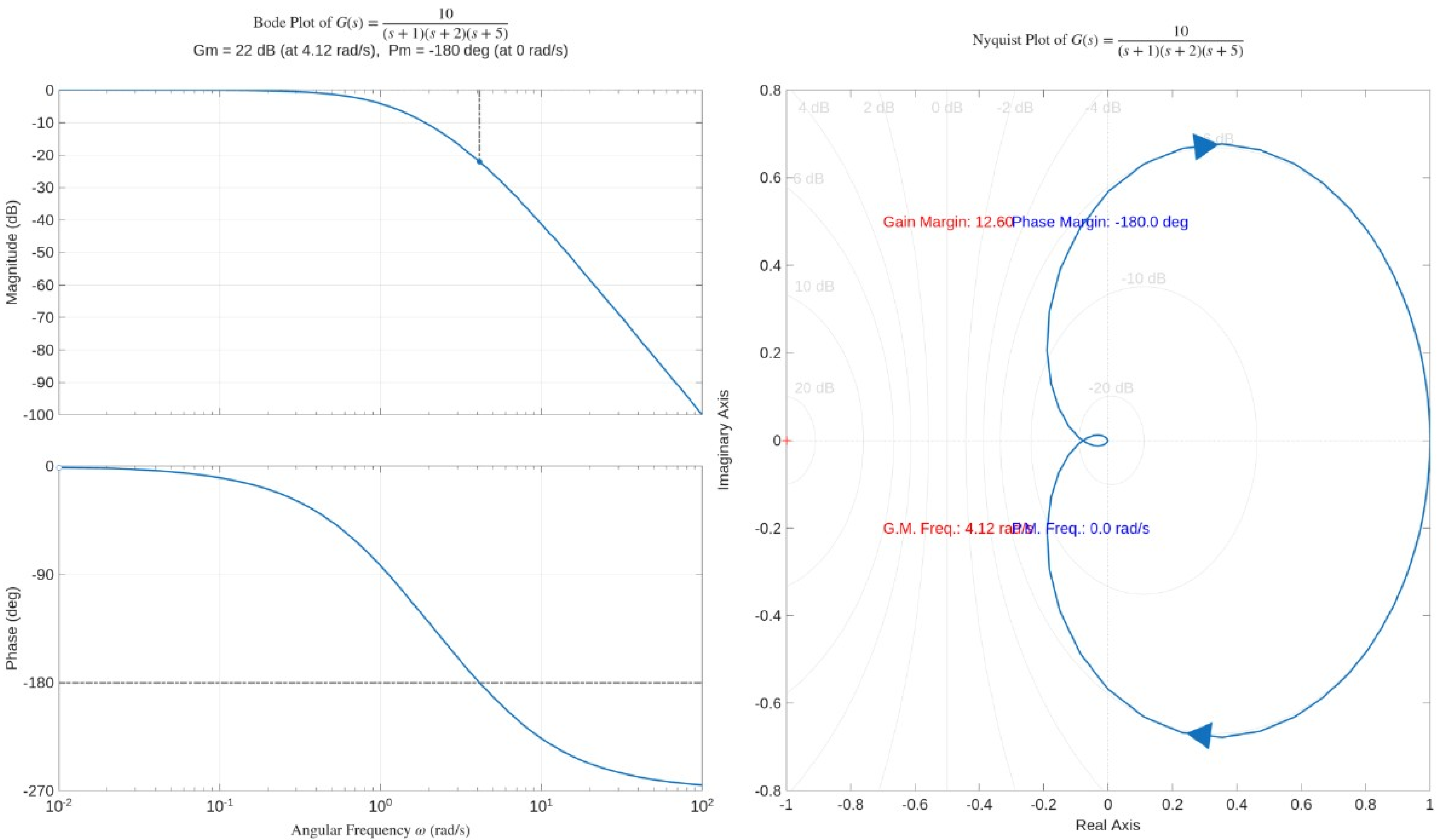
Nyquist Plot



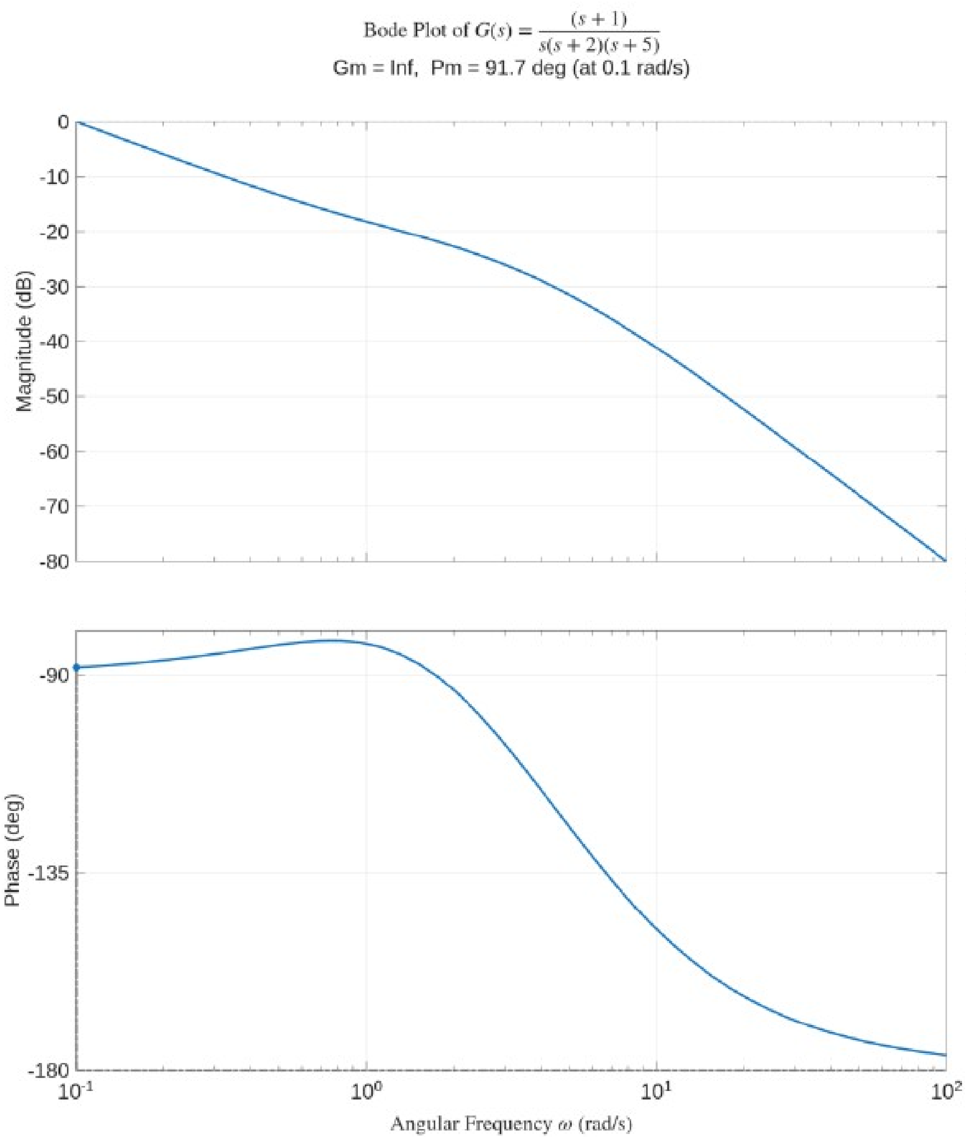


