Introduction to Continuous Control Systems

EEME E3601

Week 12

Homayoon Beigi

Homayoon.Beigi@columbia.edu

https://www.RecoTechnologies.com/beigi

Mechanical Engineering dept.

&

Electrical Engineering dept.

Columbia University, NYC, NY, U.S.A.

Root Locus Plots (Review)

$\left(K_1 + K_2 s + \frac{K_3}{s}\right)$

$K\left[K_1' + K_2' s + \frac{K_3'}{s}\right]$

Closed-loop transfer function $$\frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1+C(s)G(s)H(s)}$$

Closed-loop characteristic equation $$1+C(s)G(s)H(s) = 0$$

$(C'G'H')$ $\frac{Q}{P}$

Assume C(s)G(s)H(s) has a free parameter, K $$C(s)G(s)H(s) = \frac{KQ(s)}{P(s)}$$

$C'G'H'$

$1+CGH$

$$\begin{aligned} 1+C(s)G(s)H(s) &= 1+\frac{KQ(s)}{P(s)} \\ &= \frac{P(s)+KQ(s)}{P(s)} \\ &= 0 \end{aligned}$$

$$F(s) = P(s)+KQ(s) = 0$$

U

Root Locus Plots (Review)

$$C(s)G(s)H(s) = KC_1(s)G_1(s)H_1(s)$$
$$= \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$\frac{\rho_1 e^{i\theta_1}}{\rho_2 e^{i\theta_2}}$$

$$\frac{\rho_1}{\rho_2} e^{i(\theta_1-\theta_2)}$$

Use to draw Root Loci

$$\angle C_1(s)G_1(s)H_1(s) = \sum_{j=1}^{m} \angle(s+z_j) - \sum_{k=1}^{n} \angle(s+p_k)$$
$$= (2l+1)\pi \quad \text{where } K \geq 0$$

$$\angle C_1(s)G_1(s)H_1(s) = \sum_{j=1}^{m} \angle(s+z_j) - \sum_{k=1}^{n} \angle(s+p_k)$$
$$= 2l\pi \quad \text{where } K \leq 0$$

where $l \in \{0, \pm 1, \pm 2, \cdots\}$

homayoon.beigi@columbia.edu

Root Locus Plots

Example

Take an arbitrary point, s_1 , then if s_1 is a point on the Root Locus, the following equation must be satisfied for the case where $K \geq 0$

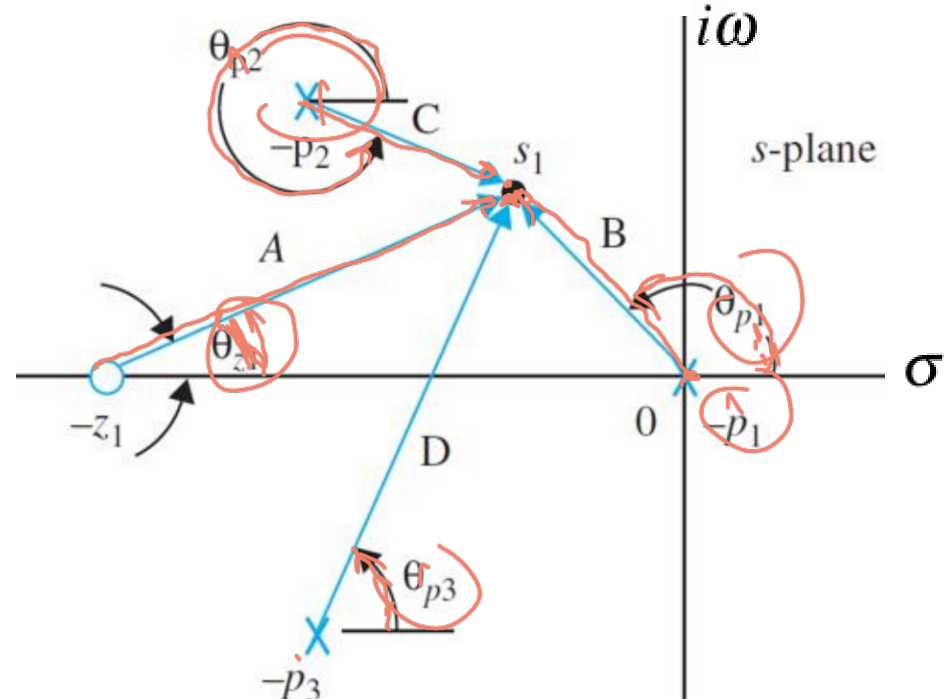
$$\begin{aligned}
\angle(s_1 + z_1) - \angle s_1 - \angle(s_1 + p_2) - \angle(s_1 + p_3) &= \theta_{z_1} - \theta_{p_1} - \theta_{p_2} - \theta_{p_3} \\
&= (2l+1)\pi
\end{aligned}$$

$s + z_1$

where $l \in \{0, \pm 1, \pm 2, \cdots\}$

$s_1 + z_1$

$s_1 - (-z_1)$



Root Locus Plots (Review)

$\rho e^{i\theta}$

$$F(s) = P(s) + KQ(s) = 0$$

$$C_1(s)G_1(s)H_1(s) = -\frac{1}{K} \iff 1 + KC_1G_1H_1 = 0$$

$$C(s)G(s)H(s) = KC_1(s)G_1(s)H_1(s) \qquad \text{where } -\infty < K \leq \infty$$

$\theta_z - \theta_p$

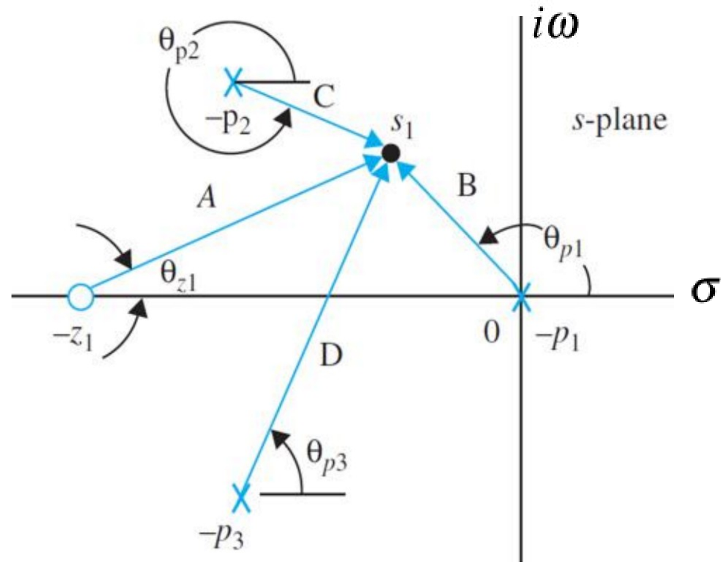
$$\begin{aligned} |C_1(s)G_1(s)H_1(s)| &= \frac{\prod_{j=1}^{m} |s+z_j|}{\prod_{k=1}^{n} |s+p_k|} \\ &= \frac{1}{|K|} \end{aligned}$$

$$|K| = \frac{\prod_{k=1}^{n} |s+p_k|}{\prod_{j=1}^{m} |s+z_j|}$$

homayoon.beigi@columbia.edu

Root Locus Plots

Example



Once s_1 satisfies the angle relation, the gain, K, may be found by the following,

$$|K| = \frac{|s_1|\,|s_1+p_2|\,|s_1+p_3|}{|s_1+z_1|}$$

$$\frac{|s|\,|(s+p_2)|\,|(s+p_3)|}{|(s+z_1)|}$$

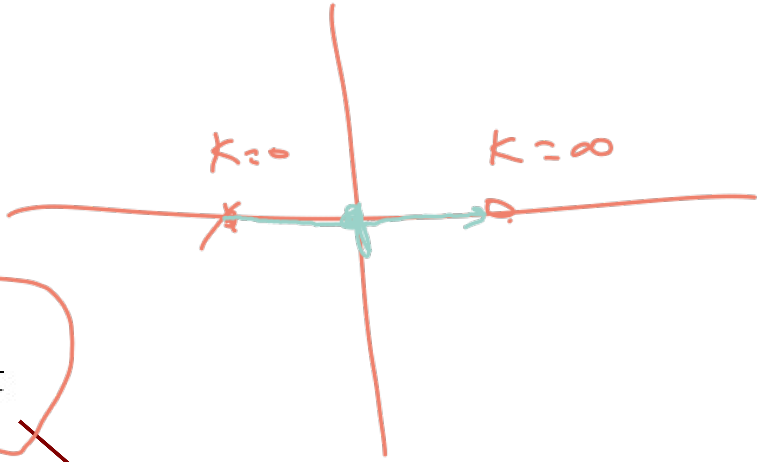
$$= \frac{B\times C\times D}{A}$$

The sign of K is determined by which angle relation the point satisfies (odd multiple of π : positive, even multiple of π : negative)

homayoon.beigi@columbia.edu

Root Locus Plots

Rule 1



∞ (poles)

$$C_1(s)G_1(s)H_1(s) = -\frac{1}{K}$$

0

∞

$$C_1(s)G_1(s)H_1(s) = -\frac{1}{K}$$

0 (zeros)

homayoon.beigi@columbia.edu

Root Locus Plots

Example

$$\frac{CG}{1+CGH} \to 0$$

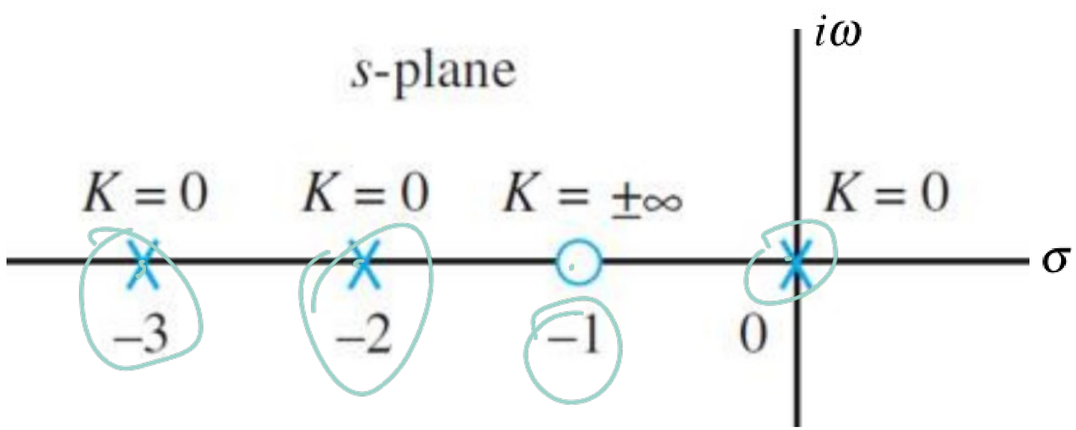
$$
\begin{aligned}
1+C(s)G(s)H(s) &= s(s+2)(s+3)+K(s+1) \\
&= 0 \\
&= 1+\frac{K(s+1)}{s(s+2)(s+3)}
\end{aligned}
$$

$$= 1 + \frac{KQ}{P}$$

$$\frac{Q}{P} = C_1 G_1 H_1$$

$$C(s)G(s)H(s) = \frac{K(s+1)}{s(s+2)(s+3)}$$

$$\boxed{C_1(s)G_1(s)H_1(s) = -\frac{1}{K}}$$



Root Locus Plots

Rule 2
Number of Branches

Number of branches of Root Loci is equal to the degree of the polynomial

Example

$$
\begin{aligned}
1 + C(s)G(s)H(s) &= s(s+2)(s+3) + K(s+1) \\
&= 0 \\
&= 1 + \frac{K(s+1)}{s(s+2)(s+3)}
\end{aligned}
$$

Number of branches = degree of the above polynomial = 3

Namely, it is *max(m,n)*

homayoon.beigi@columbia.edu

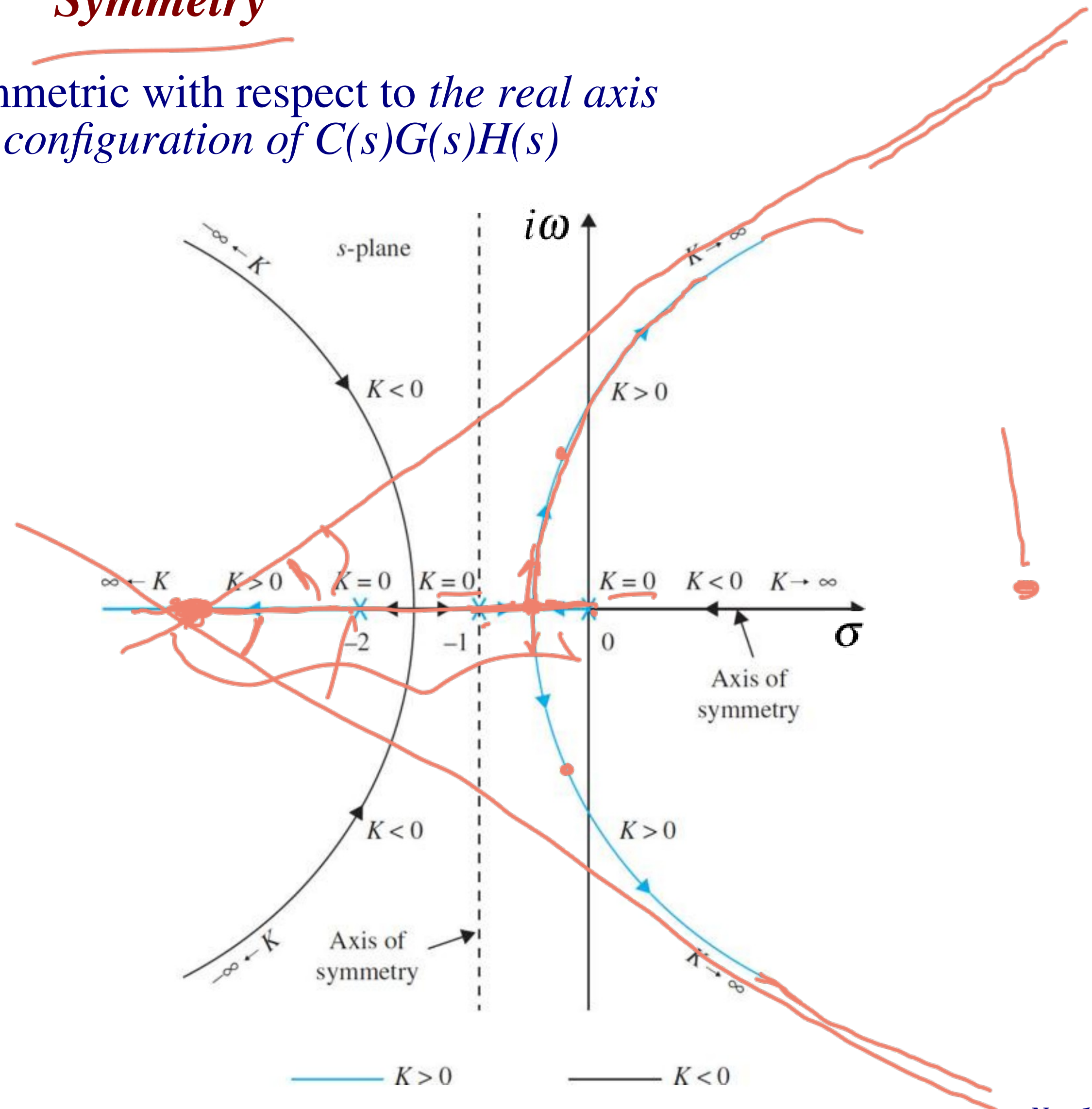
Root Locus Plots

Rule 3
Symmetry

Root Loci are symmetric with respect to *the real axis and the pole-zero configuration of C(s)G(s)H(s)*

