

Introduction to Continuous Control Systems

EEME E3601



Week 12

Homayoon Beigi

Homayoon.Beigi@columbia.edu

<https://www.RecoTechnologies.com/beigi>

Mechanical Engineering dept.
&
Electrical Engineering dept.

Columbia University, NYC, NY, U.S.A.



Root Locus Plots (Review)

Closed-loop transfer function

$$\frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)H(s)}$$

Closed-loop characteristic equation

$$1 + C(s)G(s)H(s) = 0$$

Assume $C(s)G(s)H(s)$ has a free parameter, K $C(s)G(s)H(s) = \frac{KQ(s)}{P(s)}$

$$\begin{aligned} 1 + C(s)G(s)H(s) &= 1 + \frac{KQ(s)}{P(s)} \\ &= \frac{P(s) + KQ(s)}{P(s)} \quad \leftarrow F(s) = P(s) + KQ(s) = 0 \\ &= 0 \end{aligned}$$



Root Locus Plots (Review)

$$\begin{aligned} C(s)G(s)H(s) &= KC_1(s)G_1(s)H_1(s) \\ &= \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \end{aligned}$$

Use to draw
Root Loci

$$\left\{ \begin{aligned} \angle C_1(s)G_1(s)H_1(s) &= \sum_{j=1}^m \angle(s+z_j) - \sum_{k=1}^n \angle(s+p_k) \\ &= (2l+1)\pi \quad \text{where } K \geq 0 \\ \\ \angle C_1(s)G_1(s)H_1(s) &= \sum_{j=1}^m \angle(s+z_j) - \sum_{k=1}^n \angle(s+p_k) \\ &= 2l\pi \quad \text{where } K \leq 0 \end{aligned} \right.$$

where $l \in \{0, \pm 1, \pm 2, \dots\}$

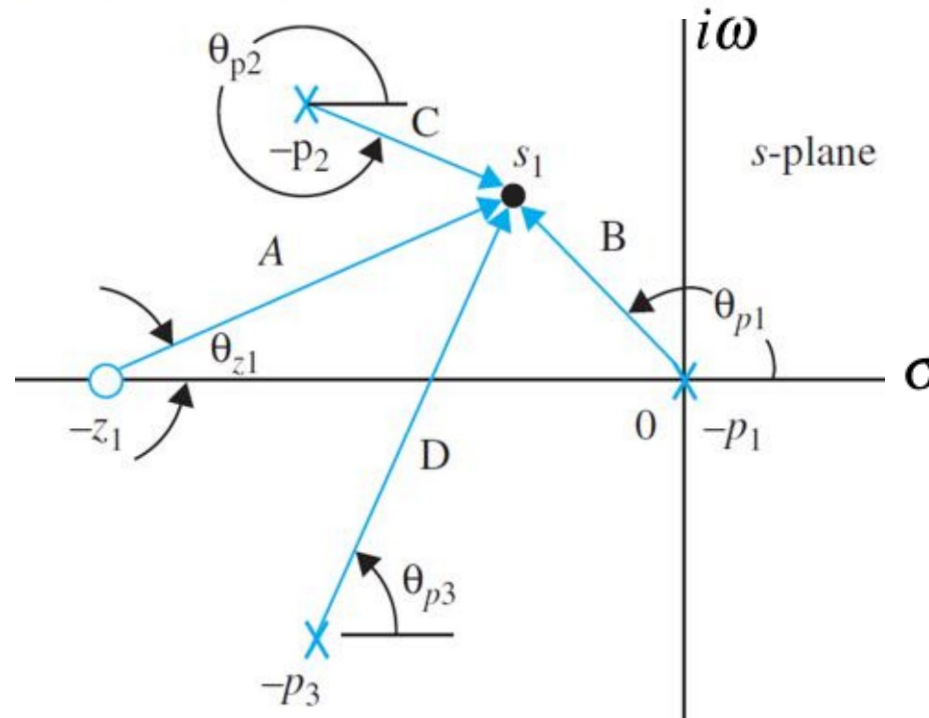
Root Locus Plots

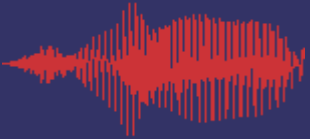
Example

Take an arbitrary point, s_1 , then if s_1 is a point on the Root Locus, the following equation must be satisfied for the case where $K \geq 0$

$$\begin{aligned} \angle(s_1 + z_1) - \angle s_1 - \angle(s_1 + p_2) - \angle(s_1 + p_3) &= \theta_{z_1} - \theta_{p_1} - \theta_{p_2} - \theta_{p_3} \\ &= (2l + 1)\pi \end{aligned}$$

where $l \in \{0, \pm 1, \pm 2, \dots\}$





Root Locus Plots (Review)

$$F(s) = P(s) + KQ(s) = 0$$

$$C_1(s)G_1(s)H_1(s) = -\frac{1}{K}$$

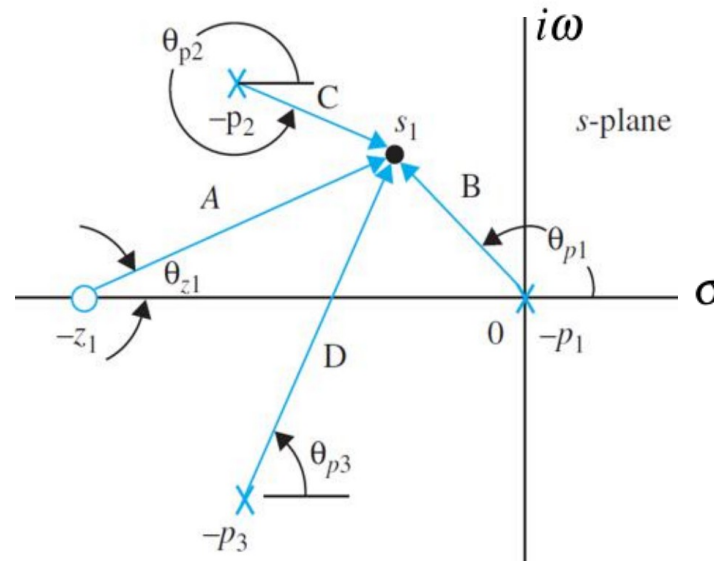
$$C(s)G(s)H(s) = KC_1(s)G_1(s)H_1(s) \quad \text{where } -\infty < K < \infty$$

$$\begin{aligned} |C_1(s)G_1(s)H_1(s)| &= \frac{\prod_{j=1}^m |s + z_j|}{\prod_{k=1}^n |s + p_k|} \\ &= \frac{1}{|K|} \end{aligned}$$

$$|K| = \frac{\prod_{k=1}^n |s + p_k|}{\prod_{j=1}^m |s + z_j|}$$

Root Locus Plots

Example



Once s_1 satisfies the angle relation, the gain, K , may be found by the following,

$$|K| = \frac{|s_1| |s_1 + p_2| |s_1 + p_3|}{|s_1 + z_1|}$$

$$= \frac{B \times C \times D}{A}$$

The sign of K is determined by which angle relation the point satisfies (odd multiple of π : positive, even multiple of π : negative)



Root Locus Plots

Rule 1

∞ (poles)

$$C_1(s)G_1(s)H_1(s) = -\frac{1}{K}$$

Diagram illustrating Rule 1 for Root Locus Plots. The equation is shown with two red arrows: one pointing from the left side of the equation to ∞ (poles) and another pointing from the right side to 0.

$$C_1(s)G_1(s)H_1(s) = -\frac{1}{K}$$

Diagram illustrating Rule 1 for Root Locus Plots. The equation is shown with two red arrows: one pointing from the left side of the equation to 0 (zeros) and another pointing from the right side to ∞ .



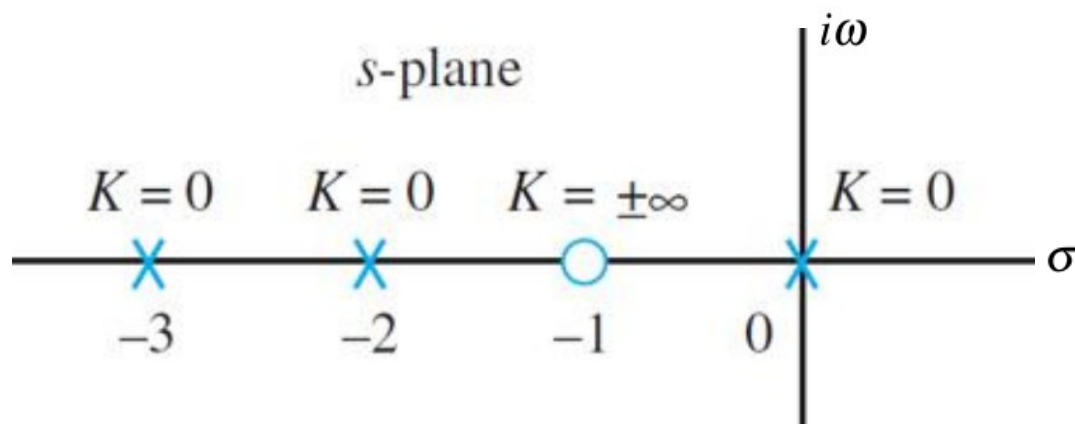
Root Locus Plots

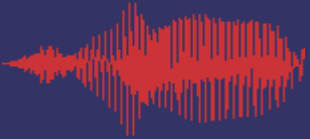
Example

$$\begin{aligned} 1 + C(s)G(s)H(s) &= s(s+2)(s+3) + K(s+1) \\ &= 0 \\ &= 1 + \frac{K(s+1)}{s(s+2)(s+3)} \end{aligned}$$

$$C(s)G(s)H(s) = \frac{K(s+1)}{s(s+2)(s+3)}$$

$$C_1(s)G_1(s)H_1(s) = -\frac{1}{K}$$





Root Locus Plots

Rule 2

Number of Branches

Number of branches of Root Loci is equal to the degree of the polynomial

Example

$$\begin{aligned} 1 + C(s)G(s)H(s) &= s(s+2)(s+3) + K(s+1) \\ &= 0 \\ &= 1 + \frac{K(s+1)}{s(s+2)(s+3)} \end{aligned}$$

Number of branches = degree of the above polynomial = 3

Namely, it is $\max(m,n)$



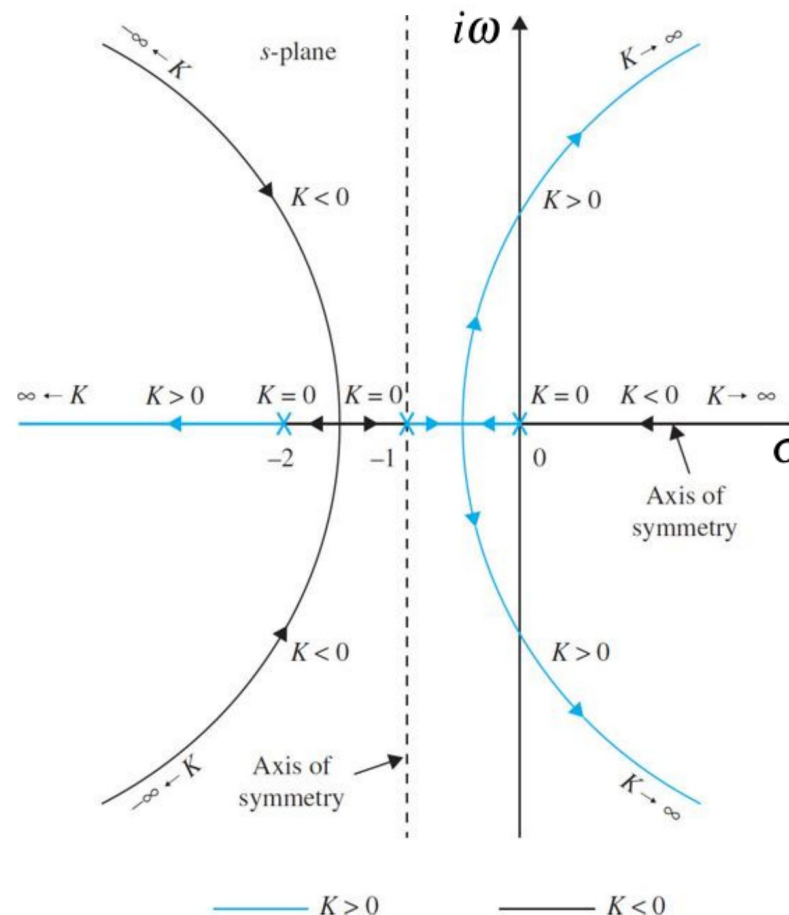
Root Locus Plots

Rule 3 *Symmetry*

Root Loci are symmetric with respect to *the real axis*
and the pole-zero configuration of $C(s)G(s)H(s)$