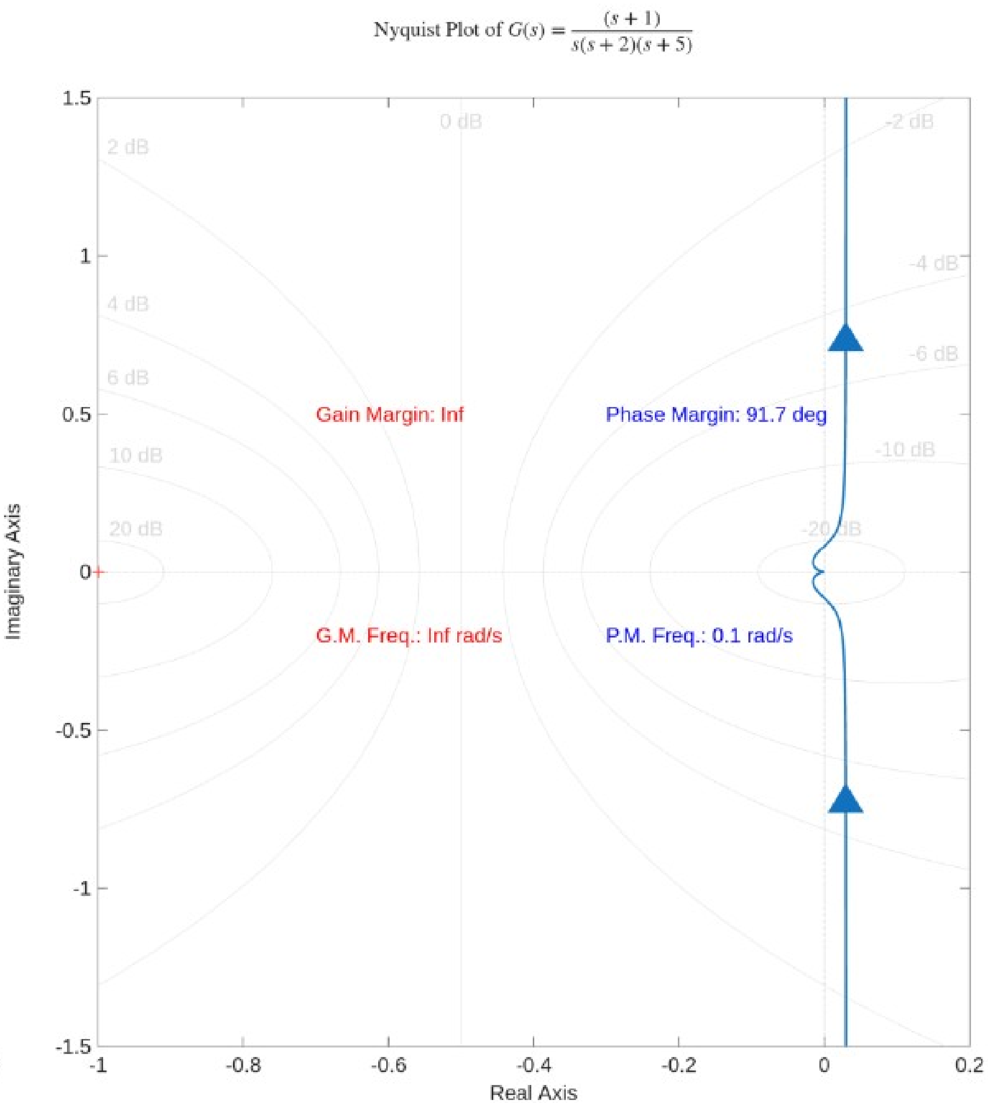
$$G(s) = \frac{10}{(s+1)(s+2)(s+3)}$$

Nyquist Plot



Nyquist Plot