Example 1

$$C(s)G(s)H(s) = \frac{K}{s(s+2)(s+3)}$$



Root Locus Plots

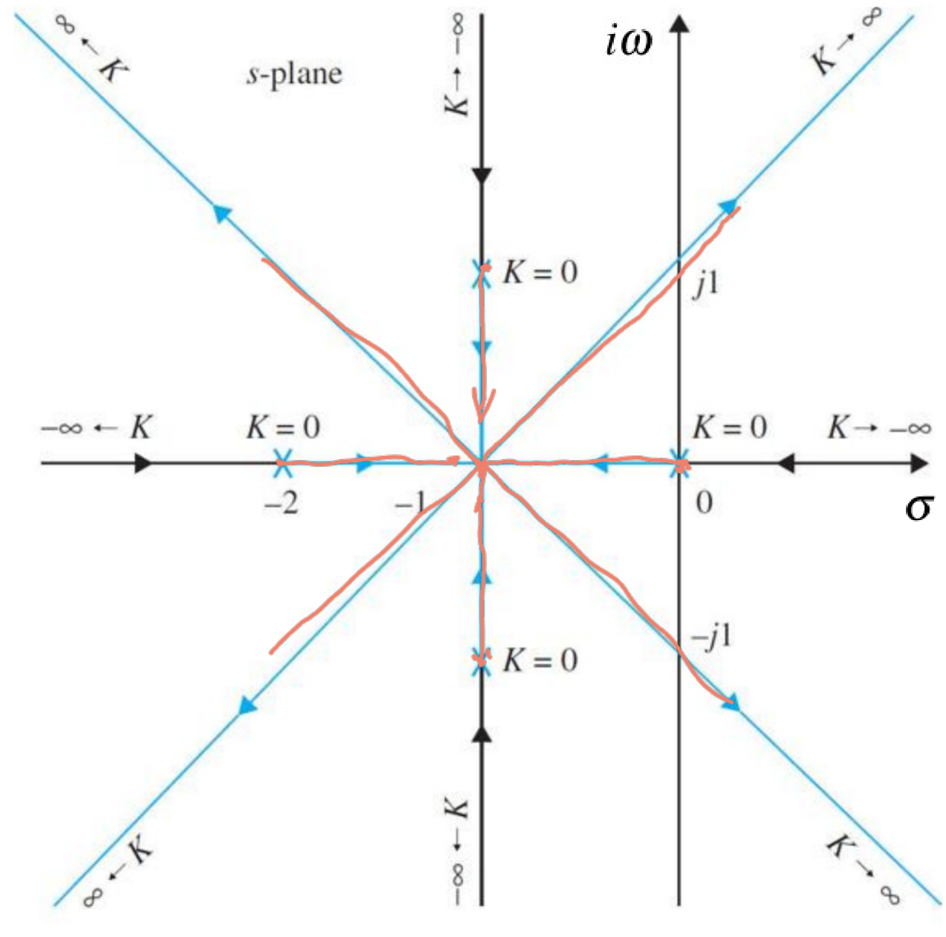
Rule 3
Symmetry

Root Loci are symmetric with respect to the real axis and the *pole-zero configuration of C(s)G(s)H(s)*

Example 2

$$\begin{aligned} 1+C(s)G(s)H(s) &= s(s+2)(s^2+2s+2)+K \\ &= 0 \\ &= s(s+2)(s+1+i)(s+1-i)+K \\ &= 1+\frac{K}{s(s+2)(s+1+i)(s+1-i)} \end{aligned}$$

$$C(s)G(s)H(s) = \frac{K}{s(s+2)(s+1+i)(s+1-i)}$$



Root Locus Plots

Rule 4

Angles of Asymptotes with the real axis

Asymptotes of Root Loci (behavior of root loci at |s| = ∞)

Asymptotes angles for $K \geq 0$ are given by

$$\theta_l = \frac{(2l+1)}{|n-m|} \times \pi \quad \forall\, n \neq m \qquad \text{where } k \geq 0$$

$$\theta_l = \frac{2l}{|n-m|} \times \pi \quad \forall\, n \neq m \qquad \text{where } k \leq 0$$

$$l \in \{0, 1, \cdots, |n-m|-1\}$$

There will be 2|n-m| asymptotes for $n \neq m$

Root Locus Plots

Rule 5

Intersection of the Asymptotes with the real axis

Asymptotes of Root Loci (behavior of root loci at $|s| = \infty$)

$$\begin{aligned}\sigma_1 &= \frac{\sum \text{Finite Poles of CGH} - \sum \text{Finite Zeros of CGH}}{n-m} \\ &= \frac{\sum \mathscr{R}e\{\text{Poles of CGH}\} - \sum \mathscr{R}e\{\text{Zeros of CGH}\}}{n-m}\end{aligned}$$

Center of Gravity of Root Loci (Always Real)
Complex conjugate imaginary parts sum to 0

Root Locus Plots

Rule 5

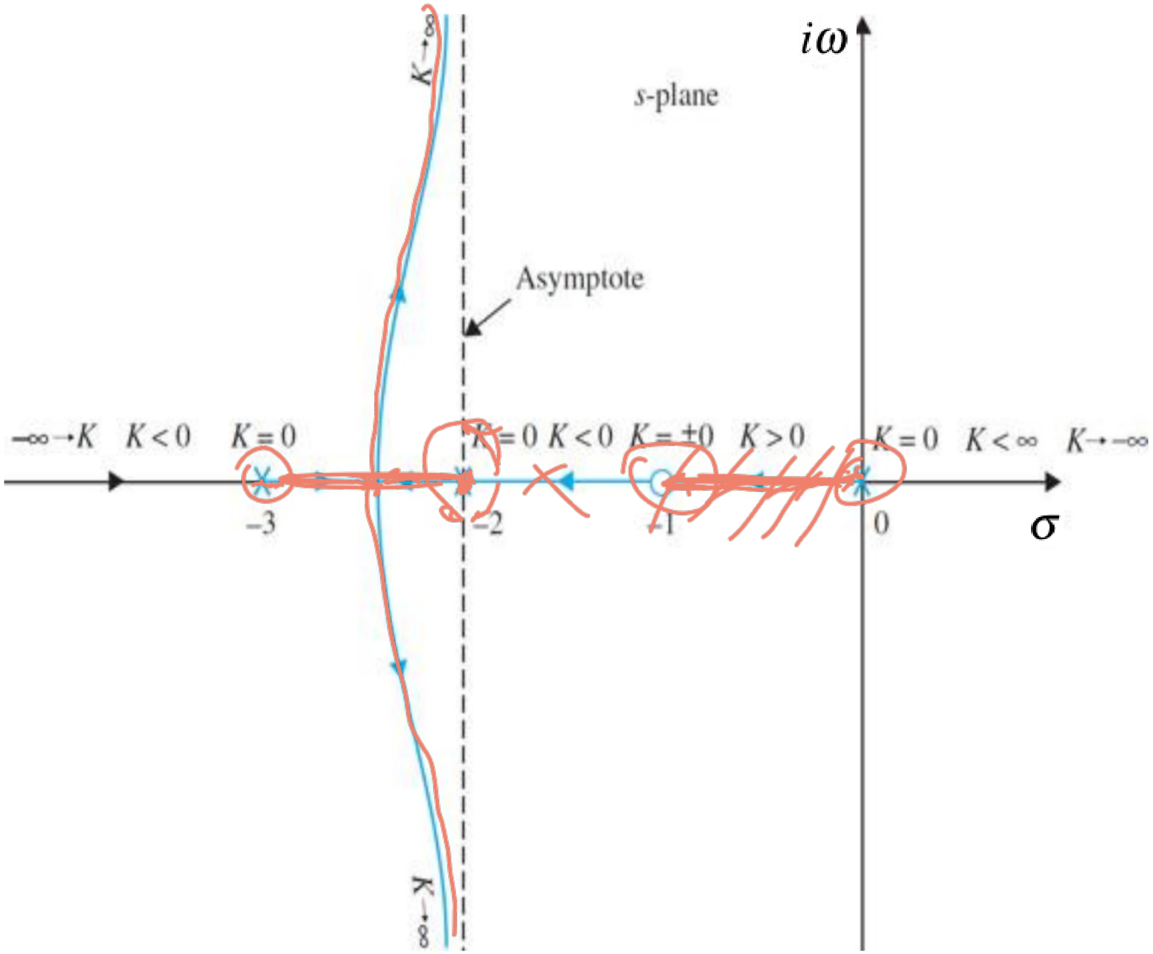
Intersection of the Asymptotes with the real axis

Asymptotes of Root Loci (behavior of root loci at $|s| = \infty$)

Example

$$\begin{aligned} 1+C(s)G(s)H(s) &= s(s+2)(s+3)+K(s+1) \\ &= 0 \\ &= 1+\frac{K(s+1)}{s(s+2)(s+3)} \end{aligned}$$

$$C(s)G(s)H(s) = \frac{K(s+1)}{s(s+2)(s+3)}$$



homayoon.beigi@columbia.edu

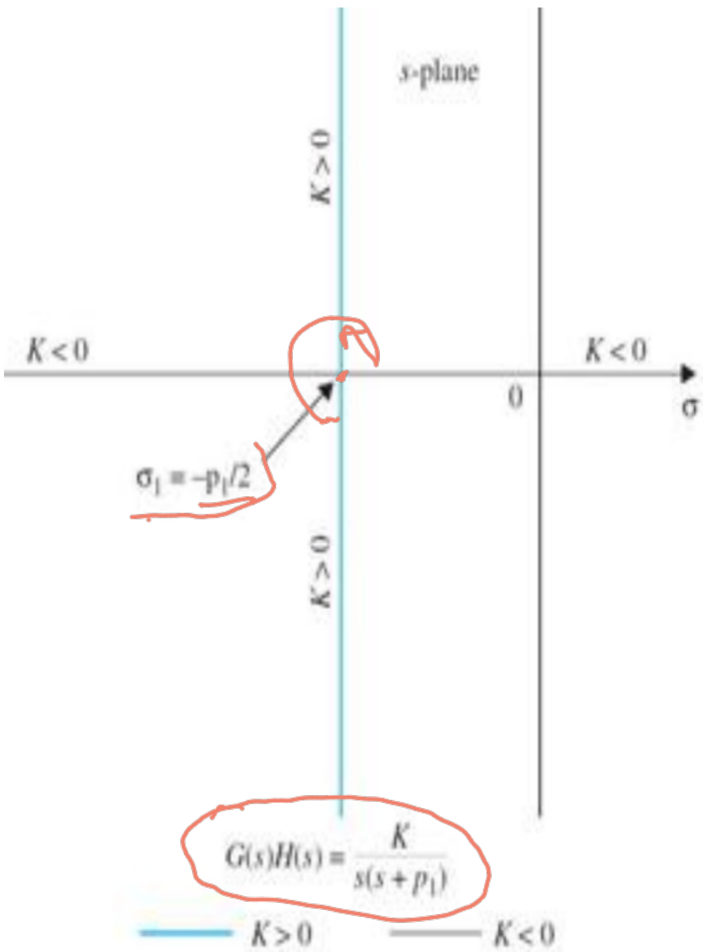
$$\sigma_1 = \frac{\sum \text{Finite Poles of CGH} - \sum \text{Finite Zeros of CGH}}{n-m} = \frac{\sum \mathscr{Re}\{\text{Poles of CGH}\} - \sum \mathscr{Re}\{\text{Zeros of CGH}\}}{n-m}$$