Example 1

$$C(s)G(s)H(s) = \frac{K}{s(s+2)(s+3)}$$





Root Locus Plots

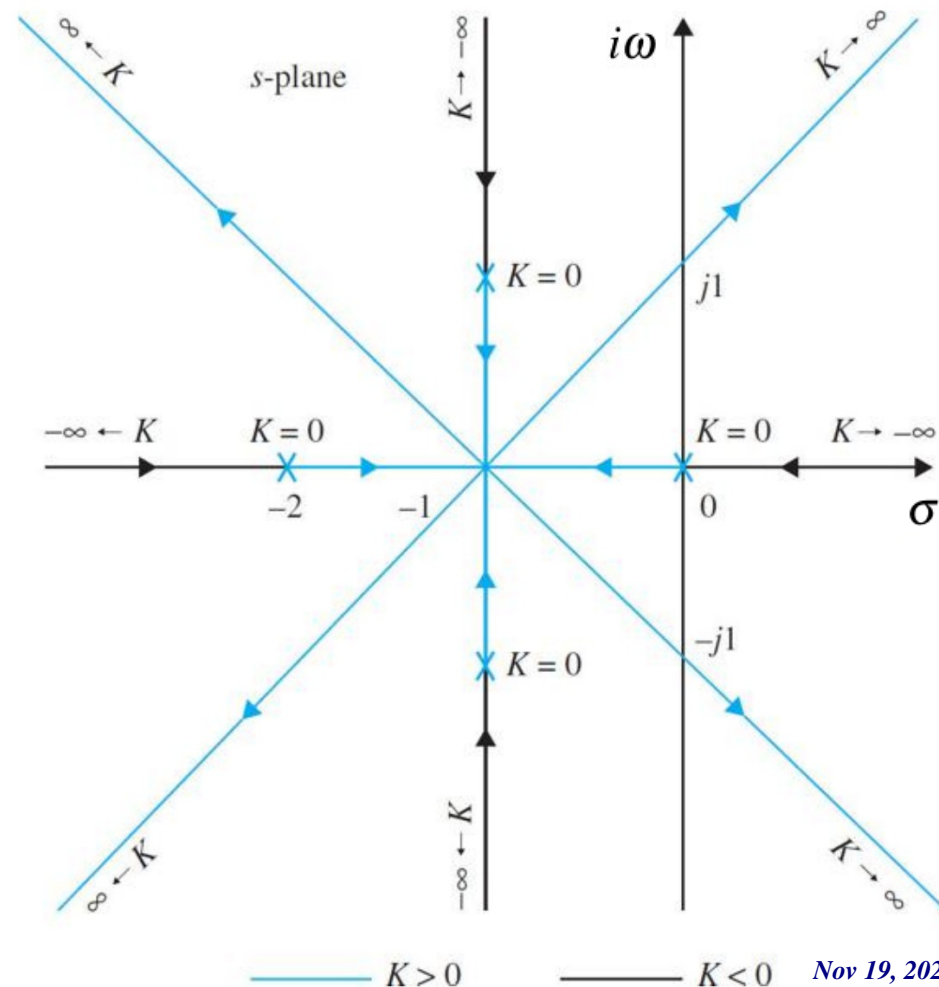
Rule 3 Symmetry

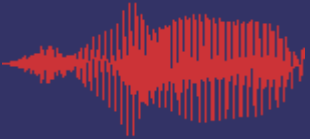
Root Loci are symmetric with respect to the real axis and the pole-zero configuration of $C(s)G(s)H(s)$

Example 2

$$\begin{aligned} 1 + C(s)G(s)H(s) &= s(s+2)(s^2 + 2s + 2) + K \\ &= 0 \\ &= s(s+2)(s+1+i)(s+1-i) + K \\ &= 1 + \frac{K}{s(s+2)(s+1+i)(s+1-i)} \end{aligned}$$

$$C(s)G(s)H(s) = \frac{K}{s(s+2)(s+1+i)(s+1-i)}$$





Root Locus Plots

Rule 4

Angles of Asymptotes with the real axis

Asymptotes of Root Loci (behavior of root loci at $|s| = \infty$)

Asymptotes angles for $K \geq 0$ are given by

$$\theta_l = \frac{(2l+1)}{|n-m|} \times \pi \quad \forall n \neq m \quad \text{where } k \geq 0$$

$$\theta_l = \frac{2l}{|n-m|} \times \pi \quad \forall n \neq m \quad \text{where } k \leq 0$$

$$l \in \{0, 1, \dots, |n-m|-1\}$$

There will be $2|n-m|$ asymptotes for $n \neq m$



Root Locus Plots

Rule 5

Intersection of the Asymptotes with the real axis

Asymptotes of Root Loci (behavior of root loci at $|s| = \infty$)

$$\begin{aligned}\sigma_1 &= \frac{\sum \text{Finite Poles of CGH} - \sum \text{Finite Zeros of CGH}}{n - m} \\ &= \frac{\sum \Re\{\text{Poles of CGH}\} - \sum \Re\{\text{Zeros of CGH}\}}{n - m}\end{aligned}$$

Center of Gravity of Root Loci (Always Real)
Complex conjugate imaginary parts sum to 0

Root Locus Plots

Rule 5

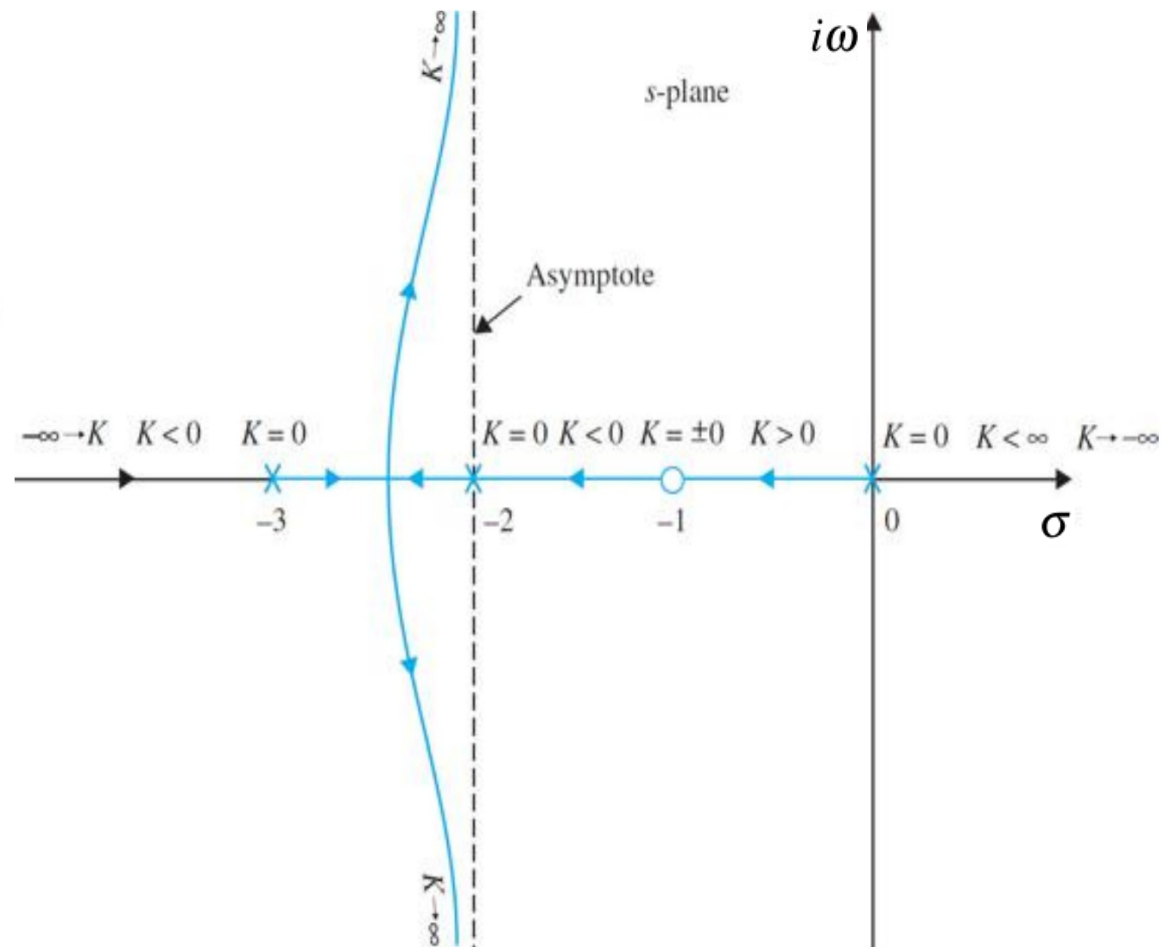
Intersection of the Asymptotes with the real axis

Asymptotes of Root Loci (behavior of root loci at $|s| = \infty$)

Example

$$\begin{aligned} 1 + C(s)G(s)H(s) &= s(s+2)(s+3) + K(s+1) \\ &= 0 \\ &= 1 + \frac{K(s+1)}{s(s+2)(s+3)} \end{aligned}$$

$$C(s)G(s)H(s) = \frac{K(s+1)}{s(s+2)(s+3)}$$



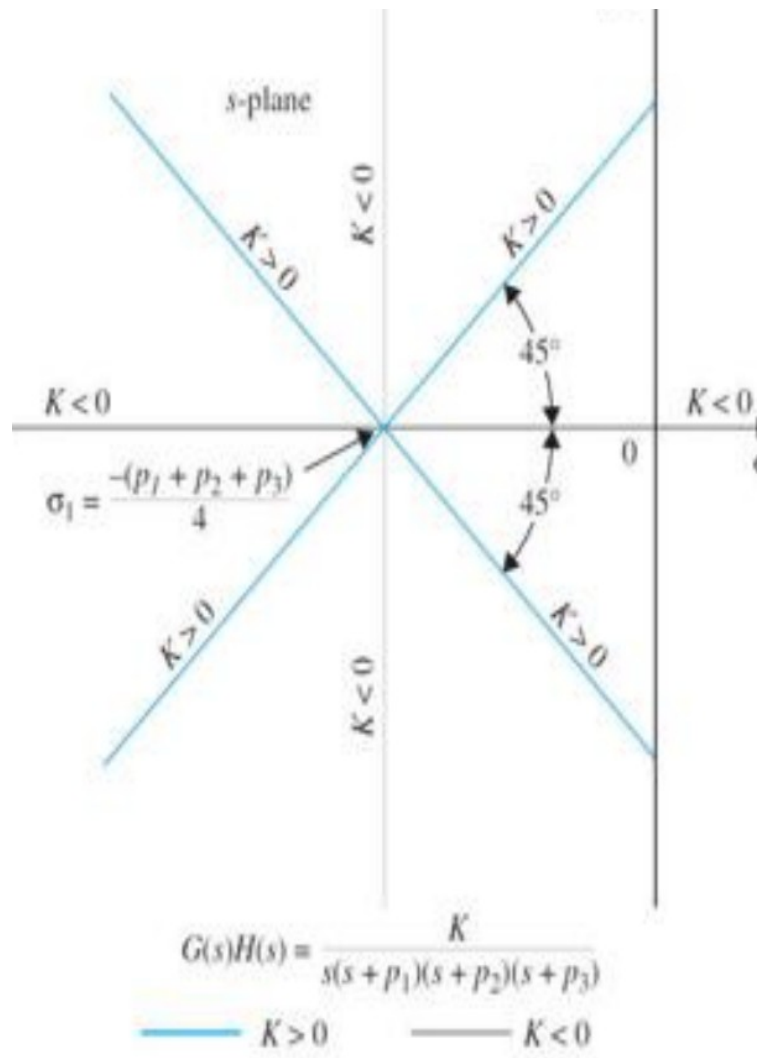
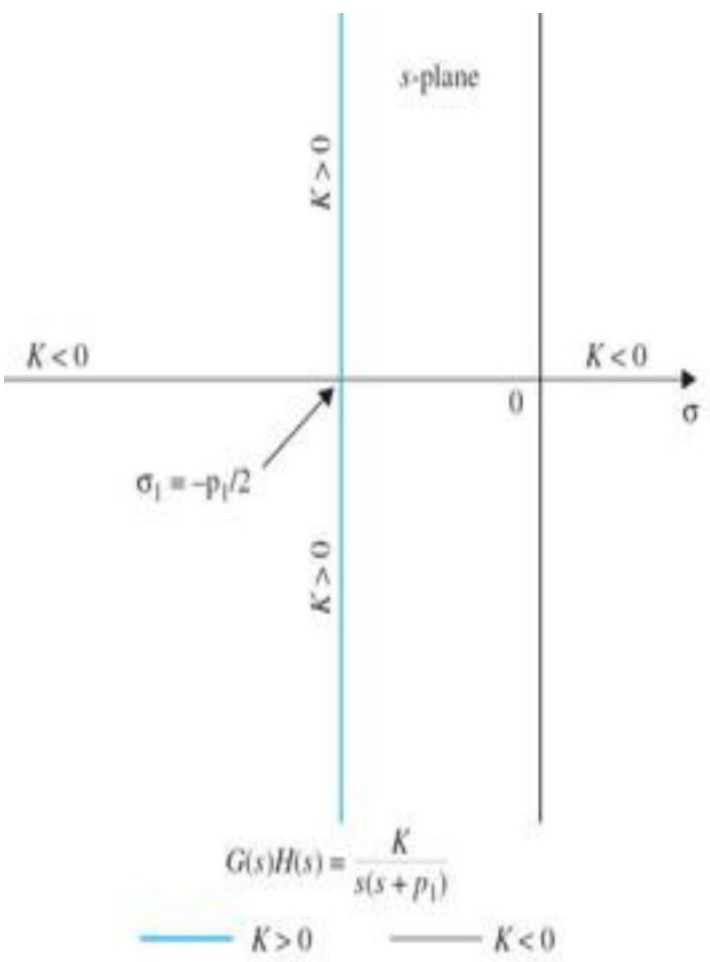


$$\sigma_1 = \frac{\sum \text{Finite Poles of CGH} - \sum \text{Finite Zeros of CGH}}{n - m}$$
$$= \frac{\sum \Re\{\text{Poles of CGH}\} - \sum \Re\{\text{Zeros of CGH}\}}{n - m}$$