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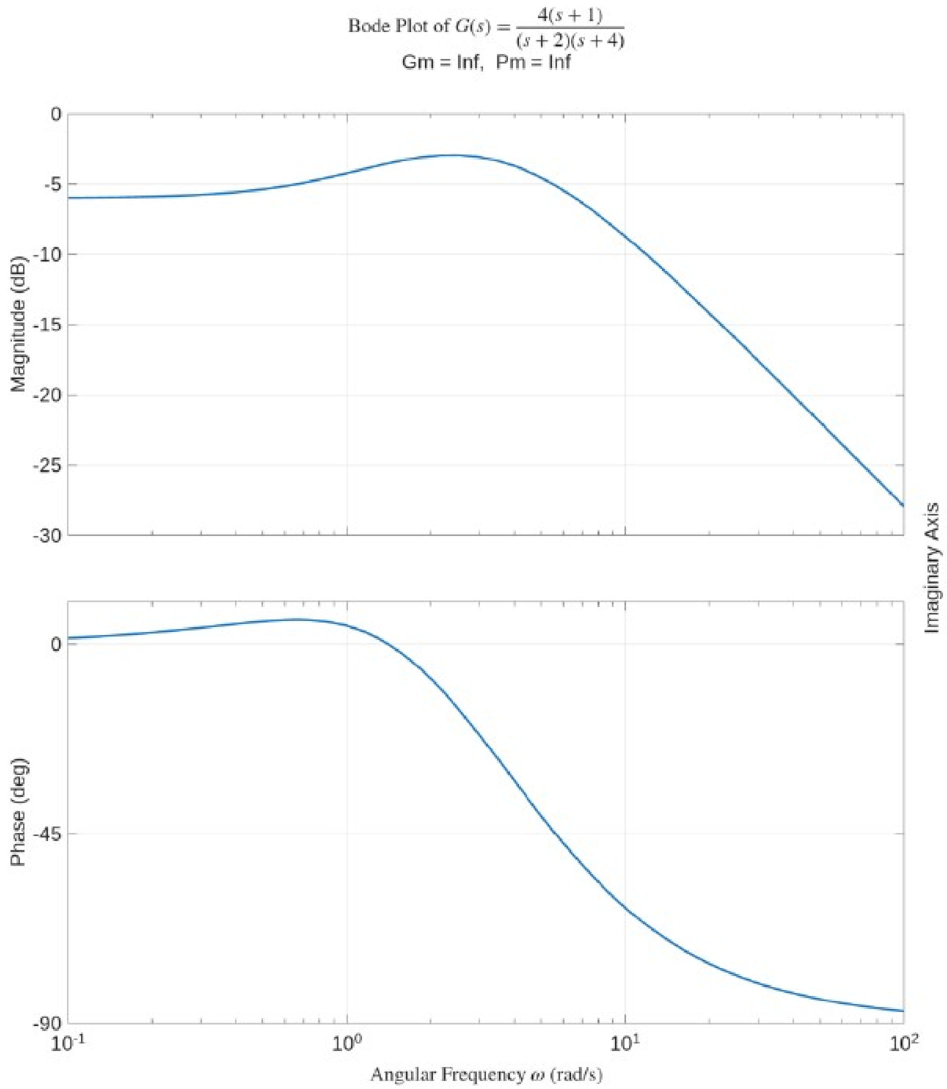
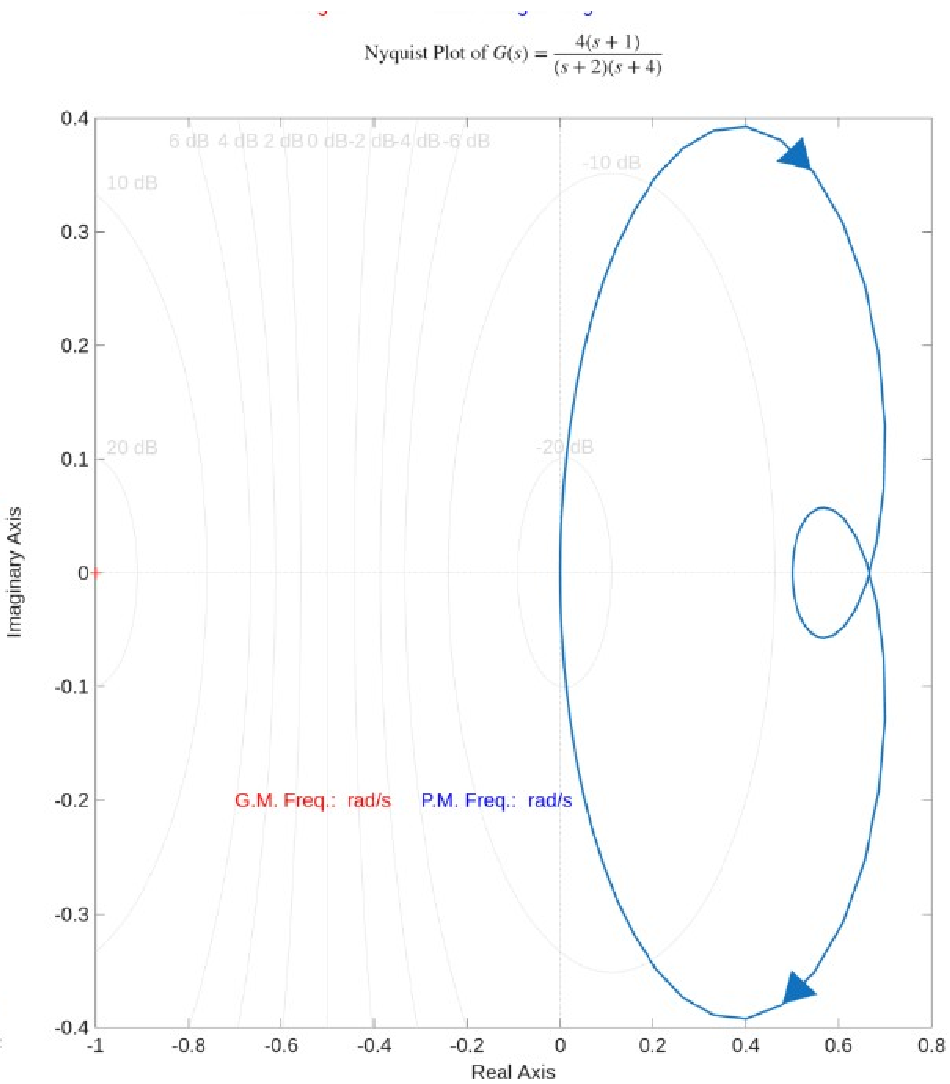
Nyquist Plot





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$$G(s) = \frac{K}{s(s+1)(s+10)}$$

Nyquist Plot