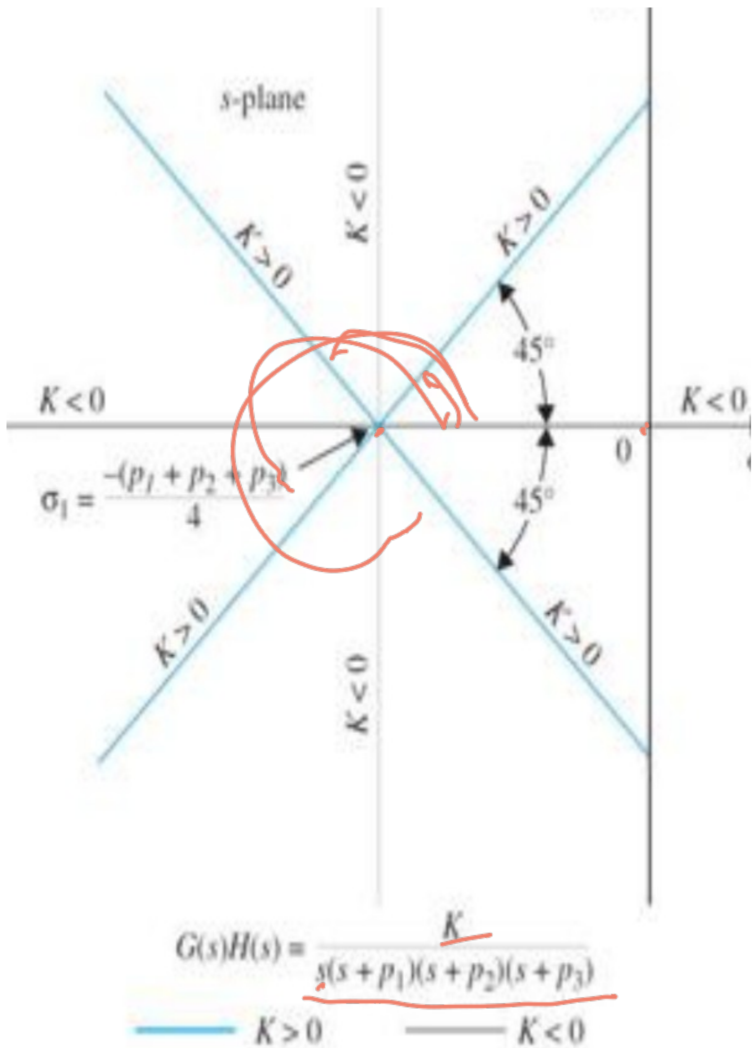
Root Locus Plots

Rules 4,5

$$\theta_l = \frac{(2l+1)}{|n-m|} \times \pi \quad \forall\, n \neq m$$

Intersection of the Asymptotes with the real axis





homayoon.beigi@columbia.edu

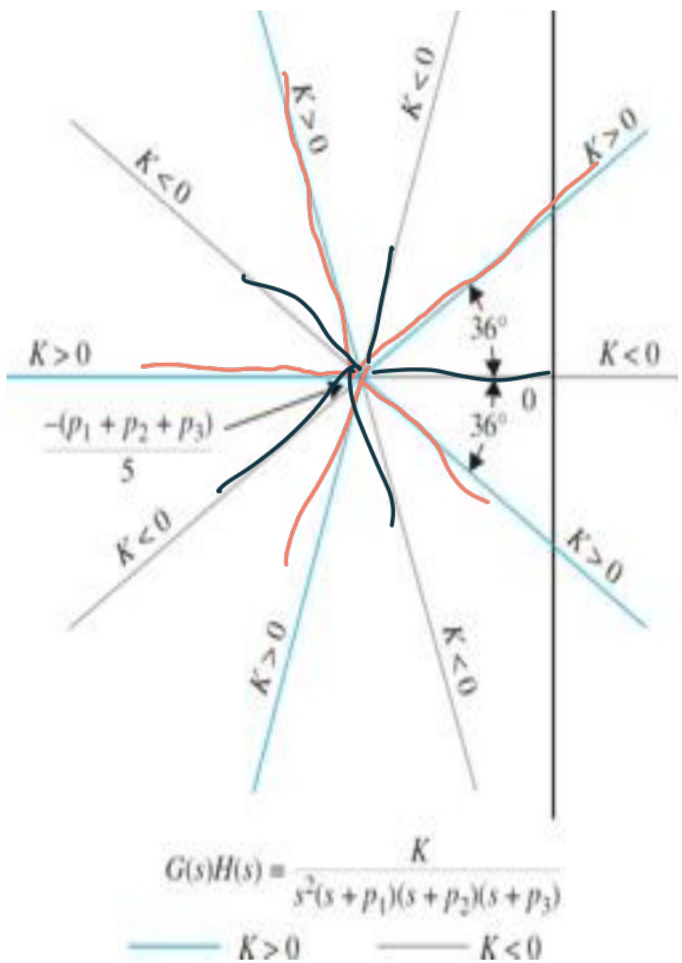
$$\sigma_1 = \frac{\sum \text{Finite Poles of CGH} - \sum \text{Finite Zeros of CGH}}{n-m}$$
$$= \frac{\sum \mathscr{R}e\{\text{Poles of CGH}\} - \sum \mathscr{R}e\{\text{Zeros of CGH}\}}{n-m}$$

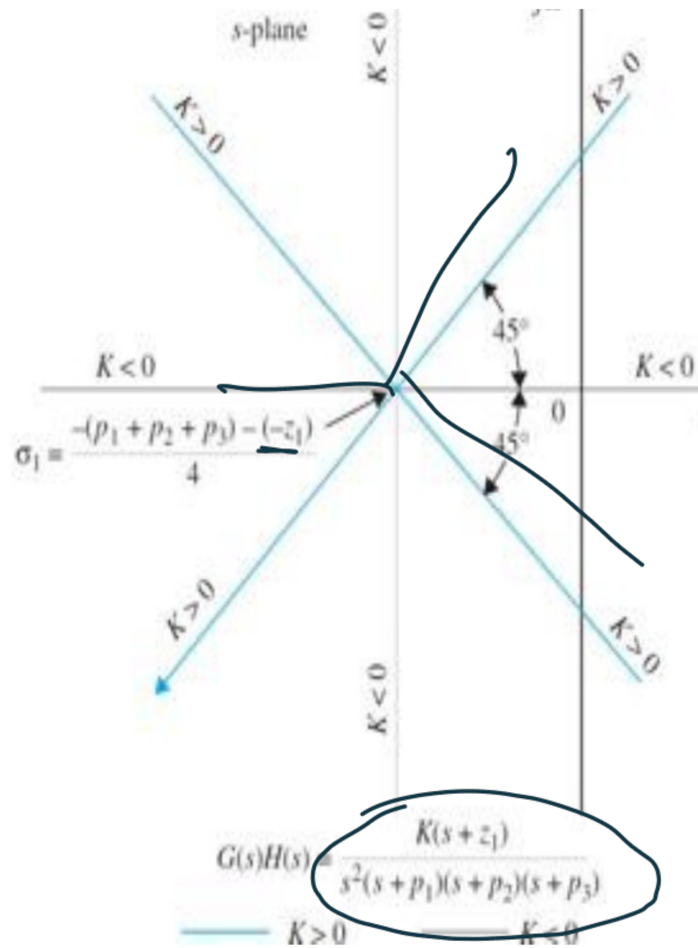
Root Locus Plots

Rules 4,5

$$\theta_l = \frac{(2l+1)}{|n-m|} \times \pi \quad \forall\, n \neq m$$

Intersection of the Asymptotes with the real axis





Root Locus Plots

Rules 4,5

Intersection of the Asymptotes with the real axis

Example Asymptotes of the root loci of $s(s+4)(s^2+2s+2)+K(s+1)=0$

$$s(s+4)(s^2+2s+2)+K(s+1)=0$$

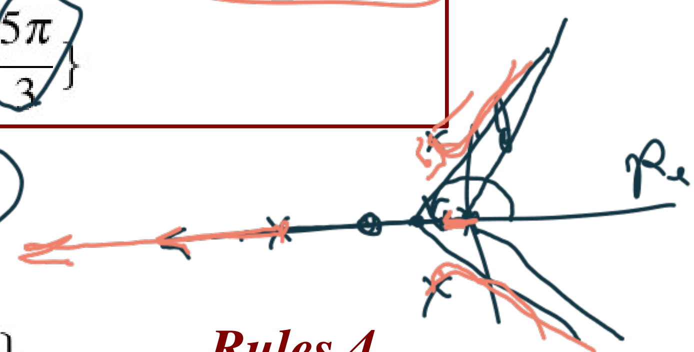
$$1+\frac{K(s+1)}{s(s+4)(s^2+2s+2)}=0$$

$$\sigma_1 = 0-4-4-1-(-1) = -\frac{5}{3}$$

$$\theta_l = \{\frac{\pi}{3}, \pi, \frac{5\pi}{3}\}$$

$$G_1(s)H_1(s) = \frac{s+1}{s(s+4)(s^2+2s+2)}$$

$(s+1+i)(s+1-i)$



$$\theta_l = \frac{(2l+1)}{|n-m|}\times\pi \quad \forall\, n \neq m \qquad l \in \{0,1,\cdots,|n-m|-1\}$$

Rules 4

$$\sigma_1 = \frac{\sum \text{Finite Poles of CGH} - \sum \text{Finite Zeros of CGH}}{n-m}$$

$$= \frac{\sum \mathscr{Re}\{\text{Poles of CGH}\} - \sum \mathscr{Re}\{\text{Zeros of CGH}\}}{n-m}$$

Rules 5

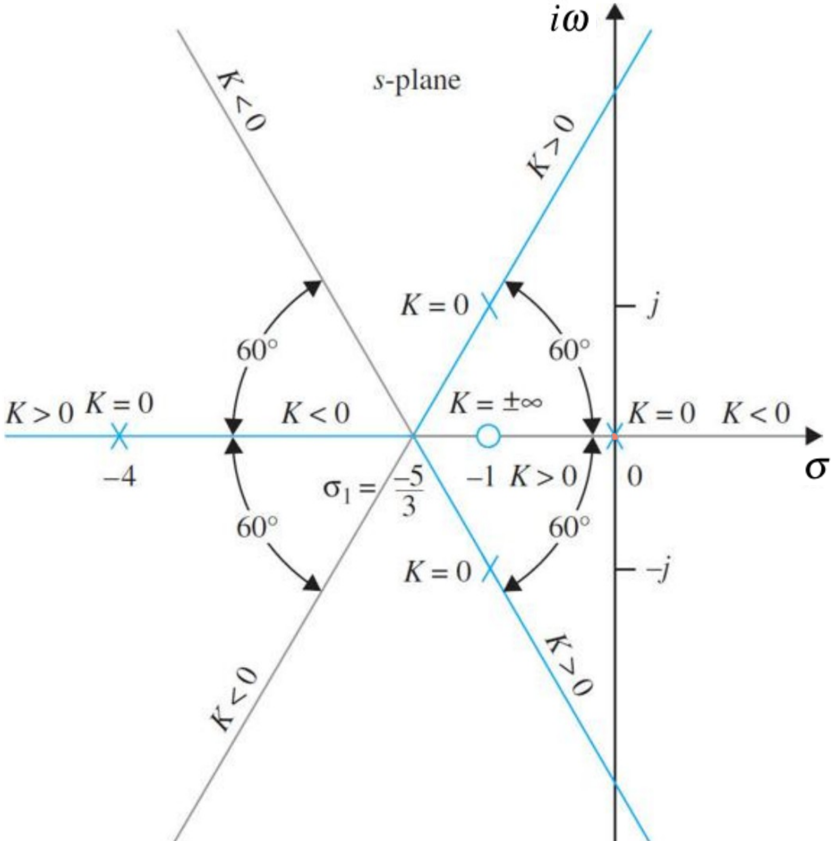
Root Locus Plots

Rules 4,5

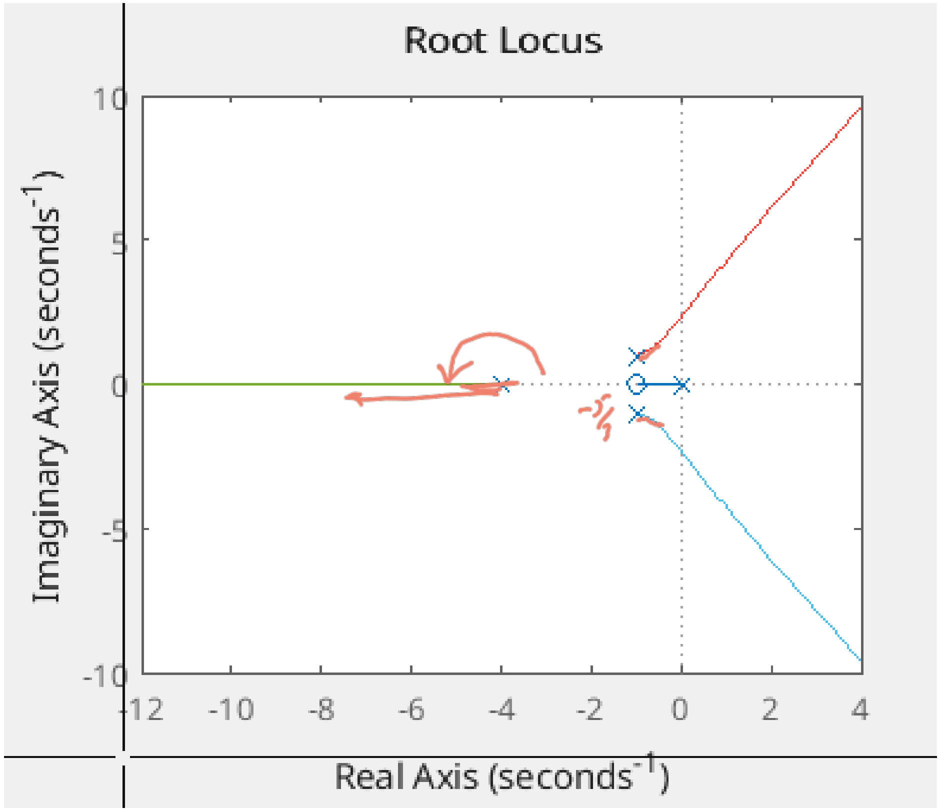
Intersection of the Asymptotes with the real axis

Example

Asymptotes of the root loci of $s(s+4)(s^2+2s+2)+K(s+1)=0$



Error in fig. 9-6 of book



Correct Root Locus

Root Locus Plots

Rule 5

Intersection of the Asymptotes with the real axis

Asymptotes of Root Loci (behavior of root loci at $|s| = \infty$)

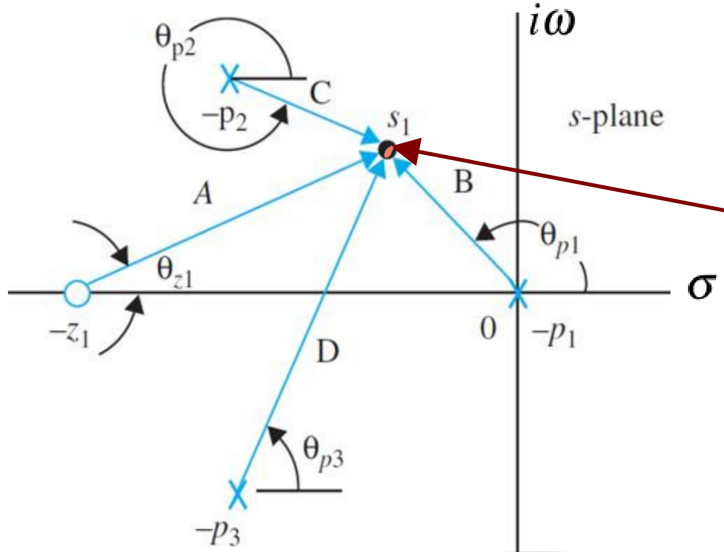
rlocus_example_rule5-1.m

```
% This example is based on Golnaraghi-Kuo Edition 10 p.532 Toolbox Fig. 9-5

PolynQ=[1 1];            % Numerator
PolynP=conv([1 0],[1 2])        % Denominator
PolynP=conv(PolynP,[1 3])      % Denominator

%\frac{Q(s)}{P(s)}
TFG = tf(PolynQ,PolynP); % Open loop transfer function

rlocus(TFG);
axis([-3 0 -8 8])

[K,poles] = rlocfind(TFG)   %rcocfind allow us to choose desired poles on the root locus
```