Root Locus Plots
Rules 4,5

$$\theta_l = \frac{(2l + 1)}{|n - m|} \times \pi \quad \forall n \neq m$$

Intersection of the Asymptotes with the real axis





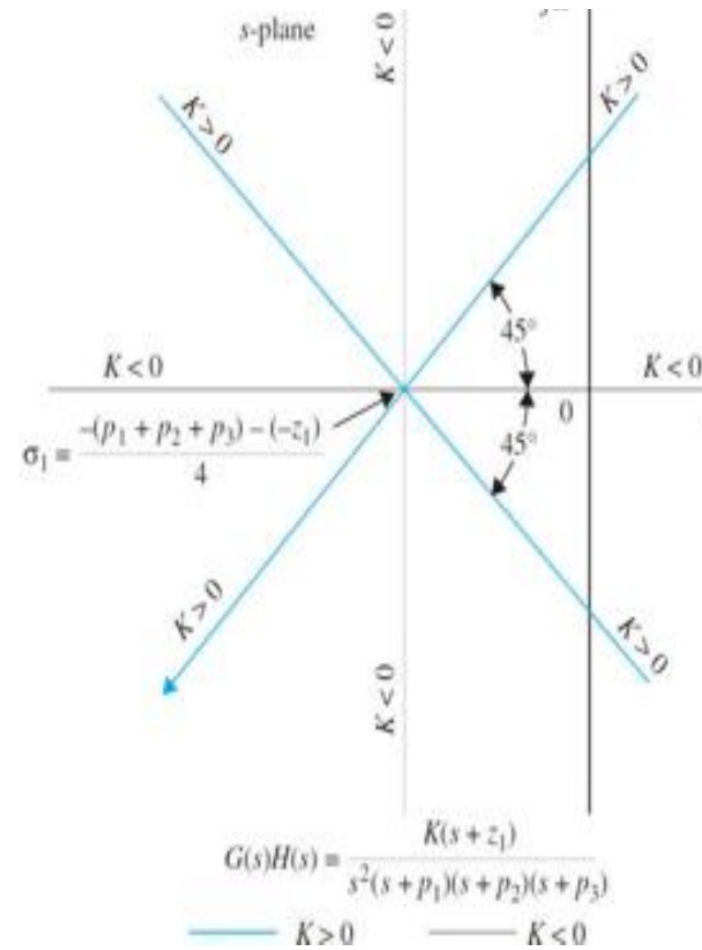
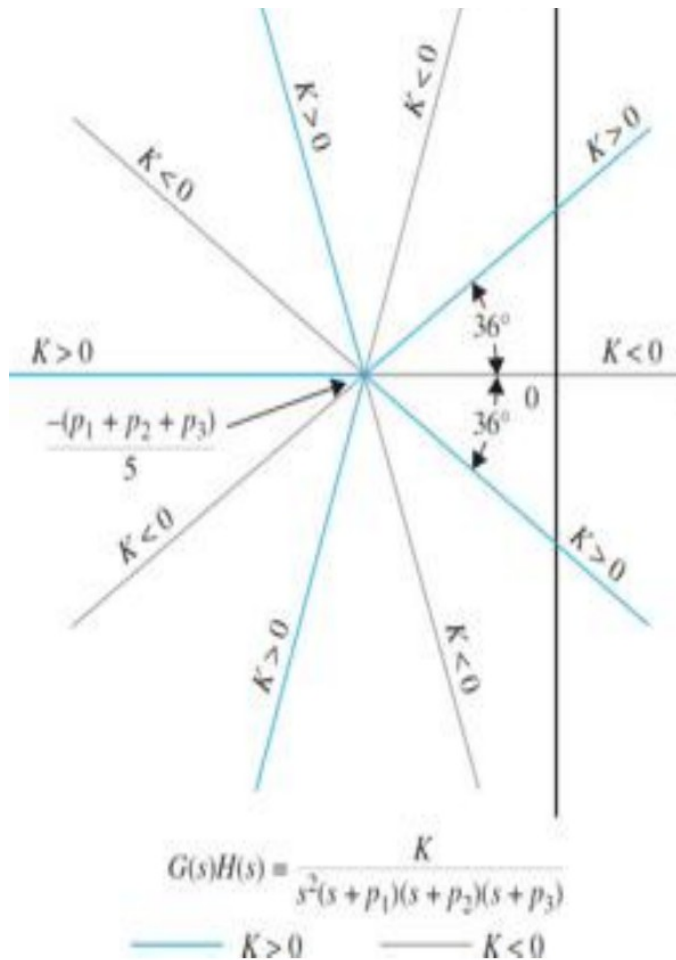
$$\begin{aligned}\sigma_1 &= \frac{\sum \text{Finite Poles of CGH} - \sum \text{Finite Zeros of CGH}}{n - m} \\ &= \frac{\sum \Re\{\text{Poles of CGH}\} - \sum \Re\{\text{Zeros of CGH}\}}{n - m}\end{aligned}$$

Root Locus Plots

Rules 4,5

Intersection of the Asymptotes with the real axis

$$\theta_l = \frac{(2l+1)}{|n-m|} \times \pi \quad \forall n \neq m$$





Root Locus Plots

Rules 4,5

Intersection of the Asymptotes with the real axis

Example Asymptotes of the root loci of $s(s+4)(s^2+2s+2)+K(s+1)=0$

$$s(s+4)(s^2+2s+2)+K(s+1)=0$$

$$1 + \frac{K(s+1)}{s(s+4)(s^2+2s+2)} = 0$$

$$G_1(s)H_1(s) = \frac{s+1}{s(s+4)(s^2+2s+2)}$$

$$\theta_l = \frac{(2l+1)}{|n-m|} \times \pi \quad \forall n \neq m \quad l \in \{0, 1, \dots, |n-m|-1\}$$

Rules 4

$$\begin{aligned} \sigma_1 &= \frac{\sum \text{Finite Poles of CGH} - \sum \text{Finite Zeros of CGH}}{n-m} \\ &= \frac{\sum \mathcal{Re}\{\text{Poles of CGH}\} - \sum \mathcal{Re}\{\text{Zeros of CGH}\}}{n-m} \end{aligned}$$

Rules 5

$$\begin{aligned} \sigma_1 &= 0 - 4 - 4 - 1 - (-1) = -\frac{5}{3} \\ \theta_l &= \left\{ \frac{\pi}{3}, \pi, \frac{5\pi}{3} \right\} \end{aligned}$$



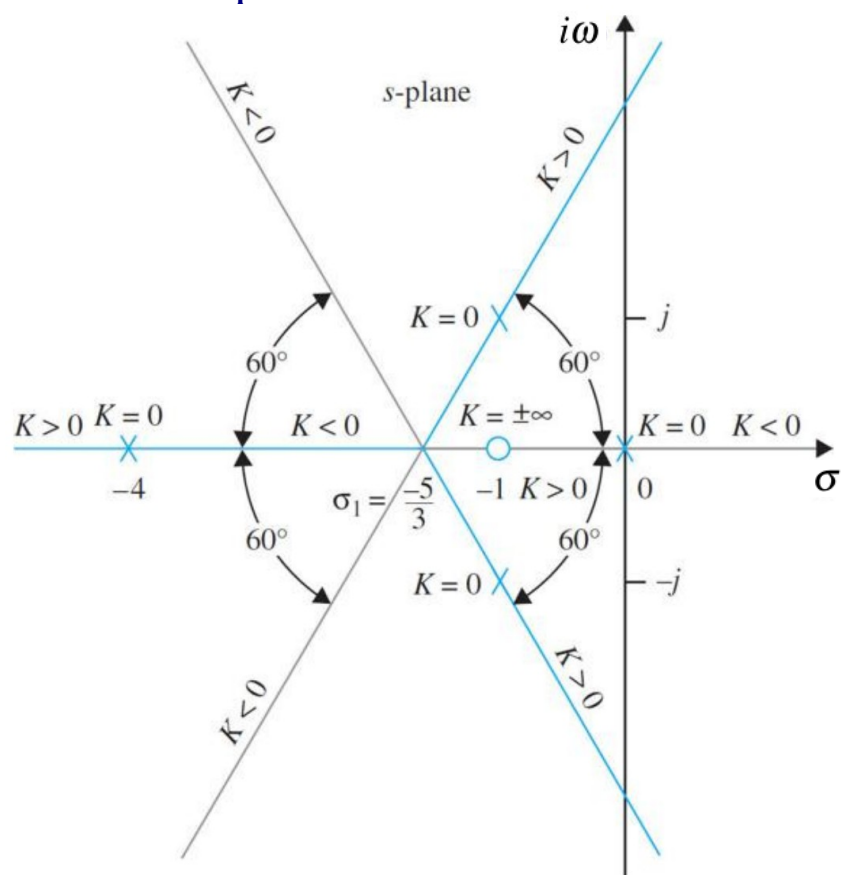
Root Locus Plots

Rules 4,5

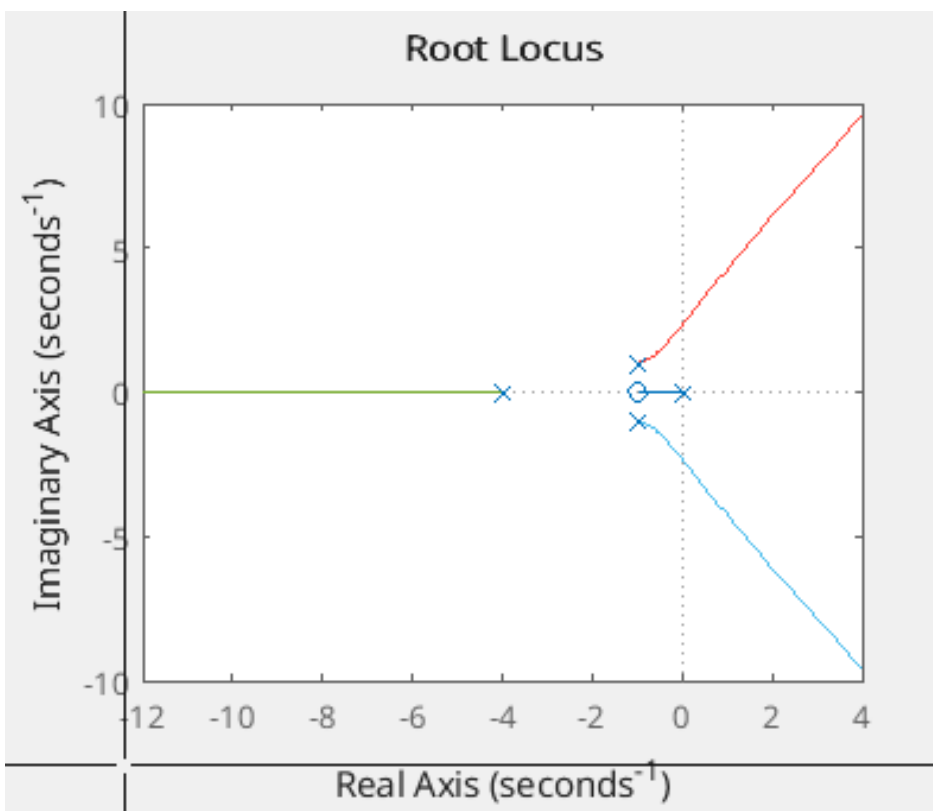
Intersection of the Asymptotes with the real axis

Example

Asymptotes of the root loci of $s(s + 4)(s^2 + 2s + 2) + K(s + 1) = 0$



Error in fig. 9-6 of book



Correct Root Locus



Root Locus Plots

Rule 5

Intersection of the Asymptotes with the real axis

Asymptotes of Root Loci (behavior of root loci at $|s| = \infty$)

rlocus_example_rule5-1.m

% This example is based on Golnaraghi-Kuo Edition 10 p.532 Toolbox Fig. 9-5

```
PolynQ=[1 1];          % Numerator
PolynP=conv([1 0],[1 2]) % Denominator
PolynP=conv(PolynP,[1 3]) % Denominator
```

```
%\frac{Q(s)}{P(s)}
TFG = tf(PolynQ,PolynP); % Open loop transfer function
```

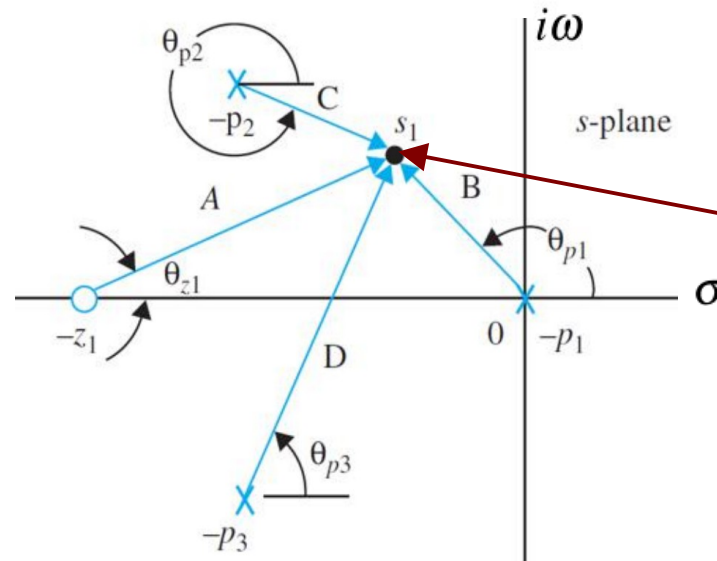
```
rlocus(TFG);
axis([-3 0 -8 8])
```

```
[K,poles] = rlocfind(TFG) %rcocfind allow us to choose desired poles on the root locus
```

Root Locus Plots

Rules 6

Angles of arrival and departure of Root Loci



Use the pole of interest in the place of s_1 to compute its angle of departure (or zero for angle of arrival)

Use to draw
Root Loci

$$\begin{aligned} \angle C_1(s)G_1(s)H_1(s) &= \sum_{j=1}^m \angle(s + z_j) - \sum_{k=1}^n \angle(s + p_k) \\ &= (2l + 1)\pi \quad \text{where } K \geq 0 \\ \angle C_1(s)G_1(s)H_1(s) &= \sum_{j=1}^m \angle(s + z_j) - \sum_{k=1}^n \angle(s + p_k) \\ &= 2l\pi \quad \text{where } K \leq 0 \end{aligned}$$

where $l \in \{0, \pm 1, \pm 2, \dots\}$

Root Locus Plots

Rules 6

Angles of arrival and departure of Root Loci

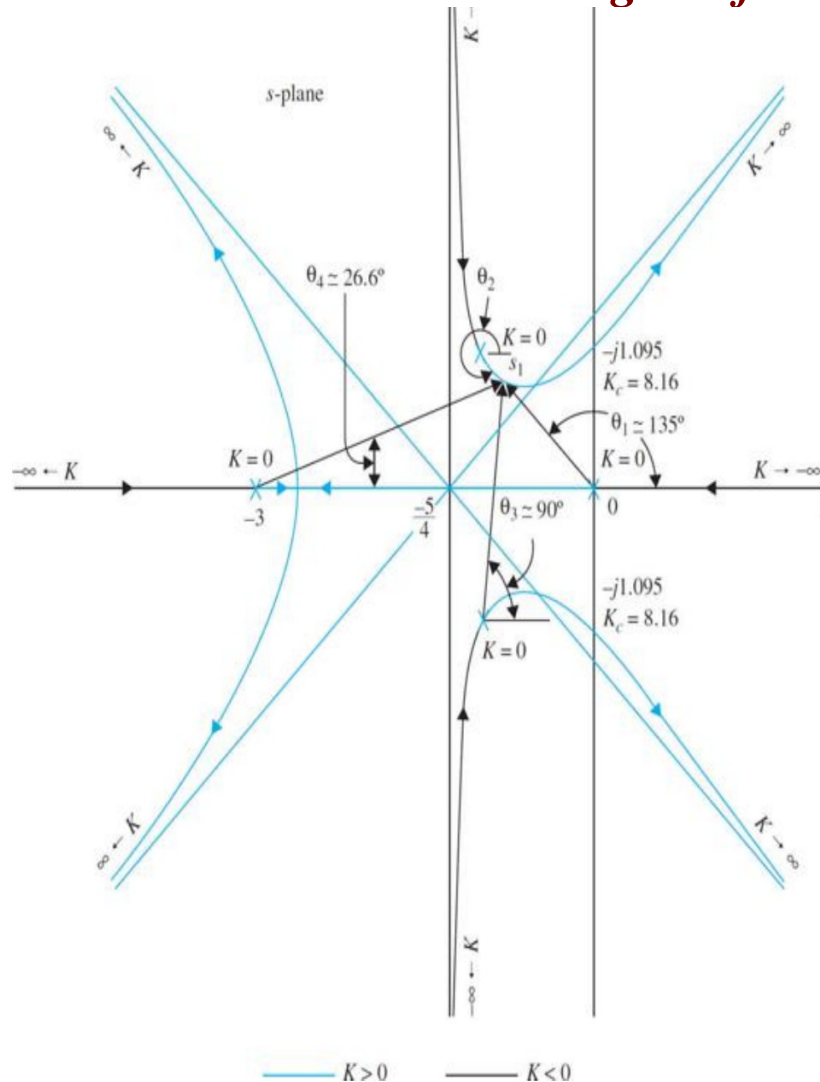
Root loci of $s(s+3)(s^2+2s+2)+K=0$ to illustrate the angles of departure or arrival

$$s(s+3)(s+1+i)(s+1-i)+K=0$$

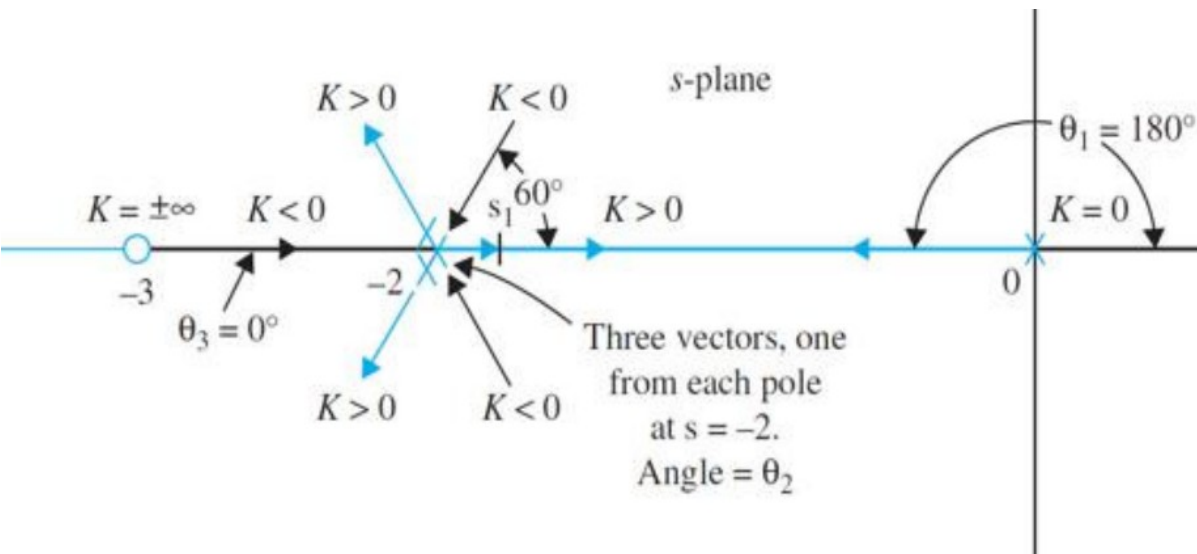
$$\begin{aligned} -\theta_2 &= (2l+1) \times 180^\circ + (135^\circ + 90^\circ + 26.6^\circ) \\ &= -(2 \times -1 + 1) \times 180^\circ + 251.60^\circ \\ &= 71.60^\circ \end{aligned}$$

$$\begin{aligned} \theta_2 &= -71.60^\circ \\ &= 288.40^\circ \end{aligned}$$

rlocus_example_rule6-1.m



Root Locus Plots

Rules 6**Angles of arrival and departure of Root Loci**

Angles of departure and arrival at a third-order pole
Three poles repeated at $s = -2$

$$s(s+2)^3 + K(s+3) = 0$$

$$1 + \frac{K(s+3)}{s(s+2)^3} = 0$$

$$G_1(s)H_1(s) = \frac{s+3}{s(s+2)^3}$$

$$\angle G_1(s)H_1(s) = \theta_{z1} - (\theta_{p1} + 3 \times \theta_{p2}) = (2l+1)\pi$$

$$\begin{aligned} 3 \times \theta_{p2} &= (2l+1) \times \pi + (\pi - \pi) \\ &= \pi, 3\pi, 5\pi \end{aligned}$$