Root Locus Plots

Rules 6

Angles of arrival and departure of Root Loci



Use the pole of interest in the place of s_1 to compute its angle of departure (or zero for angle of arrival)

Use to draw Root Loci

$$\angle C_1(s)G_1(s)H_1(s) = \sum_{j=1}^{m} \angle(s+z_j) - \sum_{k=1}^{n} \angle(s+p_k)$$
$$= (2l+1)\pi \quad \text{where} \quad K \geq 0$$

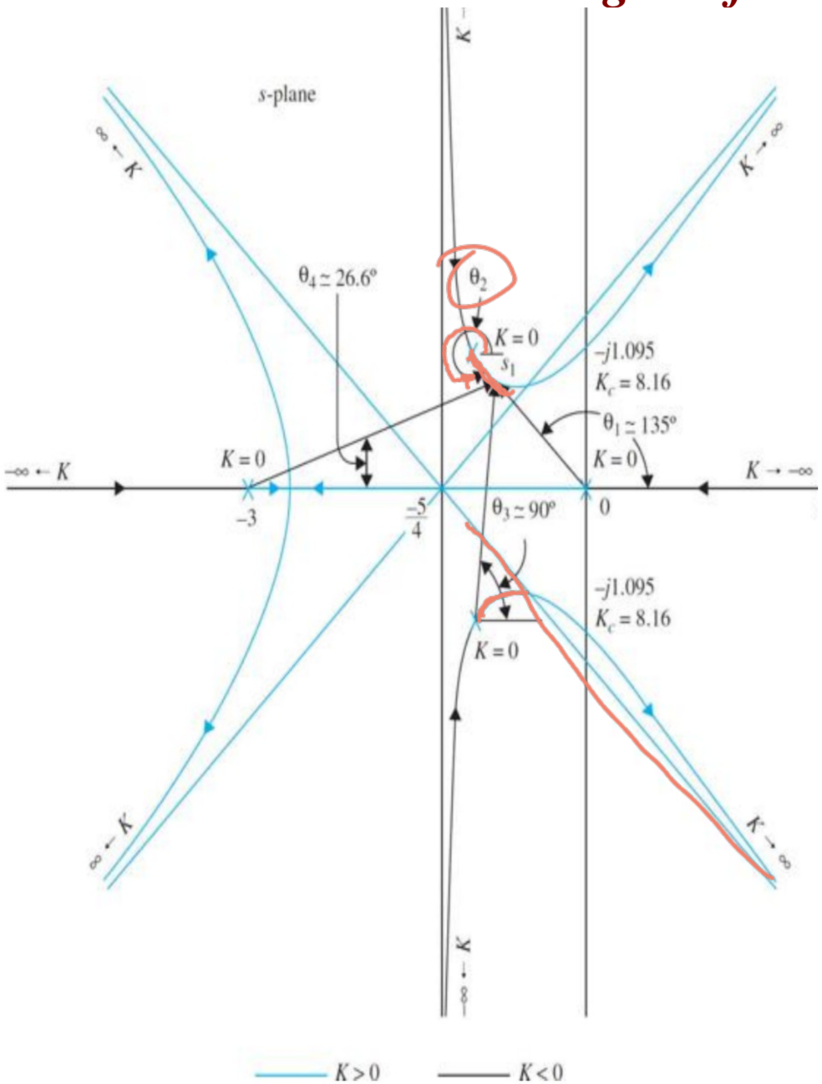
$$\angle C_1(s)G_1(s)H_1(s) = \sum_{j=1}^{m} \angle(s+z_j) - \sum_{k=1}^{n} \angle(s+p_k)$$
$$= 2l\pi \quad \text{where} \quad K \leq 0$$

where $l \in \{0, \pm 1, \pm 2, \cdots\}$

Root Locus Plots

Rules 6

Angles of arrival and departure of Root Loci



Root loci of $s(s+3)(s^2+2s+2)+K=0$ to illustrate the angles of departure or arrival

$$s(s+3)(s+1+i)(s+1-i) + K = 0$$

$$\begin{aligned} -\theta_2 &= (2l+1)\times 180^\circ + (135^\circ + 90^\circ + 26.6^\circ) \\ &= -(2\times -1+1)\times 180^\circ + 251.60^\circ \\ &= 71.60^\circ \end{aligned}$$

$$\begin{aligned} \theta_2 &= -71.60^\circ \\ &= 288.40^\circ \end{aligned}$$

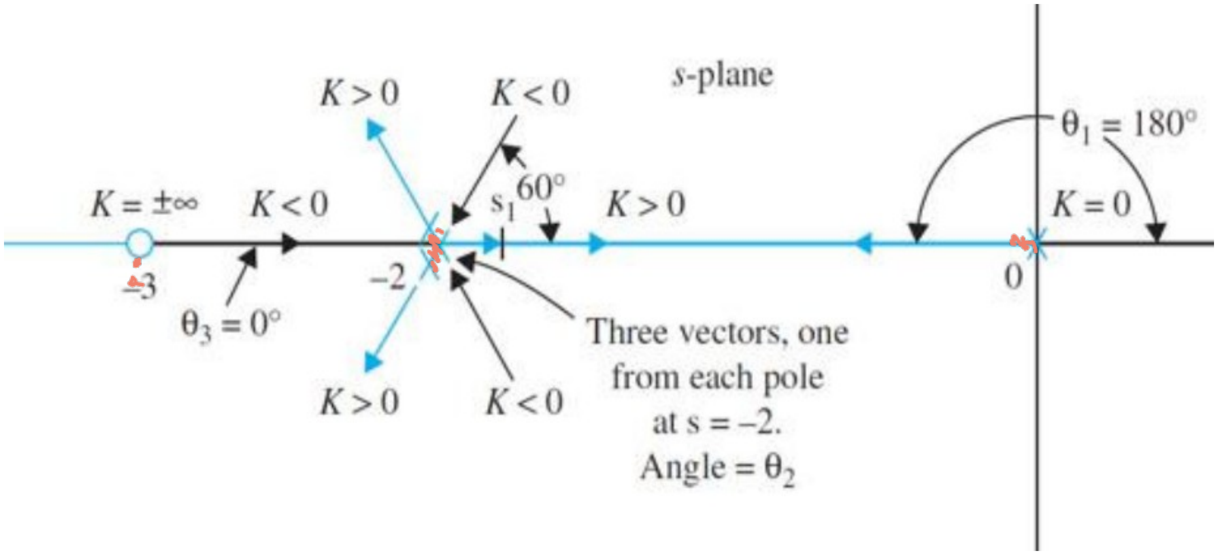
rlocus_example_rule6-1.m

homayoon.beigi@columbia.edu

Root Locus Plots

Rules 6

Angles of arrival and departure of Root Loci



Angles of departure and arrival at a third-order pole
Three poles repeated at s=-2

$$s(s+2)^3 + K(s+3) = 0$$

$$1 + \frac{K(s+3)}{s(s+2)^3} = 0$$

$$G_1(s)H_1(s) = \frac{s+3}{s(s+2)^3}$$

$$\angle G_1(s)H_1(s) = \theta_{z1} - (\theta_{p1} + 3 \times \theta_{p2}) = (2l+1)\pi$$

$$\begin{aligned} 3 \times \theta_{p2} &= (2l+1) \times \pi + (\pi - \pi) \\ &= \pi, 3\pi, 5\pi \end{aligned}$$

$$\theta_{p2} = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$$

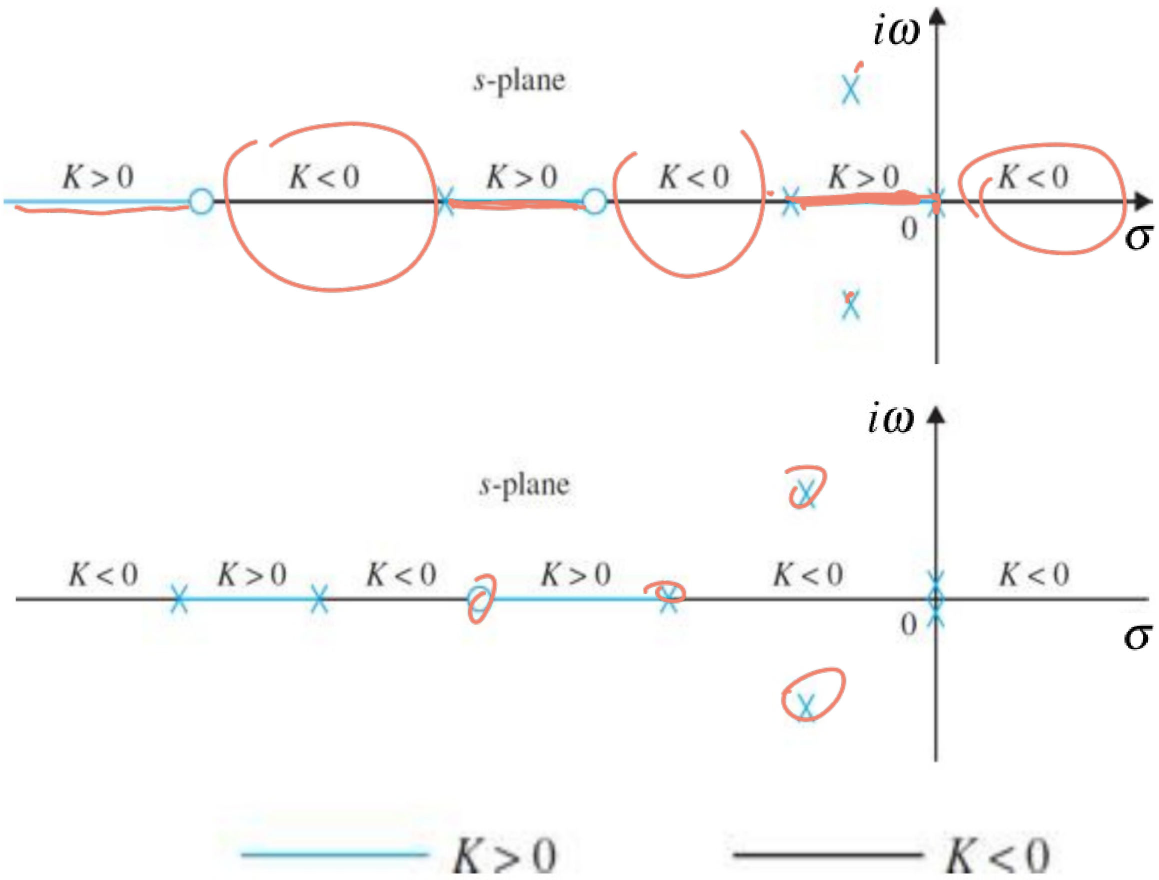
rlocus_example_rule6-2.m

homayoon.beigi@columbia.edu

Root Locus Plots

Rules 7

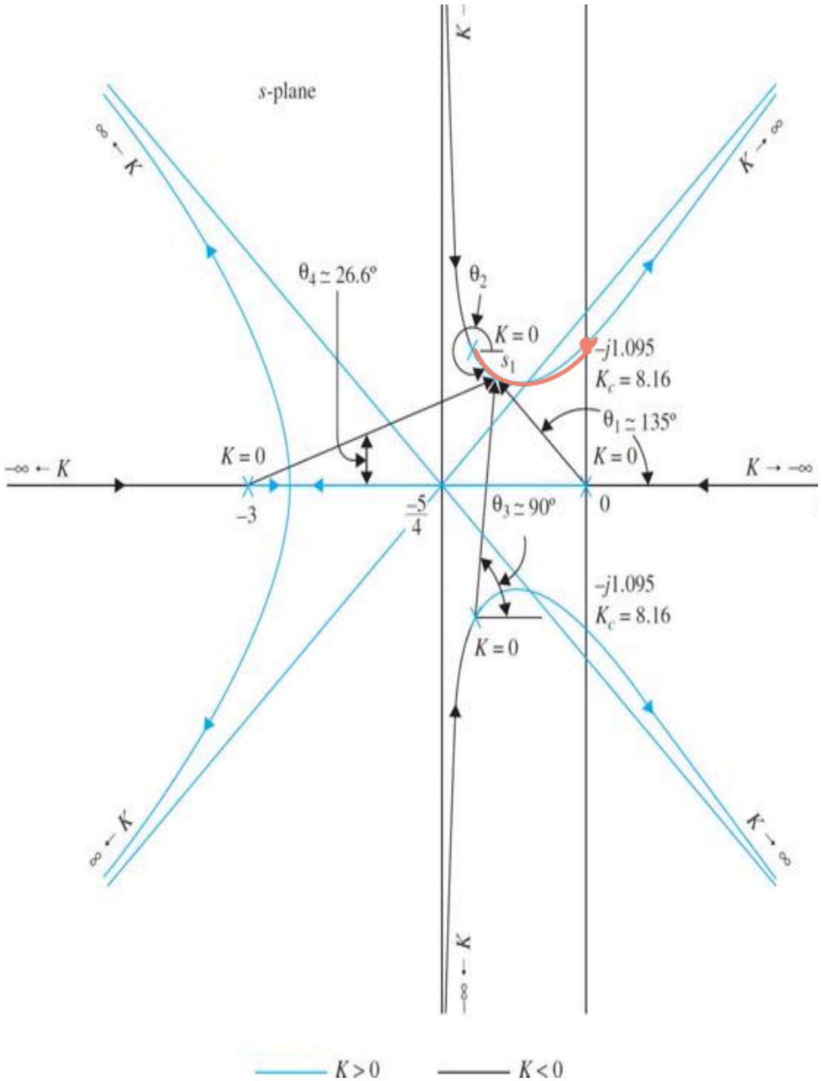
Properties on the real axis – Always to the left of odd number of poles or zeros



homayoon.beigi@columbia.edu

Root Locus Plots

Rules 8
Intersection with the Imaginary Axis



$$s(s+3)(s^2+2s+2)+K=0$$

$$s(s+3)(s^2+2s+2)=-K$$

$$i\omega(i\omega+3)((i\omega)^2+2i\omega+2)=-K$$

$$(-\omega^2+3i\omega)(-\omega^2+2i\omega+2)=-K$$

$$\omega^4-2\omega^3 i-2\omega^2-3i\omega^2-6\omega^2+6i\omega=-K$$

$$\omega^4-5\omega^3 i-8\omega^2+6\omega i=-K$$

$$\omega^2(\omega^2-8)+\omega(\cancelto{0}{-5\omega^2+6})i=-K$$

$$-5\omega^2+6=0$$

$$\omega^2=\frac{6}{5}$$

$$\omega^2=\frac{6}{5}$$

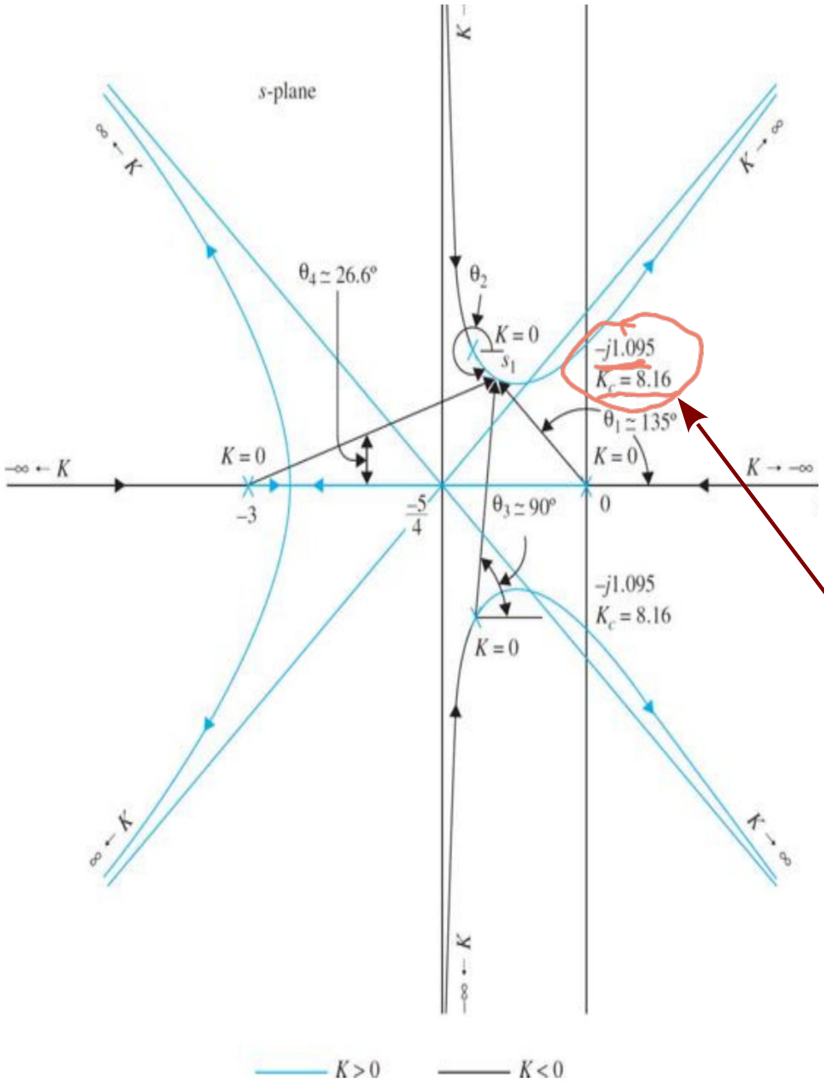
$$\boxed{\omega=\pm\sqrt{\frac{6}{5}}}$$

homayoon.beigi@columbia.edu

Root Locus Plots

Rules 8

Intersection with the Imaginary Axis



$$\boxed{\omega = \pm\sqrt{\frac{6}{5}}}$$

$$\omega^2(\omega^2 - 8)$$

$$\frac{6}{5}\left(\frac{6}{5} - 8\right) = -K$$

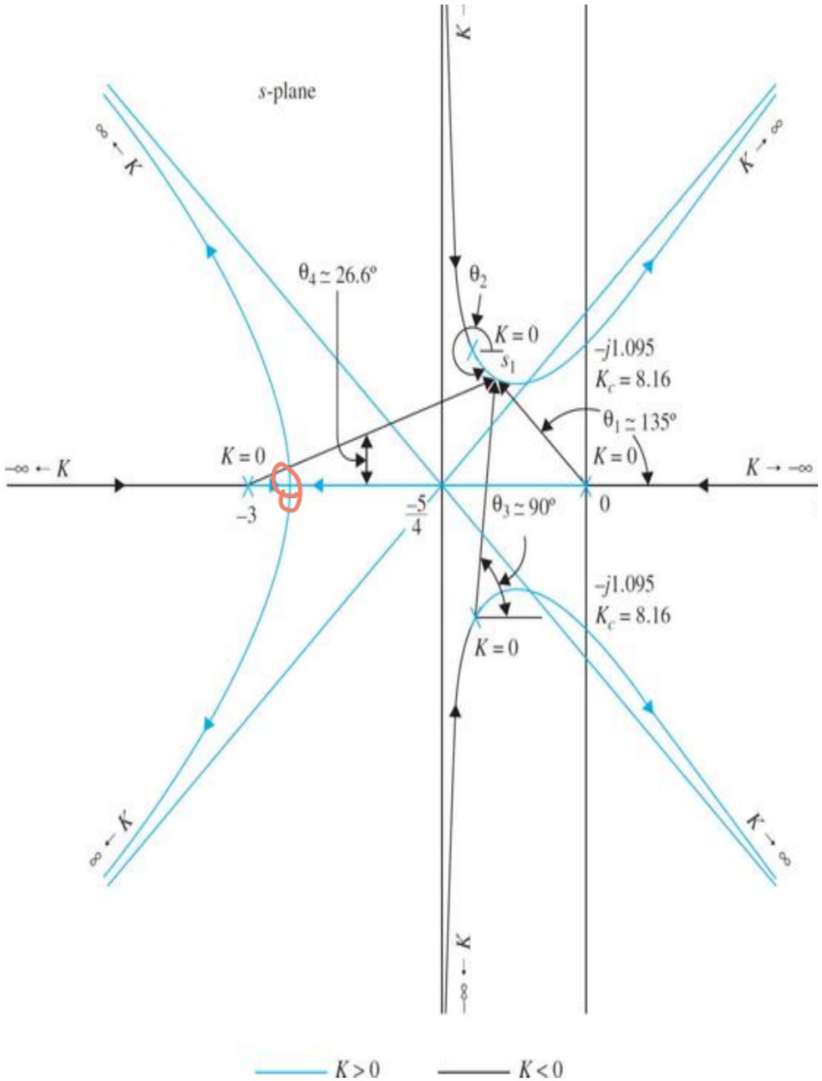
$$\frac{36}{25} - \frac{48}{5} = -K$$

$$K = \frac{48}{5} - \frac{36}{25}$$
$$= 8.16$$

rlocus_example_rule9-1.m
(program rule9-1 also
shows rule 8)

Root Locus Plots

Rules 9
Intersection with the Real Axis



The location where the root locus intersects the real axis is when,

$$\frac{dG_1(s)H_1(s))}{ds} = \frac{d(s(s+3)(s^2+2s+2))}{ds} = 0$$

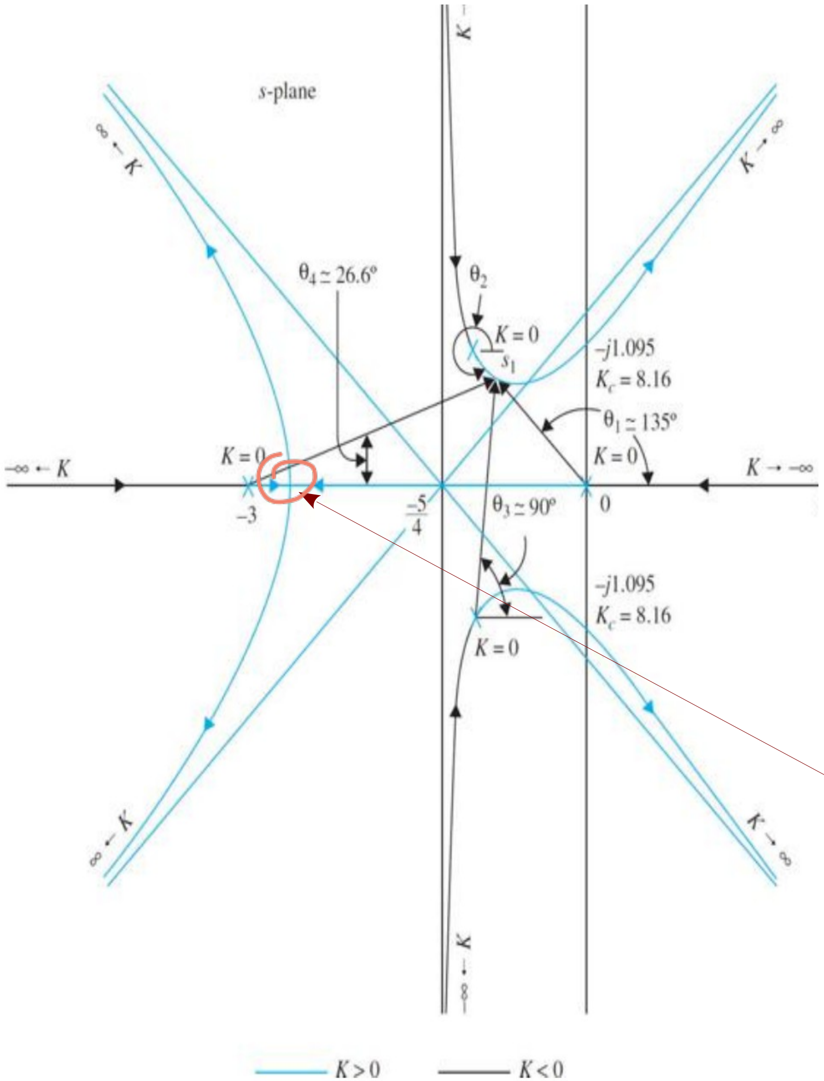
The value of K at the intersection is computed by ensuring that the solution of the above is used in the original closed loop characteristic equation,

$$1 + KG_1(s)H_1(s) = 0$$

homayoon.beigi@columbia.edu

Root Locus Plots

Rules 9
Intersection with the Real Axis



The location where the root locus intersects the real axis is when,

$$\frac{dG_1(s)H_1(s))}{ds} = \frac{d(s(s+3)(s^2+2s+2))}{ds} = 0$$

$$\begin{aligned} s(s+3)(s^2+2s+2) &= (s^2+3s)(s^2+2s+2) \\ &= s^4+2s^3+2s^2+3s^3+6s^2+6s \\ &= s^4+5s^3+8s^2+6s \end{aligned}$$

$$\frac{d(s^4+5s^3+8s^2+6s)}{ds} = 4s^3+15s^2+16s+6 = 0$$

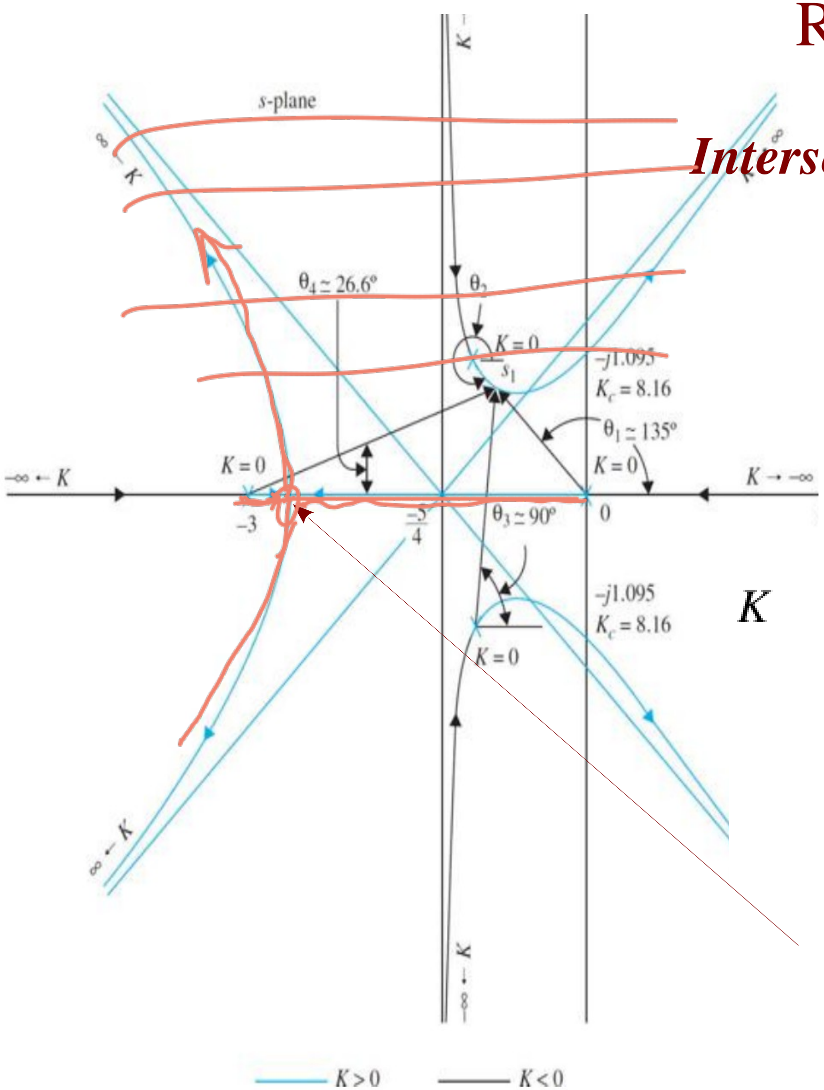
$$s_1 = -2.2886 \qquad s_{2,3} = -0.7307 \pm 0.3486i$$

Intersection point

rlocus_example_rule9-1.m

Root Locus Plots

Rules 9
Intersection with the Real Axis



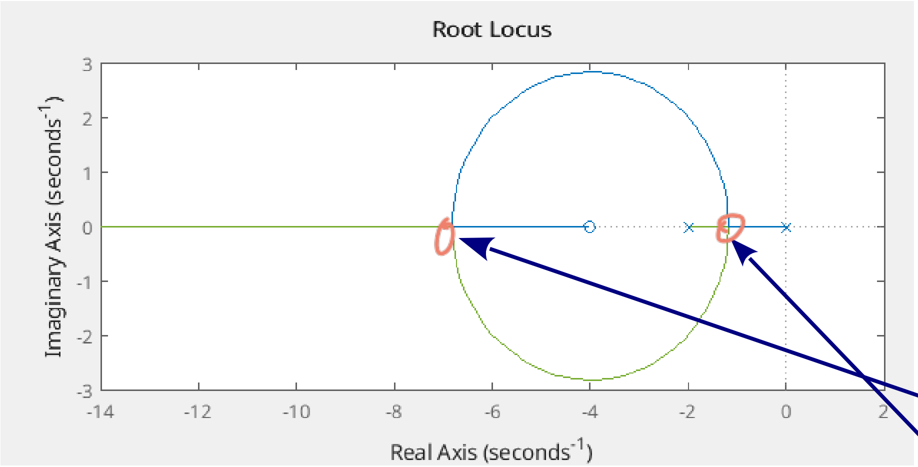
The location where the root locus intersects the real axis is when,

$$1 + KG_1(s)H_1(s) = 0$$

$$\begin{aligned}
K &= -\frac{1}{s(s+3)(s^2+2s+2)} \\
&= -\frac{1}{(-2.886)(-2.2886+3)((-22.2886)^2+2(-2.2886)+2)} \\
&= \frac{1}{4.33157} \\
&= 0.23
\end{aligned}$$