$$\theta_{p2} = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$$

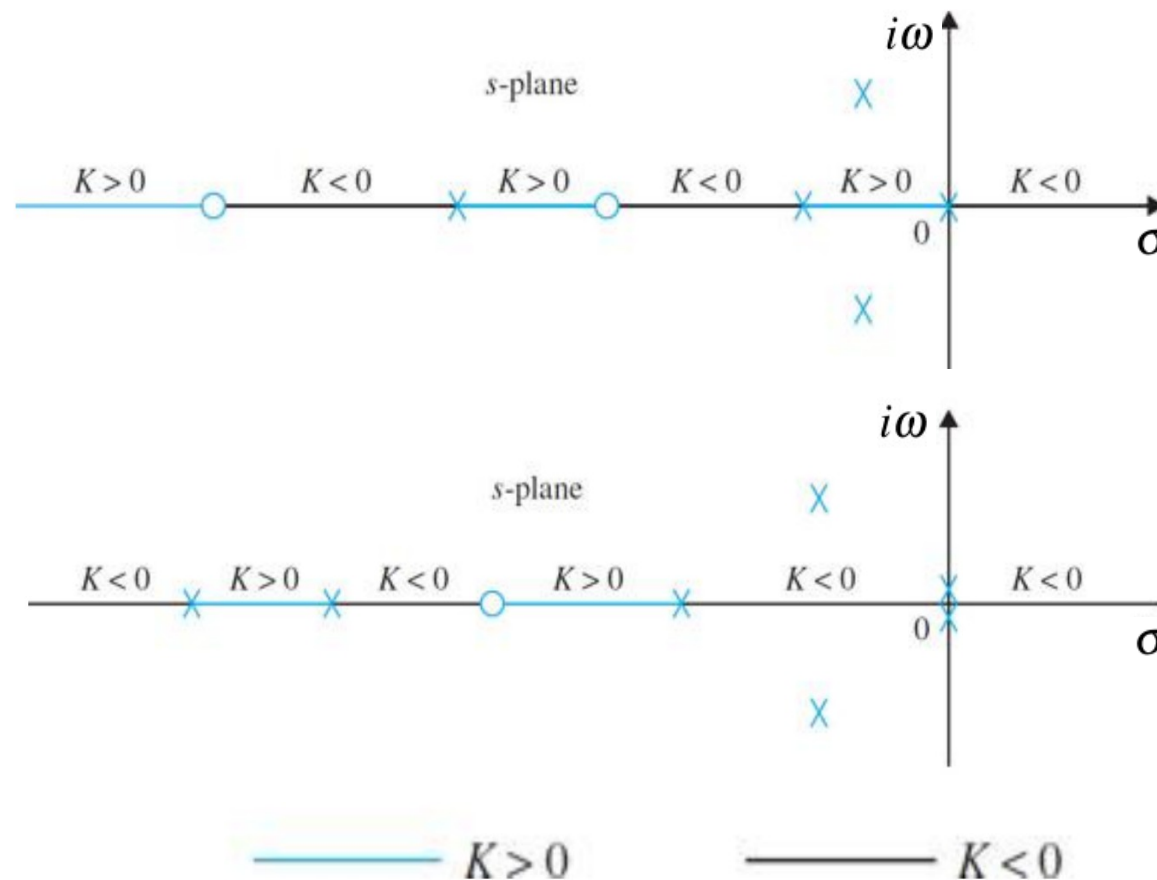
rlocus_example_rule6-2.m



Root Locus Plots

Rules 7

Properties on the real axis – Always to the left of odd number of poles or zeros

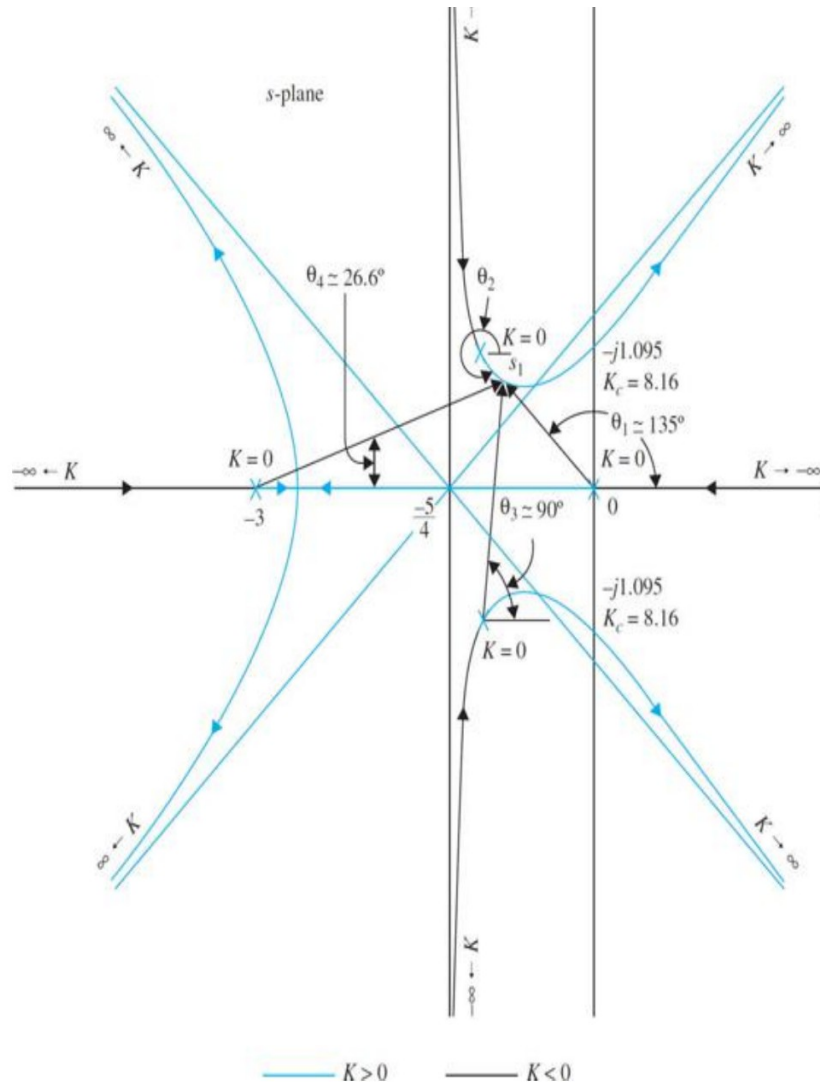




Root Locus Plots

Rules 8

Intersection with the Imaginary Axis



$$s(s+3)(s^2+2s+2)+K=0$$

$$s(s+3)(s^2+2s+2)=-K$$

$$i\omega(i\omega+3)((i\omega)^2+2i\omega+2)=-K$$

$$(-\omega^2+3i\omega)(-\omega^2+2i\omega+2)=-K$$

$$\omega^4-2\omega^3i-2\omega^2-3i\omega^2-6\omega^2+6i\omega=-K$$

$$\omega^4-5\omega^3i-8\omega^2+6\omega i=-K$$

$$\omega^2(\omega^2-8)+\omega(-5\omega^2+6)i=-K$$

$$\omega^2 = \frac{6}{5}$$

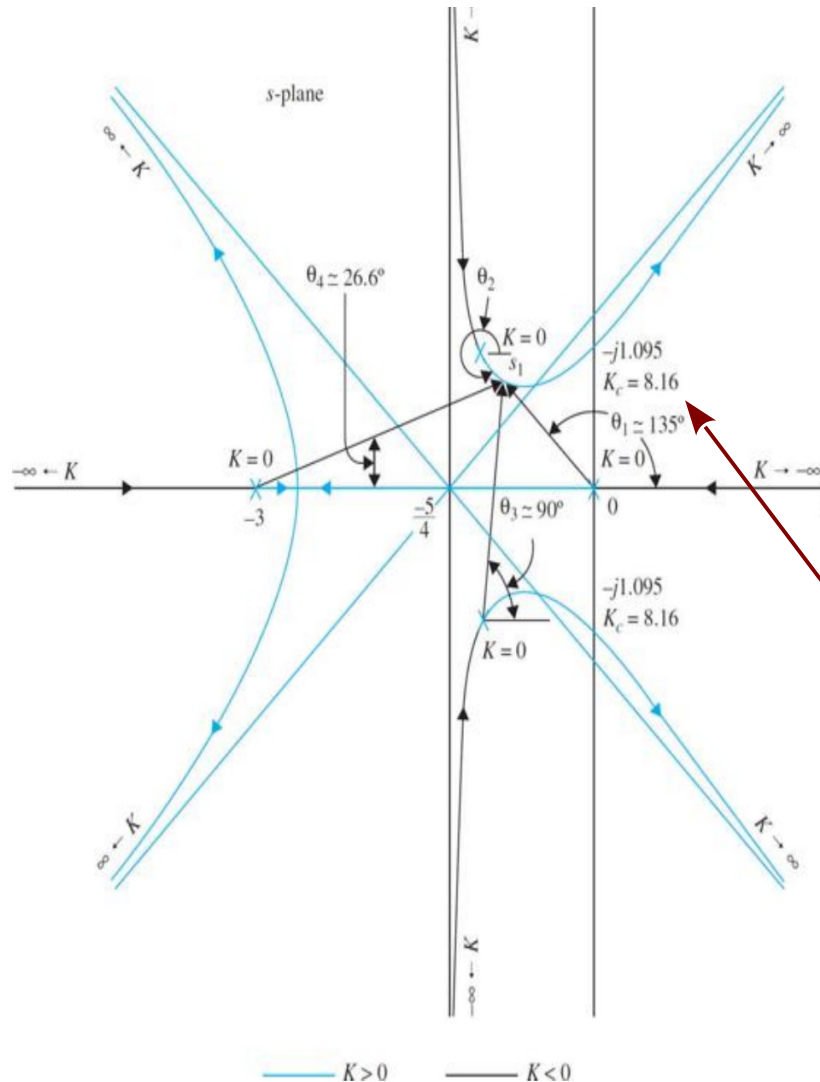
$$\boxed{\omega = \pm \sqrt{\frac{6}{5}}}$$



Root Locus Plots

Rules 8

Intersection with the Imaginary Axis



$$\omega = \pm \sqrt{\frac{6}{5}}$$

$$\omega^2(\omega^2 - 8)$$

$$\frac{6}{5}(\frac{6}{5} - 8) = -K$$

$$\frac{36}{25} - \frac{48}{5} = -K$$

$$K = \frac{48}{5} - \frac{36}{25} = 8.16$$

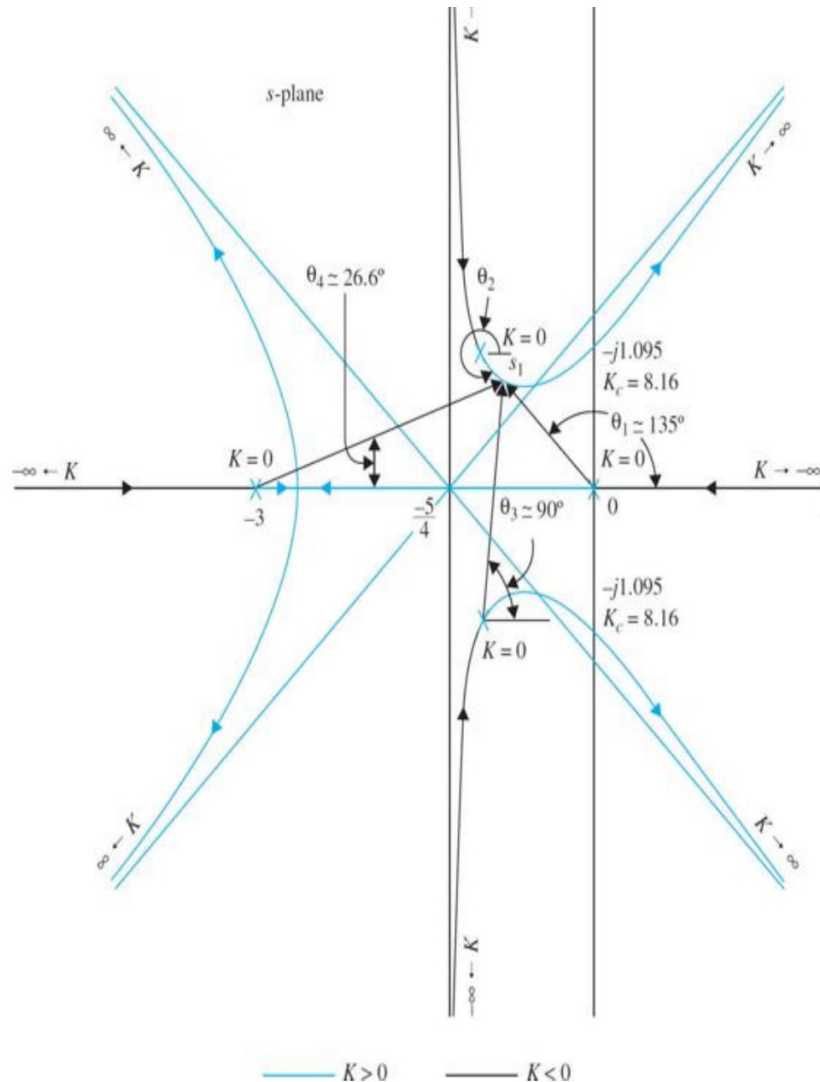
rlocus_example_rule9-1.m
(program rule9-1 also
shows rule 8)



Root Locus Plots

Rules 9

Intersection with the Real Axis



The location where the root locus intersects the real axis is when,

$$\frac{dG_1(s)H_1(s)}{ds} = \frac{d(s(s+3)(s^2+2s+2))}{ds} = 0$$

The value of K at the intersection is computed by ensuring that the solution of the above is used in the original closed loop characteristic equation,

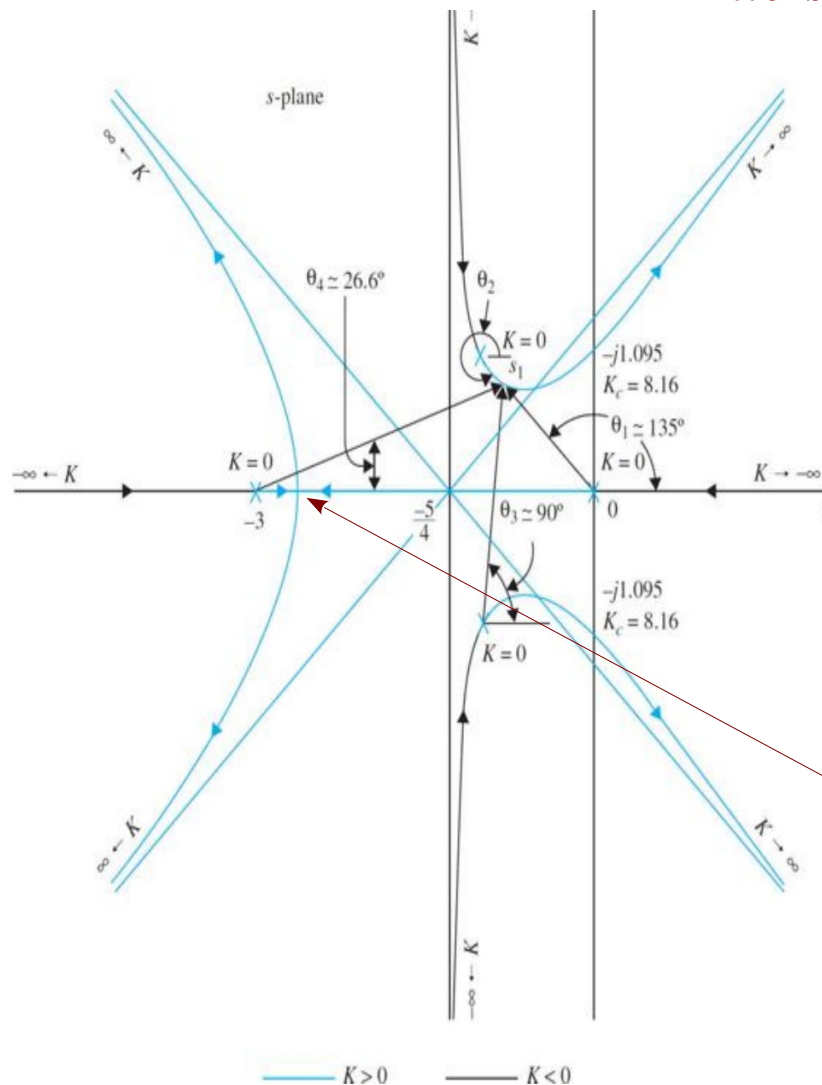
$$1 + KG_1(s)H_1(s) = 0$$



Root Locus Plots

Rules 9

Intersection with the Real Axis



The location where the root locus intersects the real axis is when,

$$\frac{dG_1(s)H_1(s)}{ds} = \frac{d(s(s+3)(s^2+2s+2))}{ds} = 0$$

$$\begin{aligned} s(s+3)(s^2+2s+2) &= (s^2+3s)(s^2+2s+2) \\ &= s^4+2s^3+2s^2+3s^3+6s^2+6s \\ &= s^4+5s^3+8s^2+6s \end{aligned}$$

$$\begin{aligned} \frac{d(s^4+5s^3+8s^2+6s)}{ds} &= 4s^3+15s^2+16s+6 \\ &= 0 \end{aligned}$$

$$s_1 = -2.2886$$

$$s_{2,3} = -0.7307 \pm 0.3486i$$

Intersection point

rlocus_example_rule9-1.m



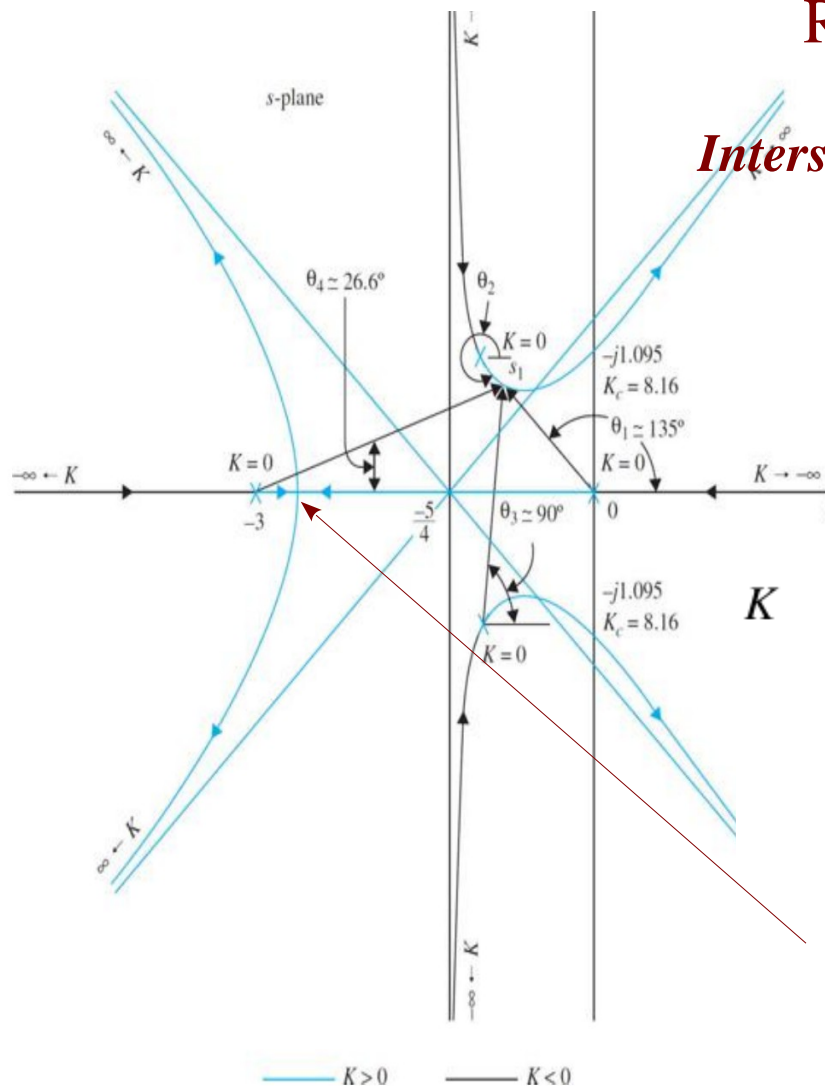
Root Locus Plots

Rules 9

Intersection with the Real Axis

The location where the root locus intersects the real axis is when,

$$1 + KG_1(s)H_1(s) = 0$$



$$\begin{aligned}
 K &= -\frac{1}{s(s+3)(s^2+2s+2)} \\
 &= -\frac{1}{(-2.886)(-2.886+3)((-2.2886)^2+2(-2.886)+2)} \\
 &= \frac{1}{4.33157} \\
 &= 0.23
 \end{aligned}$$