Root Locus Plots

Rules 9
Intersection with the Real Axis



The location where the root locus intersects the real axis is when,

$$
\begin{aligned}
\frac{dG_1(s)H_1(s))}{ds} &= \frac{d\left(\frac{s+4}{s(s+2)}\right)}{ds} \\
&= \frac{s(s+2)-2(s+1)(s+4)}{s^2(s+2)^2} \\
&= \frac{s^2+2s-2\left[s^2+5s+4\right]}{s^2(s+2)^2} \\
&= \frac{-s^2-8s-8}{s^2(s+2)^2} \\
&= 0
\end{aligned}
$$

$$\therefore \quad s^2+8s+8=0$$

$$s_1 = -1.172 \qquad s_2 = -6.6828$$

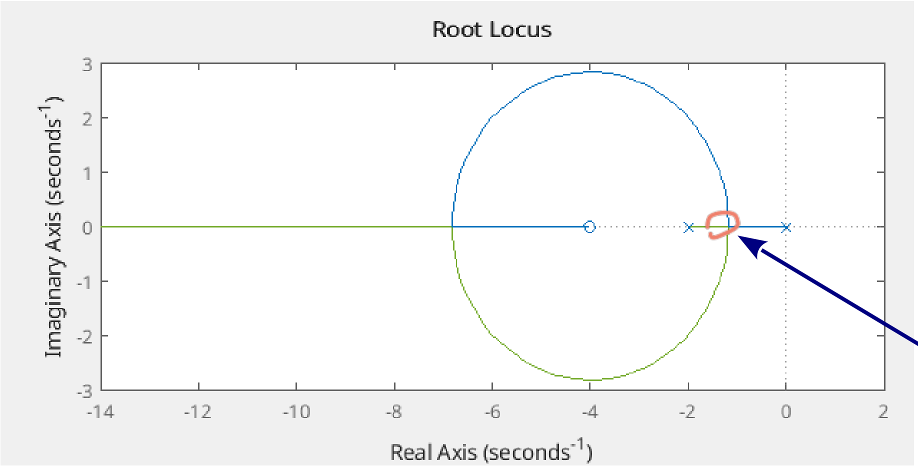
Intersection points

rlocus_example_rule9-2.m

homayoon.beigi@columbia.edu

Root Locus Plots

Rules 9
Intersection with the Real Axis



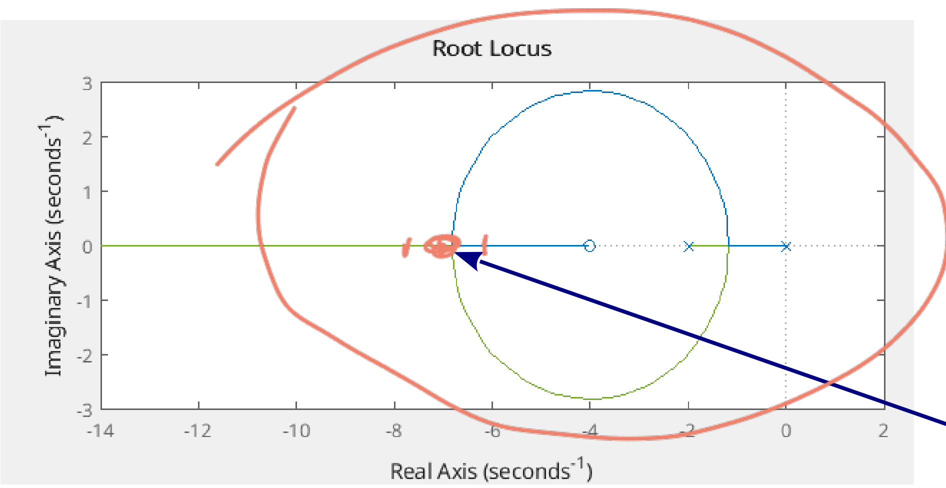
The location where the root locus intersects the real axis is when,

$$1 + K_1 G_1(s_1) H_1(s_1) = 0$$

$$\begin{aligned} K_1 &= -\frac{s_1(s_1+2)}{s_1+4} \\ &= -\frac{-1.172 \times (-1.172+2)}{(-1.172+4)} \\ &= \frac{0.97}{2.28} \\ &= 0.426 \end{aligned}$$

Root Locus Plots

Rules 9
Intersection with the Real Axis



The location where the root locus intersects the real axis is when,

$$1 + K_2 G_1(s_2) H_1(s_2) = 0$$

$$\begin{aligned} K_2 &= -\frac{s_2(s_2+2)}{s_2+4} \\ &= -\frac{-6.6828 \times (-6.6828+2)}{(-6.6828+4)} \\ &= \frac{-31.294}{-2.6828} \\ &= 11.665 \end{aligned}$$

Root Locus Plots

Rules 10
Arrival and Departure Angles from the Real Axis

The n Root Loci arrive and depart
to/from the root axis at 180/n degrees

Root Locus Plots

Rules 11
Root Sensitivity

Root sensitivity tends toward infinity at the break-away points

Break-away points transition between real, repeated, and complex roots

Robust System $\longleftrightarrow$ Low Root Sensitivity

Break-away points may be computed as follows,

$$\begin{aligned} S_K &= \frac{\frac{ds}{s}}{\frac{dK}{K}} \\ &= \frac{K}{s}\frac{ds}{dK} \end{aligned}$$

homayoon.beigi@columbia.edu

Bode Plot

$P(t) = A\sin(\Omega t)$
$y(t) = C\sin(\Omega t) + D\cos(\Omega t)$

Concerned with Steady-State Response of the system
We assume that the transients have died out – stable system

Forcing Function	Particular Solution Form
α (Constant)	C (Constant)
Polynomial of order n in t	Polynomial of order n in t
$\alpha\cos(\Omega t)+\beta\sin(\Omega t)$	$C\cos(\Omega t)+D\sin(\Omega t)$
$\beta e^{\alpha t}$	$Ce^{\alpha t}$
$e^{\alpha t}(\alpha\cos(\Omega t)+\beta\sin(\Omega t))$	$e^{\alpha t}(C\cos(\Omega t)+D\sin(\Omega t))$
Product of the Above	*Product of the Above*

For an oscillatory forcing function, steady state
frequency stays the same, different amplitude and phase

Bode Plot

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \underbrace{\sin(3t)}$$

$$y(0) = -1$$

$$\dot{y}(0) = 0$$

$$\boxed{y(t) = \underbrace{\frac{10}{13}e^{-2t} - \frac{17}{10}e^{-t}}_{\text{Transience}} \boxed{-\frac{9}{130}\cos(3t) - \frac{7}{130}\sin(3t)}}$$

frequency stays the same, different amplitude and phase

$$\ddot{y}(t) + 4\dot{y}(t) + 4y(t) = \underbrace{\cos(4t)}$$

$$y(0) = 1$$

$$\dot{y}(0) = 1$$

$$\boxed{y(t) = \underbrace{\frac{103}{100}e^{-2t} + \frac{29}{10}te^{-2t}}_{\text{Transience}} \boxed{-\frac{3}{100}\cos(4t) + \frac{1}{25}\sin(4t)}}$$

frequency stays the same, different amplitude and phase

homayoon.beigi@columbia.edu

Bode Plot

$$\ddot{y}(t)+\dot{y}(t)+y(t) = \underbrace{1+\cos(6t)}$$

$$
\begin{aligned}
y(0) &= 0 \\
\dot{y}(0) &= 0
\end{aligned}
$$

$$y(t) = \boxed{1 - \frac{35}{1261}\cos(6t) + \frac{6}{1261}\sin(6t)} \underbrace{- \frac{2}{1261}e^{-t/2}\left(613\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1262}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right)}_{\text{Transience}}$$

frequency stays the same,
different amplitude and phase

$$\ddot{y}(t)+y(t) = \underbrace{1+e^{(-2t)}}$$

$$
\begin{aligned}
y(0) &= 0 \\
\dot{y}(0) &= 0
\end{aligned}
$$

$$y(t) = \boxed{1 + \frac{1}{5}e^{-2t}} \underbrace{- \frac{6}{5}\cos(t) + \frac{2}{5}\sin(t)}_{\text{Transience}}$$

homayoon.beigi@columbia.edu

Bode Plot

The objective:

$$C\cos(\omega t) + D\sin(\omega t) = A\sin(\omega t + \phi)$$

Note that,

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

Therefore,

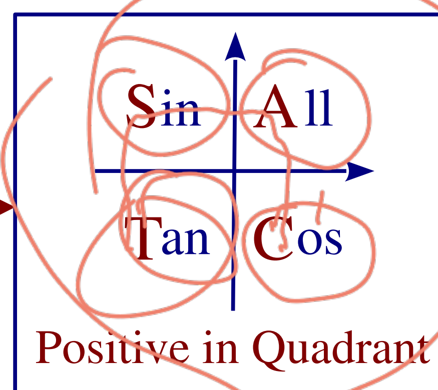
$$C\cos(\omega t) + D\sin(\omega t) = A\cos(\phi)\sin(\omega t) + A\sin(\phi)\cos(\omega t)$$

Since sine and cosine are linearly independent, the coefficients must match.

$$\begin{aligned} C &= A\sin(\phi) \\ D &= A\cos(\phi) \end{aligned}$$

$$\boxed{\begin{aligned} \sin(\phi) &= \frac{C}{A} \\ \cos(\phi) &= \frac{D}{A} \end{aligned}}$$

Use to find quadrant



homayoon.beigi@columbia.edu

Bode Plot

Special case: If D = 0, then there is only cos, in which case,

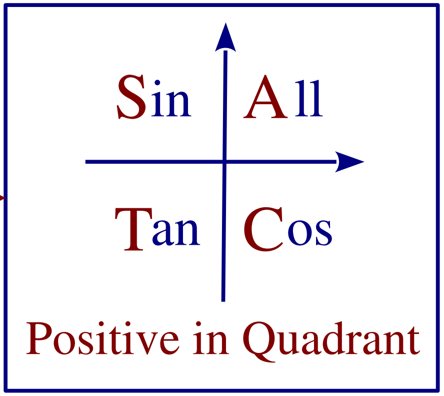
$$
\begin{aligned}
A &= C \\
\phi &= \frac{\pi}{2}
\end{aligned}
$$

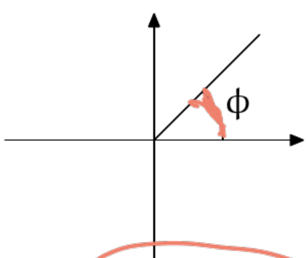
Otherwise,

$$
\begin{aligned}
A &= \sqrt{C^2 + D^2} \\
\tan(\phi) &= \frac{C}{D}
\end{aligned}
$$

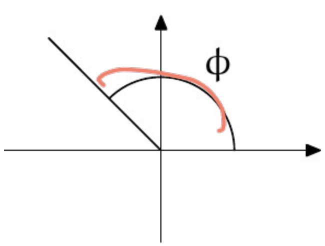
$$
\boxed{
\begin{aligned}
\sin(\phi) &= \frac{C}{A} \\
\cos(\phi) &= \frac{D}{A}
\end{aligned}
}
$$

Use to find quadrant

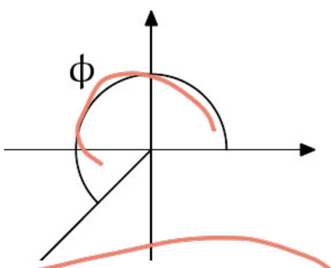




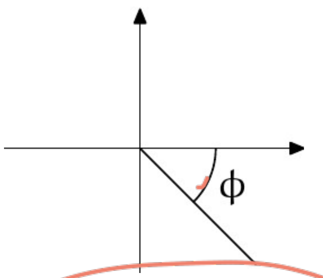
$$\phi = \tan^{-1}\left(\frac{C}{D}\right)$$



$$\phi = \tan^{-1}\left(\frac{C}{D}\right) + \pi$$



$$\phi = \tan^{-1}\left(\frac{C}{D}\right) + \pi$$



$$\phi = \tan^{-1}\left(\frac{C}{D}\right)$$

Homework 10

See Courseworks

Intensity

Intensity **–** Power per unit area. $\frac{W}{m^2}$ $\left(\frac{J/s}{m^2}\right.$ or $\left.\frac{N}{ms}\right)$

$$\text{Unit of } \zeta = \left(\frac{kg}{m^3}\right)\left(\frac{m}{s}\right)\left(\frac{s}{s}\right) = \frac{kgm}{s^2}\frac{s}{m^3} = \frac{Ns}{m^3}$$

$$I = \frac{P^2}{\zeta} = \frac{P^2}{\rho c}$$

pressure differential

specific acoustic impedance

$$\zeta = \rho c$$

Dry air at 1 atm and 20°C

$$\rho = 1.204 \ \frac{kg}{m^3}, \quad c = 343.2 \ \frac{m}{s} \quad \Longrightarrow \quad \zeta = 413.21 \ \frac{Ns}{m^3}$$

$$P_0 = 2 \times 10^{-5} \ \frac{N}{m^2} \quad \textit{(RMS of P for 1kHz)}$$

$$I_0 = \frac{P_0^2}{\zeta} \approx 10^{-12} \ \frac{W}{m^2} \quad \textit{(Intensity 1kHz)}$$

Pressure and Intensity Threshold to hear 1 kHz

homayoon.beigi@columbia.edu

Relative Intensity

$$
\begin{aligned}
I_r &= 10 \log\left(\frac{I}{I_0}\right) \\
&= 10 \log\left(\frac{P^2}{P_0^2}\right) \\
&= 20 \log\left(\frac{P}{P_0}\right)
\end{aligned}
$$

Dimensionless (in dB)

– 3 kHz – 4 kHz – around the resonance freq. of the ear canal

– 10 dB – 80 dB

Quiet Library: *40 – 60 dB*

Loud Rock Concert: *110 dB*

Average Relative Intensity: *58 dB*

Male Speakers are about *4.5 dB* louder than females

homayoon.beigi@columbia.edu

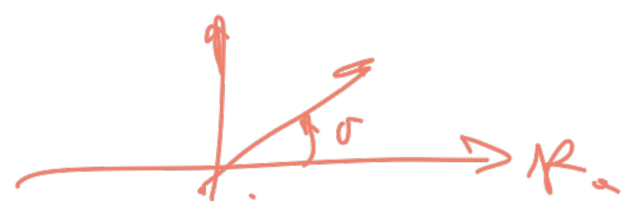
Fourier Series Expansion

We can write any function in terms
of a set of sinusoidal functions.