Root Locus Plots

Rules 9

Intersection with the Real Axis

The location where the root locus intersects the real axis is when,

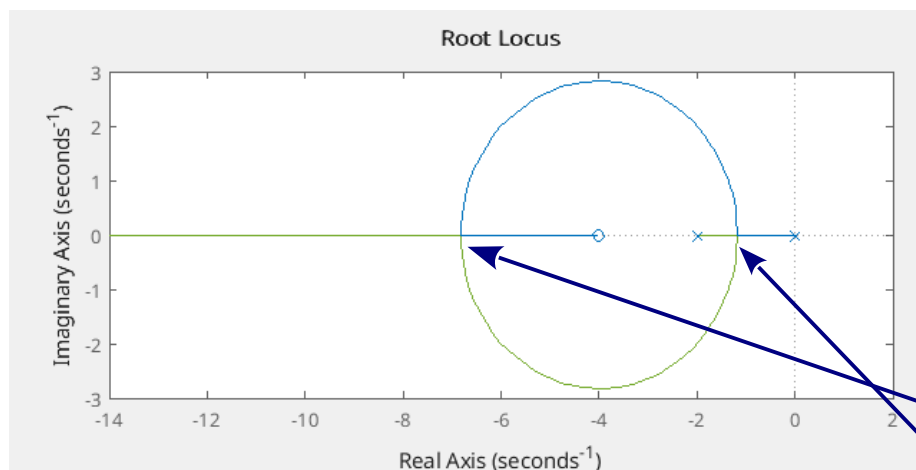
$$\begin{aligned}\frac{dG_1(s)H_1(s)}{ds} &= \frac{d\left(\frac{s+4}{s(s+2)}\right)}{ds} \\ &= \frac{s(s+2) - 2(s+1)(s+4)}{s^2(s+2)^2} \\ &= \frac{s^2 + 2s - 2[s^2 + 5s + 4]}{s^2(s+2)^2} \\ &= \frac{-s^2 - 8s - 8}{s^2(s+2)^2} \\ &= 0\end{aligned}$$

$$\therefore s^2 + 8s + 8 = 0$$

$$s_1 = -1.172 \quad s_2 = -6.6828$$

Intersection points

rlocus_example_rule9-2.m





Root Locus Plots

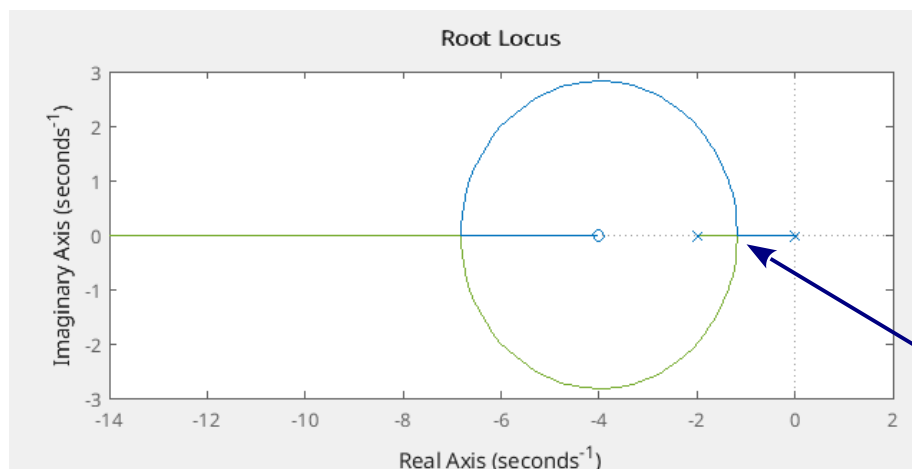
Rules 9

Intersection with the Real Axis

The location where the root locus intersects the real axis is when,

$$1 + K_1 G_1(s_1) H_1(s_1) = 0$$

$$\begin{aligned} K_1 &= -\frac{s_1(s_1 + 2)}{s_1 + 4} \\ &= -\frac{-1.172 \times (-1.172 + 2)}{(-1.172 + 4)} \\ &= \frac{0.97}{2.28} \\ &= 0.426 \end{aligned}$$





Root Locus Plots

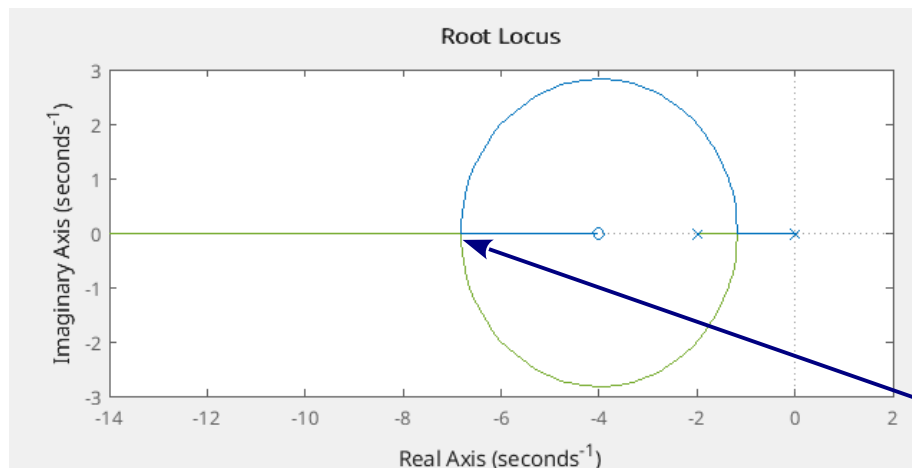
Rules 9

Intersection with the Real Axis

The location where the root locus intersects the real axis is when,

$$1 + K_2 G_1(s_2) H_1(s_2) = 0$$

$$\begin{aligned} K_2 &= -\frac{s_2(s_2 + 2)}{s_2 + 4} \\ &= -\frac{-6.6828 \times (-6.6828 + 2)}{(-6.6828 + 4)} \\ &= \frac{-31.294}{-2.6828} \\ &= 11.665 \end{aligned}$$





Root Locus Plots

Rules 10

Arrival and Departure Angles from the Real Axis

*The n Root Loci arrive and depart
to/from the root axis at $180/n$ degrees*



Root Locus Plots

Rules 11

Root Sensitivity

Root sensitivity tends toward infinity at the break-away points

Break-away points transition between real, repeated, and complex roots

Robust System \longleftrightarrow Low Root Sensitivity

Break-away points may be computed as follows,

$$\begin{aligned} S_K &= \frac{\frac{ds}{s}}{\frac{dK}{K}} \\ &= \frac{K}{s} \frac{ds}{dK} \end{aligned}$$



Bode Plot

Concerned with Steady-State Response of the system
We assume that the transients have died out – stable system

Forcing Function	Particular Solution Form
α (Constant)	C (Constant)
Polynomial of order n in t	Polynomial of order n in t
$\alpha \cos(\Omega t) + \beta \sin(\Omega t)$	$C \cos(\Omega t) + D \sin(\Omega t)$
$\beta e^{\alpha t}$	$C e^{\alpha t}$
$e^{\alpha t} (\alpha \cos(\Omega t) + \beta \sin(\Omega t))$	$e^{\alpha t} (C \cos(\Omega t) + D \sin(\Omega t))$
<i>Product of the Above</i>	<i>Product of the Above</i>

For an oscillatory forcing function, steady state frequency stays the same, different amplitude and phase



Bode Plot

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \sin(3t)$$

$$y(0) = -1$$

$$\dot{y}(0) = 0$$

$$y(t) = \underbrace{\frac{10}{13}e^{-2t} - \frac{17}{10}e^{-t}}_{\text{Transience}} - \frac{9}{130}\cos(3t) - \frac{7}{130}\sin(3t)$$

frequency stays the same,
different amplitude and phase

$$\ddot{y}(t) + 4\dot{y}(t) + 4y(t) = \cos(4t)$$

$$y(0) = 1$$

$$\dot{y}(0) = 1$$

$$y(t) = \underbrace{\frac{103}{100}e^{-2t} + \frac{29}{10}te^{-2t}}_{\text{Transience}} - \frac{3}{100}\cos(4t) + \frac{1}{25}\sin(4t)$$

frequency stays the same,
different amplitude and phase

Transience

Bode Plot

$$\ddot{y}(t) + \dot{y}(t) + y(t) = 1 + \cos(6t)$$

$$y(0) = 0$$

$$\dot{y}(0) = 0$$

$$y(t) = \boxed{1 - \frac{35}{1261} \cos(6t) + \frac{6}{1261} \sin(6t)} - \frac{2}{1261} e^{-t/2} \left(613 \cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1262}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}t\right) \right)$$

frequency stays the same,
different amplitude and phase

Transience

$$\ddot{y}(t) + y(t) = 1 + e^{(-2t)}$$

$$y(0) = 0$$

$$\dot{y}(0) = 0$$

$$y(t) = \boxed{1 + \frac{1}{5} e^{-2t}} - \frac{6}{5} \cos(t) + \frac{2}{5} \sin(t)$$

Transience



Bode Plot

The objective:

$$C \cos(\omega t) + D \sin(\omega t) = A \sin(\omega t + \phi)$$

Note that,

$$\sin(x + y) = \sin(x) \cos(y) + \cos(x) \sin(y)$$

Therefore,

$$C \cos(\omega t) + D \sin(\omega t) = A \cos(\phi) \sin(\omega t) + A \sin(\phi) \cos(\omega t)$$

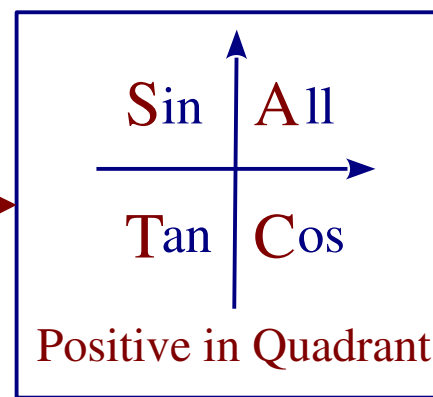
Since sine and cosine are linearly independent, the coefficients must match.

$$C = A \sin(\phi)$$

$$D = A \cos(\phi)$$

$$\begin{aligned} \sin(\phi) &= \frac{C}{A} \\ \cos(\phi) &= \frac{D}{A} \end{aligned}$$

Use to find quadrant





Bode Plot

Special case: If $D = 0$, then there is only cos, in which case,

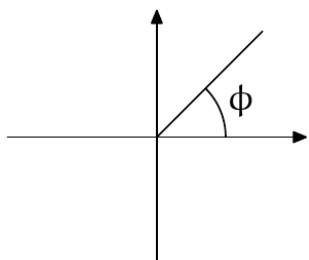
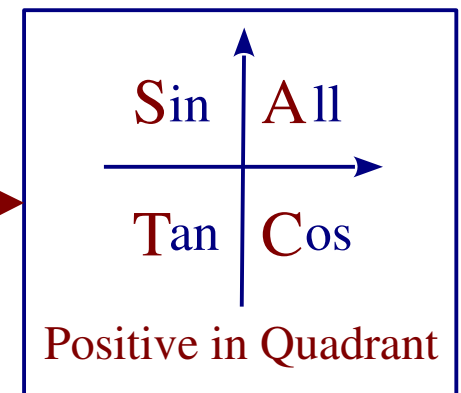
$$\begin{aligned} A &= C \\ \phi &= \frac{\pi}{2} \end{aligned}$$

Otherwise,

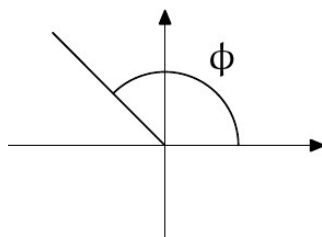
$$\begin{aligned} A &= \sqrt{C^2 + D^2} \\ \tan(\phi) &= \frac{C}{D} \end{aligned}$$

$$\begin{aligned} \sin(\phi) &= \frac{C}{A} \\ \cos(\phi) &= \frac{D}{A} \end{aligned}$$

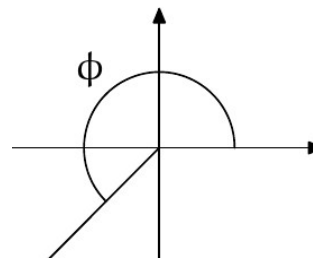
Use to find quadrant



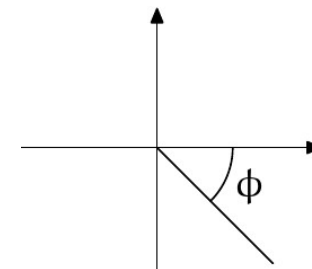
$$\phi = \tan^{-1} \left(\frac{C}{D} \right)$$