Fourier Series Expansion: Any periodic function may be written in terms of an infinite series of exponential functions (or sines and cosines)

Defined for the period: [-T, T]

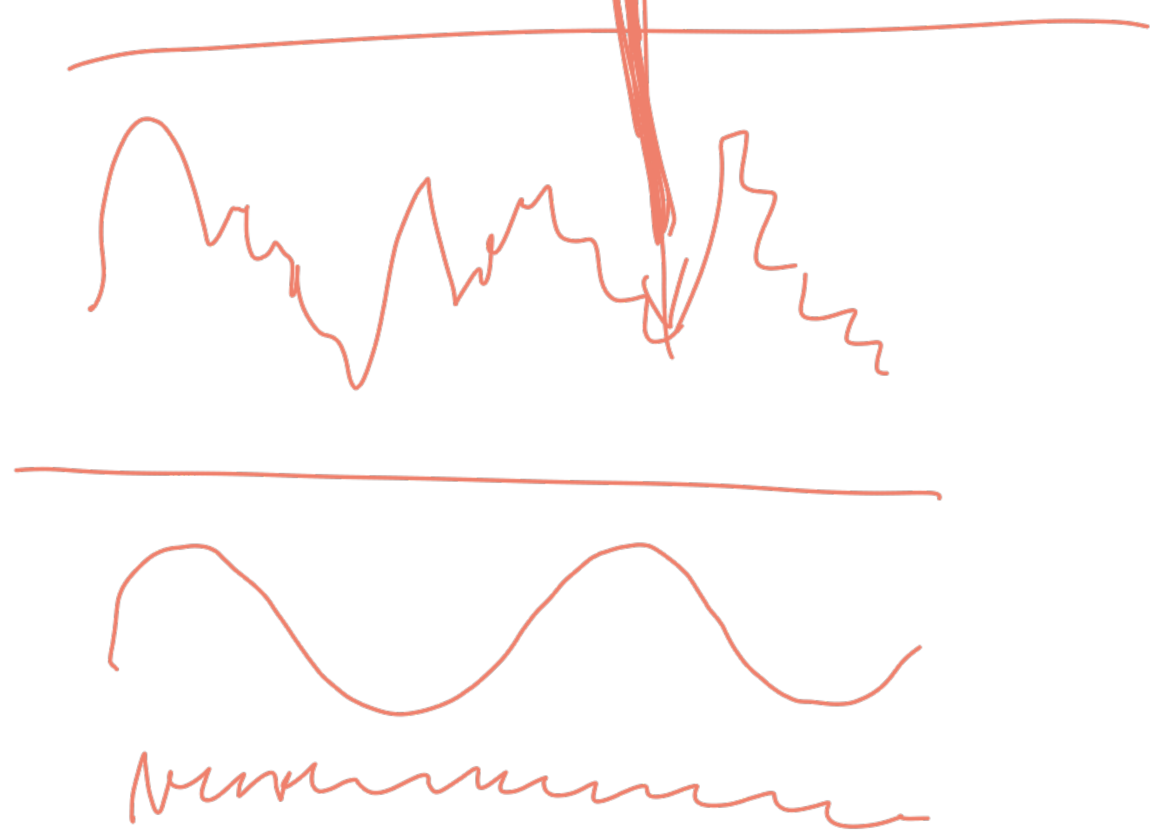
$$h(t) \approx \sum_{n=-\infty}^{\infty} c_n e^{i\left(\frac{n\pi t}{T}\right)} \qquad \boxed{c_n = \frac{1}{2T}\int_{-T}^{T} h(t)e^{-i\left(\frac{n\pi t}{T}\right)}dt}$$



Parseval's Theorem

Theorem 24.28 (Parseval's Theorem – Fourier Transform). *The Total power in a signal is the same when computed in the time or Frequency domain. In other words, the Total Power P is given by,*

$$\begin{aligned}\mathscr{P} &\stackrel{\Delta}{=} \int_{-\infty}^{\infty} |h(t)|^2 dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 d\omega\end{aligned}$$



$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$$

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t}\, dt$$

Parseval's Theorem: *proof* (Complex Fourier Transform)

$$\mathscr{P} \stackrel{\Delta}{=} \int_{-\infty}^{\infty} |h(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 d\omega$$

$$\begin{aligned}
\overline{se^{i\theta}} &= \overline{(\sigma + i\omega)e^{i\theta}} \\
&= \overline{(\sigma + i\omega)(\cos(\theta) + i\sin(\theta))} \\
&= \overline{(\sigma\cos(\theta) - \omega\sin(\theta)) + i(\sigma\sin(\theta) + \omega\cos(\theta))} \\
&= (\sigma\cos(\theta) - \omega\sin(\theta)) - i(\sigma\sin(\theta) + \omega\cos(\theta)) \\
&= \cos(\theta)(\sigma - i\omega) - \sin(\theta)(\omega + i\sigma) \\
&= \cos(\theta)(\sigma - i\omega) - i\sin(\theta)(\sigma - i\omega) \\
&= (\sigma - i\omega)(\cos(\theta) - i\sin(\theta)) \\
&= \bar{s}e^{-i\theta}
\end{aligned}$$

Then,

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{i\omega t} d\omega \longleftrightarrow \overline{h(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{H(\omega)}e^{-i\omega t} d\omega$$

homayoon.beigi@columbia.edu

Plancherel's Theorem: *proof* (Complex Fourier Transform)

$|h|^2$ $\quad h\bar{h}$

$$\hat{\mathscr{P}} \stackrel{\Delta}{=} \int_{-\infty}^{\infty} g(t)\overline{h(t)}dt$$

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{i\omega t}d\omega$$

$$\overline{h(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{H(\omega)}e^{-i\omega t}d\omega$$

$$\begin{aligned}
\hat{\mathscr{P}} &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\omega_1)e^{i\omega_1 t}d\omega_1 \int_{-\infty}^{\infty} \overline{H(\omega_2)}e^{-i\omega_2 t}d\omega_2 dt \\
&= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\omega_1)e^{i\omega_1 t}\overline{H(\omega_2)}e^{-i\omega_2 t}d\omega_2 d\omega_1 dt \\
&= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\omega_1)\overline{H(\omega_2)}e^{i\omega_1 t}e^{-i\omega_2 t}d\omega_2 d\omega_1 dt \\
&= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\omega_1)\overline{H(\omega_2)}\int_{-\infty}^{\infty} e^{i\omega_1 t}e^{-i\omega_2 t}dt\,d\omega_2 d\omega_1
\end{aligned}$$

$$\int_{-\infty}^{\infty} e^{i\omega_1 t}e^{-i\omega_2 t}dt = \begin{cases} 0 & \forall \quad \omega_1 \neq \omega_2 \\ 2\pi & \text{for} \quad \omega_1 = \omega_2 \end{cases}$$ (Orthogonality of Exponential Functions)

$$\begin{aligned}
\hat{\mathscr{P}} &= \int_{-\infty}^{\infty} g(t)\overline{h(t)}dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\overline{H(\omega)}d\omega
\end{aligned}$$

(Plancherel's Theorem)
(Swiss Mathematician:
Michel Plancherel)

Parseval's Theorem: *proof* (Complex Fourier Transform)

$$\begin{aligned}\hat{\mathscr{P}} &= \int_{-\infty}^{\infty} g(t)\overline{h(t)}dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\overline{H(\omega)}d\omega\end{aligned}$$

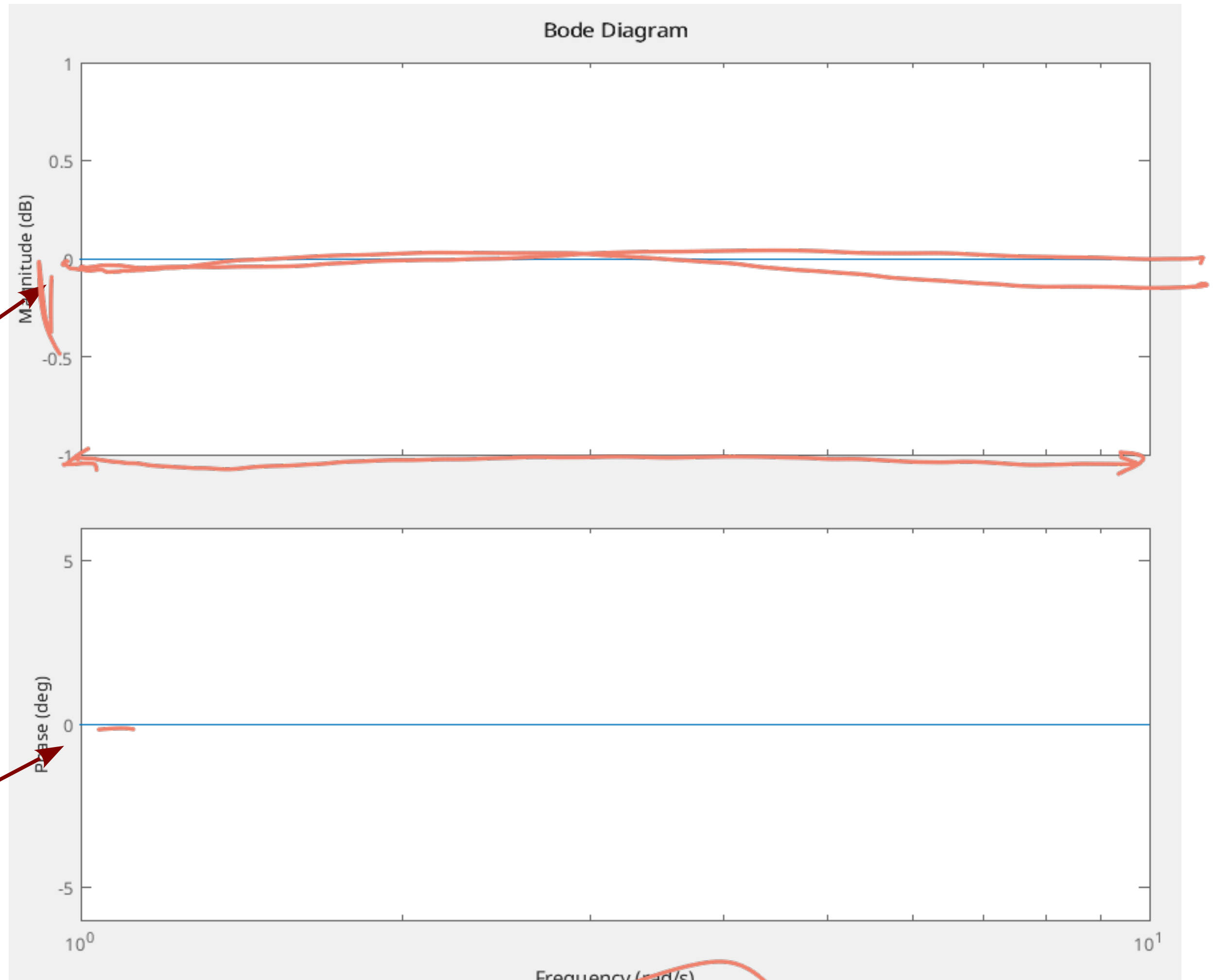
(Plancherel's Theorem)

Special case where g(t) = h(t)

$$\begin{aligned}\mathscr{P} &= \int_{-\infty}^{\infty} |h(t)|^2\,dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2\,d\omega\end{aligned}$$

(Parseval's Theorem)