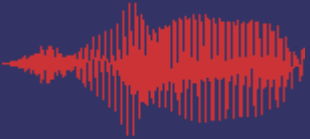
$$\phi = \tan^{-1} \left(\frac{C}{D} \right) + \pi$$



$$\phi = \tan^{-1} \left(\frac{C}{D} \right) + \pi$$



$$\phi = \tan^{-1} \left(\frac{C}{D} \right)$$



Homework 10

See Courseworks

Intensity

Intensity – Power per unit area. $\frac{W}{m^2}$ ($\frac{J/s}{m^2}$ or $\frac{N}{ms}$)

$$\begin{aligned} \text{Unit of } \zeta &= \left(\frac{kg}{m^3} \right) \left(\frac{m}{s} \right) \left(\frac{s}{s} \right) \\ &= \frac{kgm}{s^2} \frac{s}{m^3} \\ &= \frac{Ns}{m^3} \end{aligned}$$

$$\begin{aligned} I &= \frac{P^2}{\zeta} \\ &= \frac{P^2}{\rho c} \end{aligned}$$

pressure differential

specific acoustic impedance

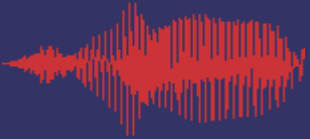
$$\zeta = \rho c$$

Dry air at 1 atm and 20°C

$$\left\{ \begin{array}{l} \rho = 1.204 \frac{kg}{m^3} \\ c = 343.2 \frac{m}{s} \end{array} \right. \Rightarrow \zeta = 413.21 \frac{Ns}{m^3}$$

$$\begin{aligned} P_0 &= 2 \times 10^{-5} \frac{N}{m^2} \quad (\text{RMS of } P \text{ for } 1kHz) \\ I_0 &= \frac{P_0^2}{\zeta} \approx 10^{-12} \frac{W}{m^2} \quad (\text{Intensity } 1kHz) \end{aligned}$$

Pressure and Intensity Threshold to hear 1 kHz



Relative Intensity

$$\begin{aligned} I_r &= 10 \log \left(\frac{I}{I_0} \right) \quad \text{Dimensionless (in dB)} \\ &= 10 \log \left(\frac{P^2}{P_0^2} \right) \\ &= 20 \log \left(\frac{P}{P_0} \right) \end{aligned}$$

Maximum Intensity – 3 kHz – 4 kHz – around the resonance freq. of the ear canal

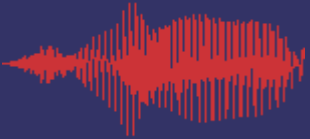
Comfort Range in Relative Intensity – 10 dB – 80 dB

Quiet Library: 40 – 60 dB

Loud Rock Concert: 110 dB

Average Relative Intensity: 58 dB

Male Speakers are about 4.5 dB louder than females



Fourier Series Expansion

We can write any function in terms of a set of sinusoidal functions.

Fourier Series Expansion: Any periodic function may be written in terms of an infinite series of exponential functions (or sines and cosines)

Defined for the period: $[-T, T]$

$$h(t) \approx \sum_{n=-\infty}^{\infty} c_n e^{i\left(\frac{n\pi t}{T}\right)} \quad c_n = \frac{1}{2T} \int_{-T}^T h(t) e^{-i\left(\frac{n\pi t}{T}\right)} dt$$



Parseval's Theorem

Theorem 24.28 (Parseval's Theorem – Fourier Transform). *The Total power in a signal is the same when computed in the time or Frequency domain. In other words, the Total Power P is given by,*

$$\begin{aligned}\mathcal{P} &\triangleq \int_{-\infty}^{\infty} |h(t)|^2 dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega\end{aligned}$$



Parseval's Theorem: *proof* (Complex Fourier Transform)

$$\mathcal{P} \triangleq \int_{-\infty}^{\infty} |h(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega$$

$$\begin{aligned} \overline{se^{i\theta}} &= \overline{(\sigma + i\omega)e^{i\theta}} \\ &= \overline{(\sigma + i\omega)(\cos(\theta) + i\sin(\theta))} \\ &= \overline{(\sigma \cos(\theta) - \omega \sin(\theta)) + i(\sigma \sin(\theta) + \omega \cos(\theta))} \\ &= (\sigma \cos(\theta) - \omega \sin(\theta)) - i(\sigma \sin(\theta) + \omega \cos(\theta)) \\ &= \cos(\theta)(\sigma - i\omega) - \sin(\theta)(\omega + i\sigma) \\ &= \cos(\theta)(\sigma - i\omega) - i\sin(\theta)(\sigma - i\omega) \\ &= (\sigma - i\omega)(\cos(\theta) - i\sin(\theta)) \\ &= \bar{s}e^{-i\theta} \end{aligned}$$

Then,

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)e^{i\omega t} d\omega \quad \longleftrightarrow \quad \overline{h(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{H(\omega)}e^{-i\omega t} d\omega$$



Plancherel's Theorem: *proof* (Complex Fourier Transform)

$$\hat{\mathcal{P}} \triangleq \int_{-\infty}^{\infty} g(t) \overline{h(t)} dt$$

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{i\omega t} d\omega$$

$$\overline{h(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{H(\omega)} e^{-i\omega t} d\omega$$

$$\hat{\mathcal{P}} = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\omega_1) e^{i\omega_1 t} d\omega_1 \int_{-\infty}^{\infty} \overline{H(\omega_2)} e^{-i\omega_2 t} d\omega_2 dt$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\omega_1) e^{i\omega_1 t} \overline{H(\omega_2)} e^{-i\omega_2 t} d\omega_2 d\omega_1 dt$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\omega_1) \overline{H(\omega_2)} e^{i\omega_1 t} e^{-i\omega_2 t} d\omega_2 d\omega_1 dt$$

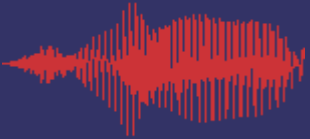
$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\omega_1) \overline{H(\omega_2)} \int_{-\infty}^{\infty} e^{i\omega_1 t} e^{-i\omega_2 t} dt d\omega_2 d\omega_1$$

$$\int_{-\infty}^{\infty} e^{i\omega_1 t} e^{-i\omega_2 t} dt = \begin{cases} 0 & \forall \omega_1 \neq \omega_2 \\ 2\pi & \text{for } \omega_1 = \omega_2 \end{cases} \quad (\text{Orthogonality of Exponential Functions})$$

$$\hat{\mathcal{P}} = \int_{-\infty}^{\infty} g(t) \overline{h(t)} dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) \overline{H(\omega)} d\omega$$

(Plancherel's Theorem)
(Swiss Mathematician:
Michel Plancherel)



Parseval's Theorem: *proof* (Complex Fourier Transform)

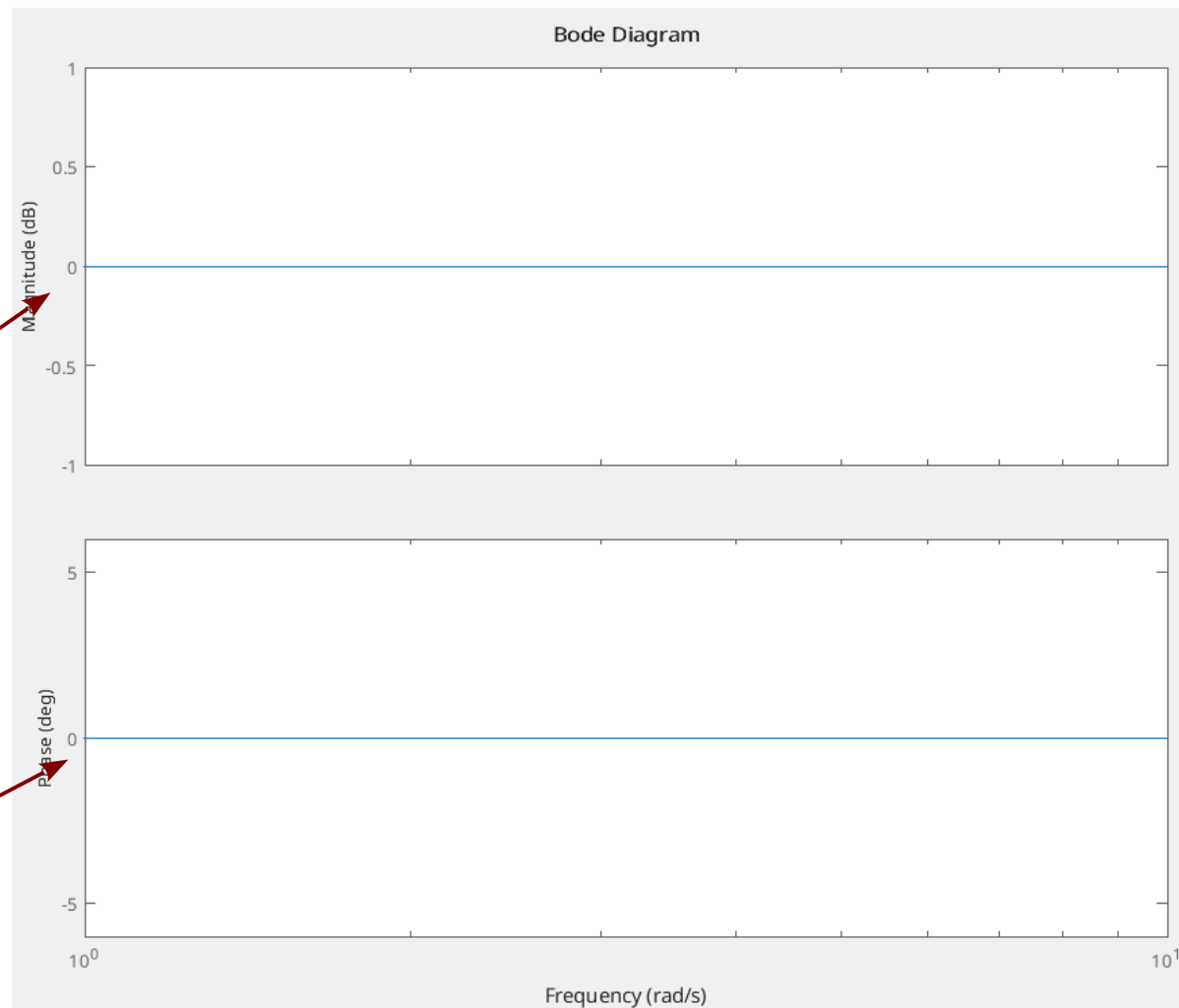
$$\begin{aligned}\hat{\mathcal{P}} &= \int_{-\infty}^{\infty} g(t) \overline{h(t)} dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) \overline{H(\omega)} d\omega\end{aligned}\quad \text{(Plancherel's Theorem)}$$

Special case where $g(t) = h(t)$

$$\begin{aligned}\mathcal{P} &= \int_{-\infty}^{\infty} |h(t)|^2 dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega\end{aligned}\quad \text{(Parseval's Theorem)}$$



Bode Plot



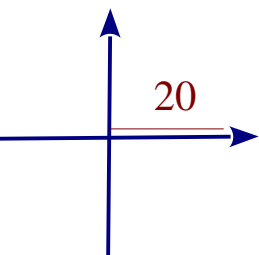
$20 \log 1 = 0$

0 phase shift for
positive gains

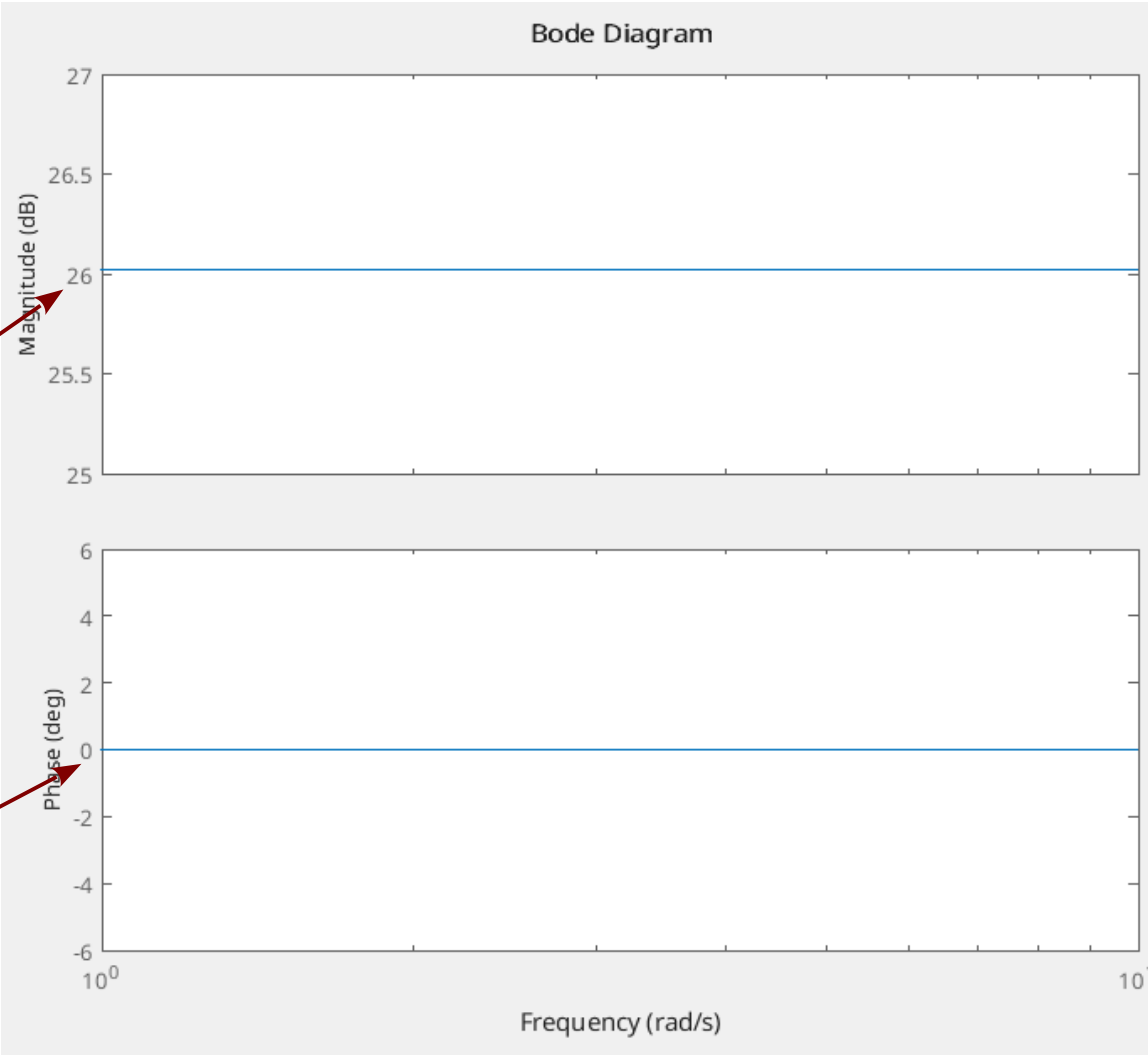
$$G = 1$$



Bode Plot



$20 \log 20$



0 phase shift for positive gains

$G = 20$

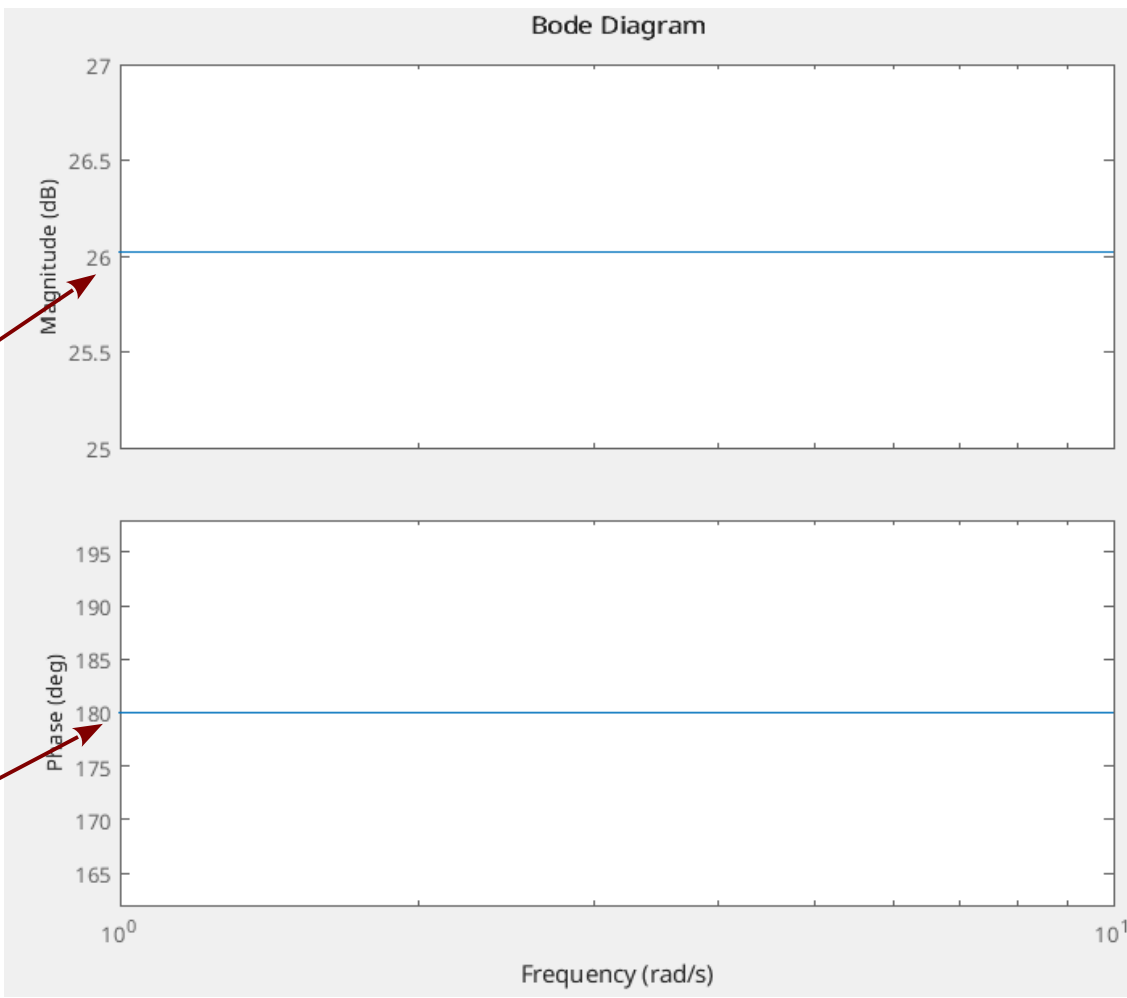


Bode Plot

-20

$20 \log 20$

Opposite (180 degree phase)
For negative gains



$$G = -20$$

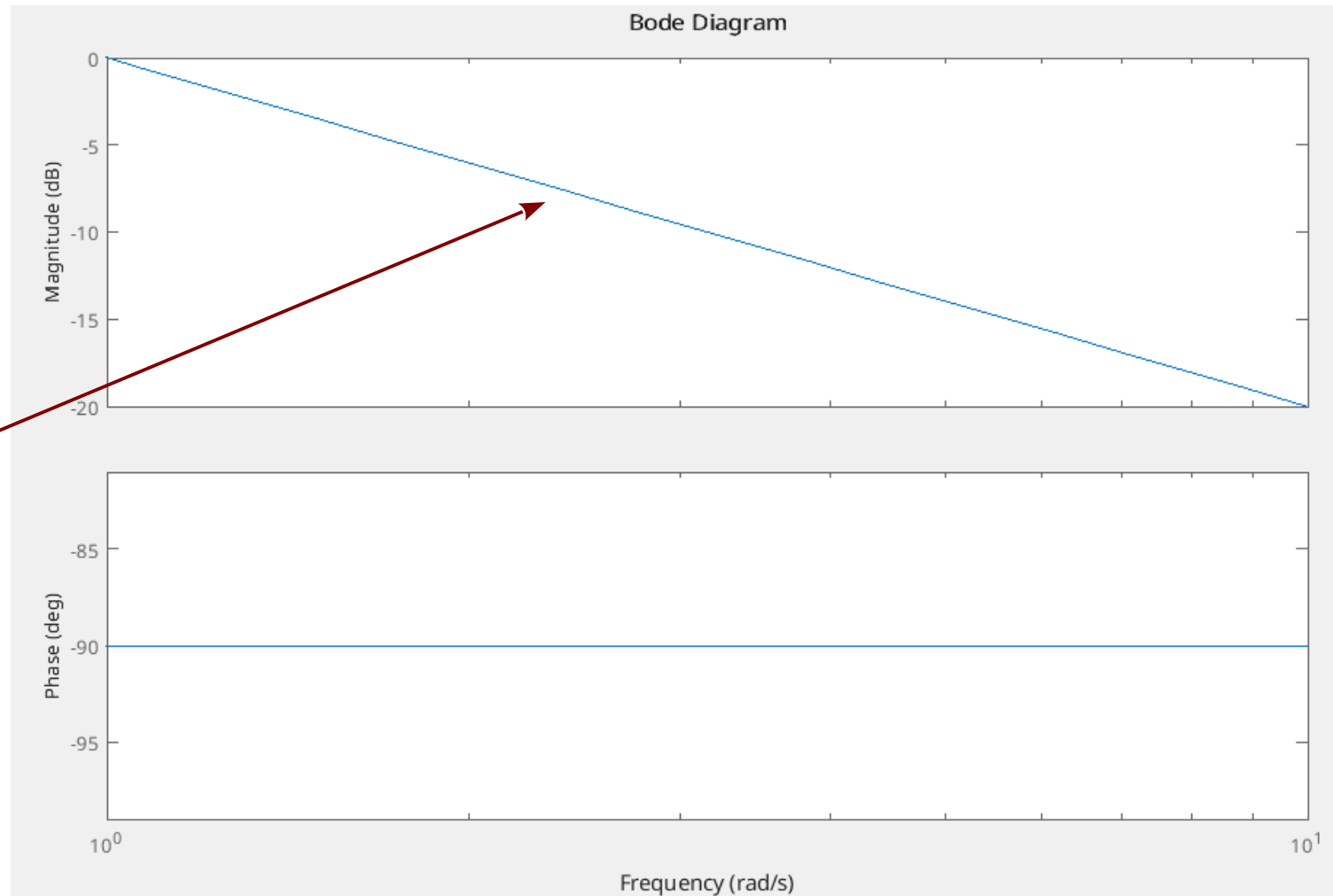


Bode Plot (Integrator)

$$\begin{aligned} G(i\omega) &= \frac{1}{i\omega} \\ &= -\frac{1}{\omega}i \\ |G(i\omega)| &= \left| -\frac{1}{\omega}i \right| \\ &= \frac{1}{\omega} \end{aligned}$$

$$\angle G(i\omega) = -90^\circ$$

Slope=20db/decade



$$G = \frac{1}{s}$$



Bode Plot

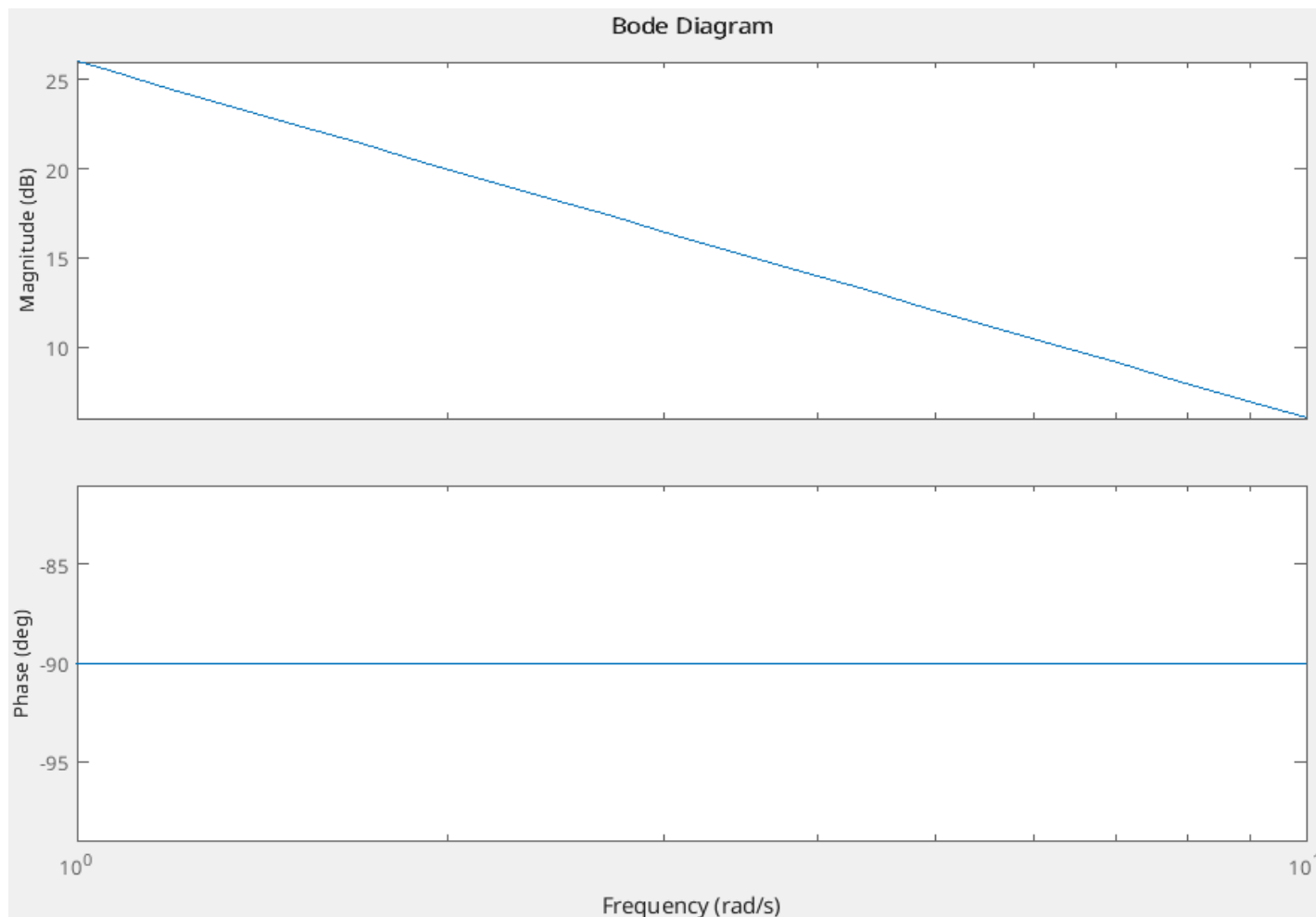
$$\begin{aligned} G(s) &= \frac{20}{s} \\ &= 20 \frac{1}{s} \end{aligned}$$

$$|G(i\omega)| = |20| \left| -\frac{1}{\omega} i \right|$$

$$20 \log(|20| \frac{1}{\omega}) =$$

$$20 \log(|20|) + 20 \log(\frac{1}{\omega})$$

$$\begin{aligned} \angle G(i\omega) &= \angle 20 + \angle -\frac{1}{\omega} i \\ &= 0^\circ - 90^\circ \\ &= -90^\circ \end{aligned}$$



$$G(s) = \frac{20}{s}$$