Bode Plot



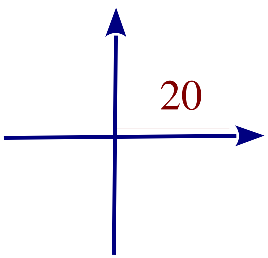
$20 \log 1 = 0$

0 phase shift for
positive gains

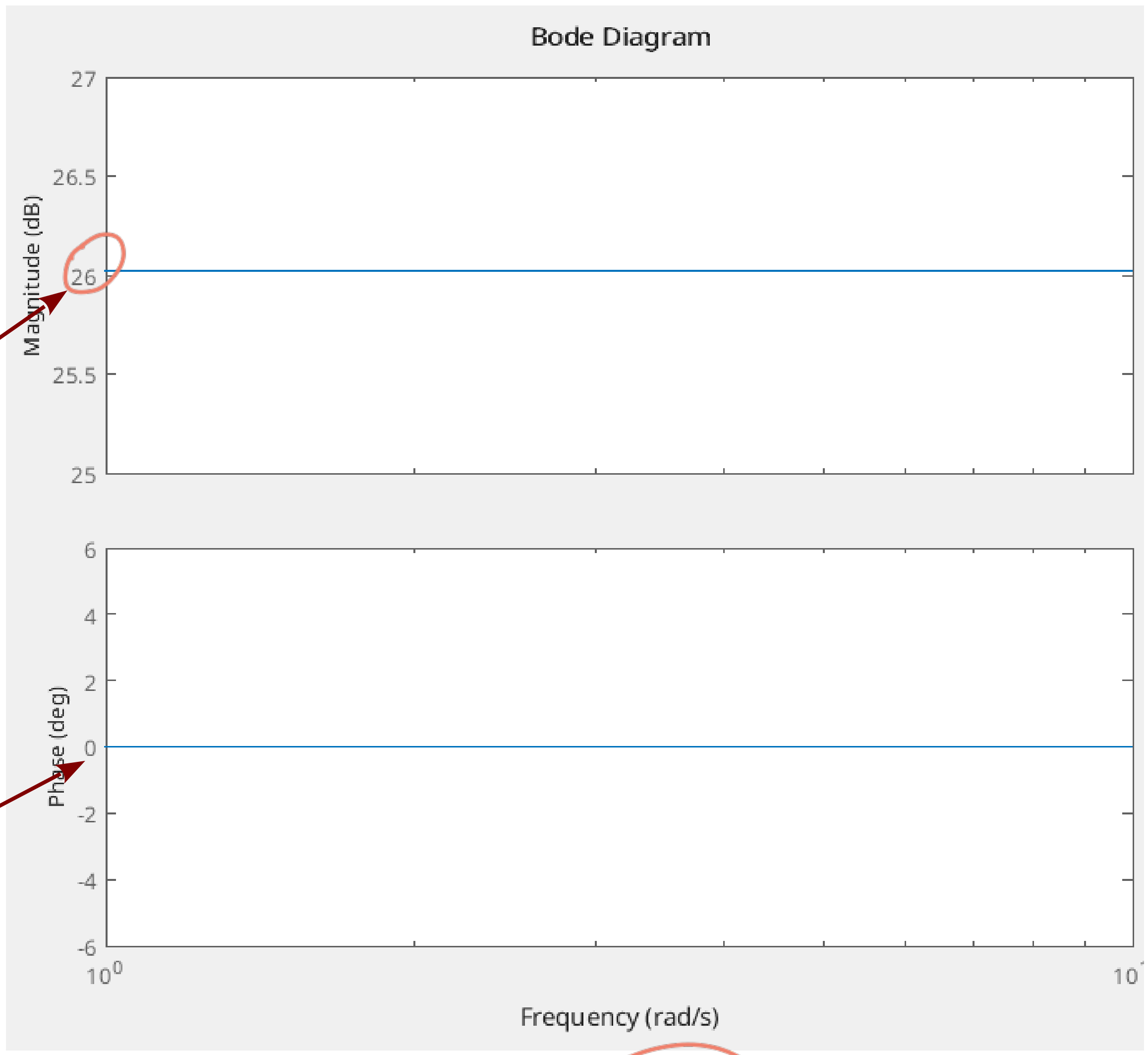
$G = 1$

homayoon.beigi@columbia.edu

Bode Plot



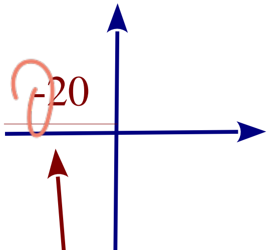
20 log 20



0 phase shift for positive gains

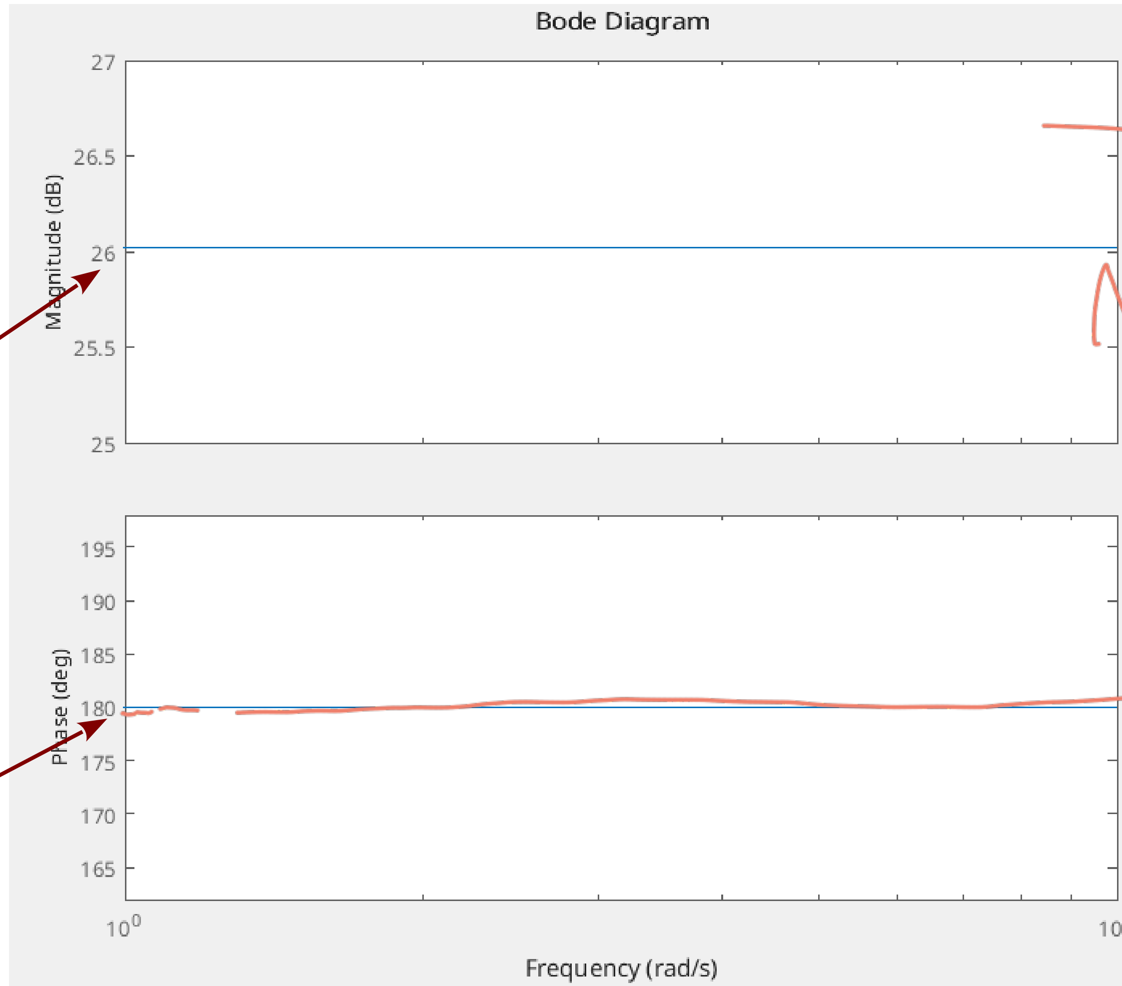
$G = 20$

Bode Plot



20 log 20

Opposite (180 degree phase)
For negative gains



$$G = -20$$

homayoon.beigi@columbia.edu

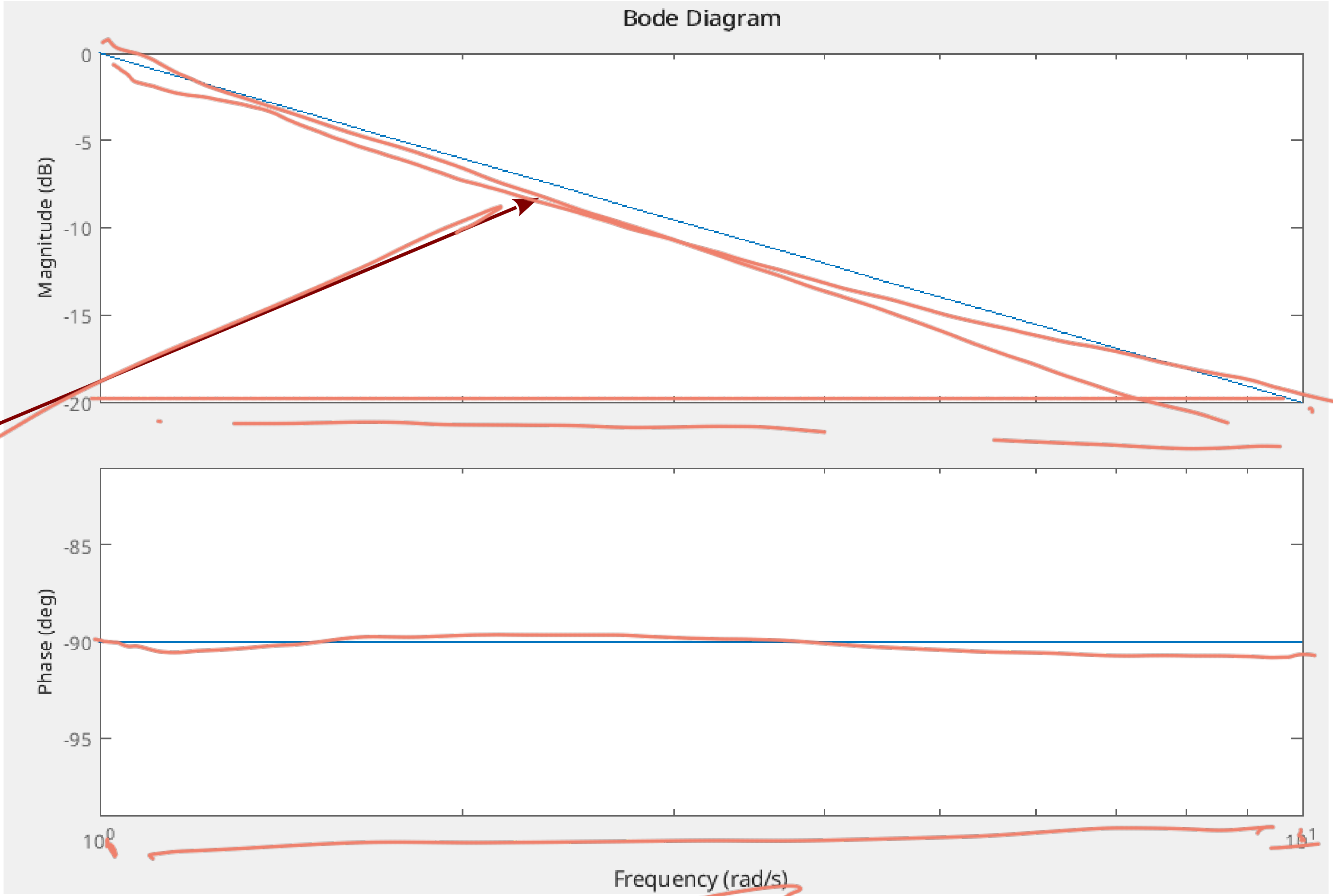
Bode Plot (Integrator)

$$G(i\omega) = \frac{1}{i\omega} = -\frac{1}{\omega} i$$

$$|G(i\omega)| = \left|-\frac{1}{\omega} i\right| = \frac{1}{\omega}$$

$$\angle G(i\omega) = -90^\circ$$

Slope=20db/decade



$$G = \frac{1}{s}$$

$i\omega$

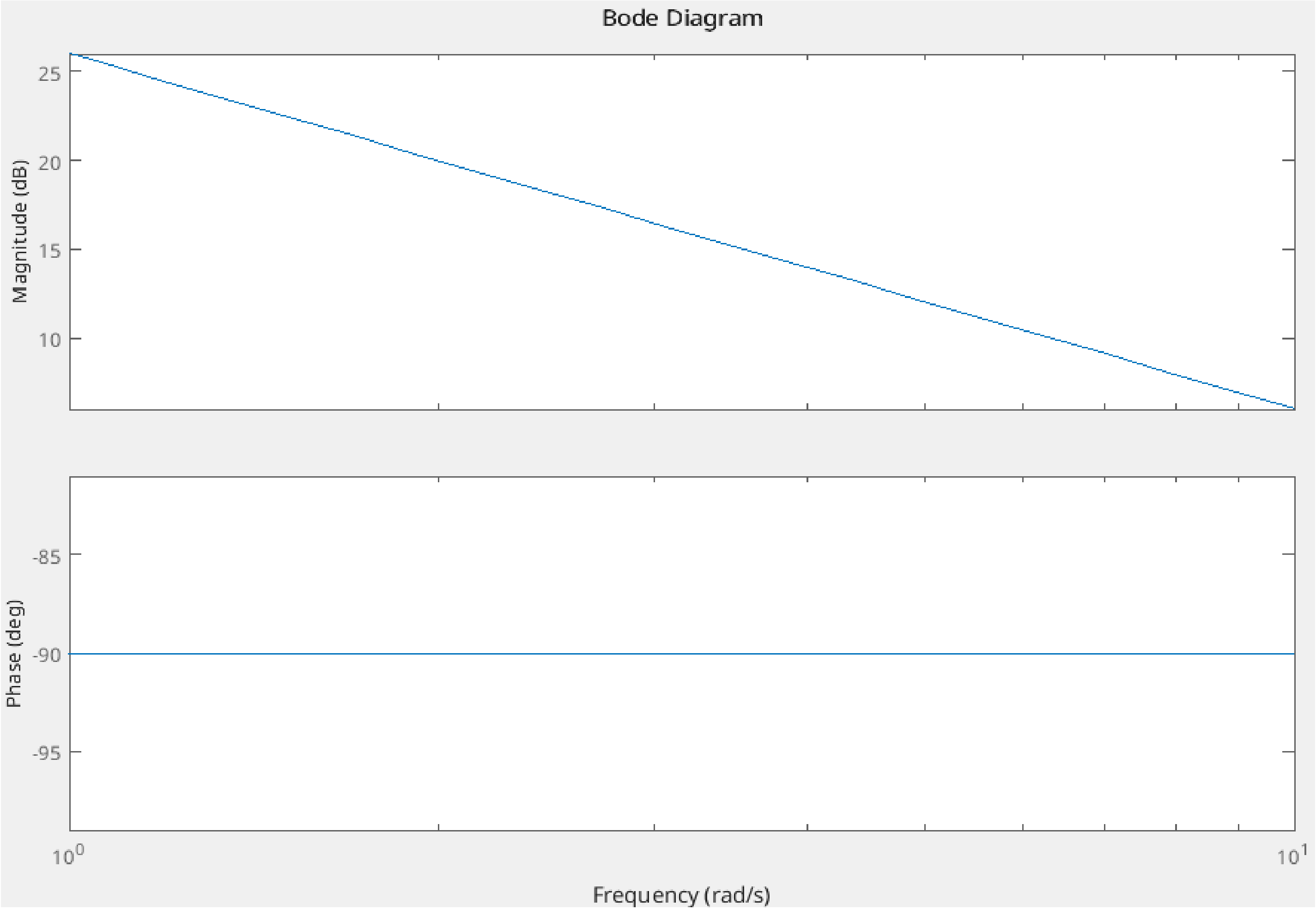
Bode Plot

$$G(s) = \frac{20}{s} = 20\frac{1}{s}$$

$$|G(i\omega)| = |20|\left|-\frac{1}{\omega}i\right|$$

$$20\log(|20|\frac{1}{\omega}) = 20\log(|20|) + 20\log(\frac{1}{\omega})$$

$$\angle G(i\omega) = \angle 20 + \angle -\frac{1}{\omega}i = 0^\circ - 90^\circ = -90^\circ$$



$$G(s) = \frac{20}{s}$$