# Introduction to Continuous Control Systems

## EEME E3601

Week 11

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# Asymptotic Stability (Testing)

**Corollary**

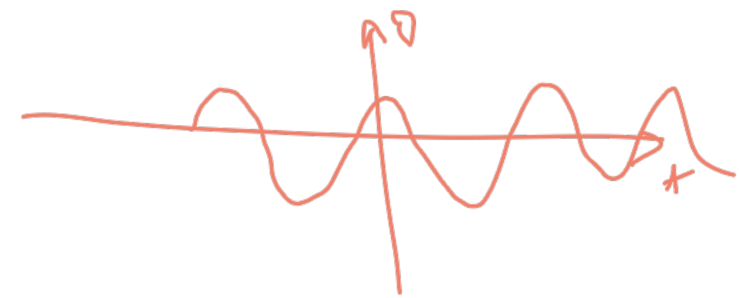
*Second, third, and fourth order polynomials are stable if and only if,*

**A.** *For a second order polynomial*

$$P(s) = s^2 + p_1 s + p_0$$

$p_1$ *and* $p_0$ *are both positive.*

$$\mathcal{L}^{-1}\left\{(s^2 + p_0)\,Y(s)\right\} = \ddot{y}(t) + \overset{\omega_n^2}{p_0}\, y(t) = 0$$



**B.** *For a third order polynomial,*

$$P(s) = s^3 + p_2 s^2 + p_1 s + p_0$$

$p_2, p_1, p_0 > 0$ *and* $p_2 p_1 > p_0$.

**C.** *For a fourth order polynomial,*

$$P(s) = s^4 + p_3 s^3 + p_2 s^2 + p_1 s + p_0$$

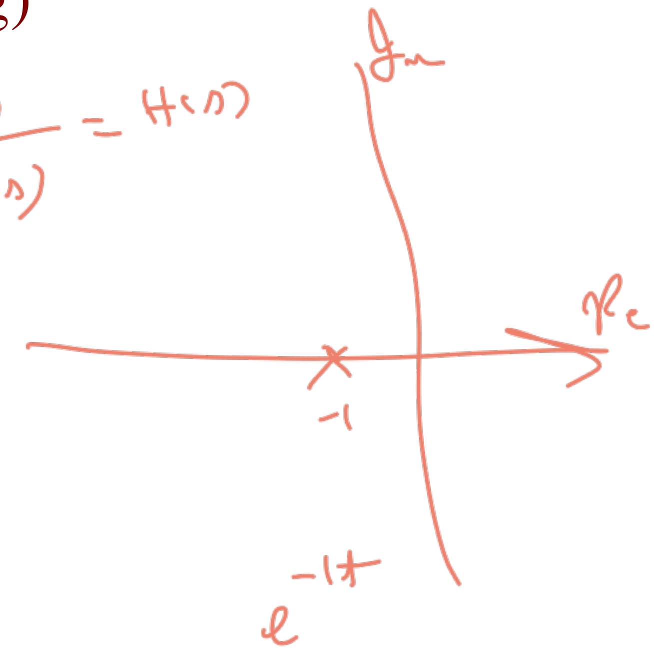
$p_3, p_2, p_1, p_0 > 0$ *and* $p_3 p_2 p_1 > p_3^2 p_0 + p_1^2$.

*Proof.* The proof is easily done by applying the Liénard-Chipart Theorem directly on these specific polynomials. □

# Asymptotic Stability (Testing)

$$\frac{Y(s)}{U(s)} = H(s)$$

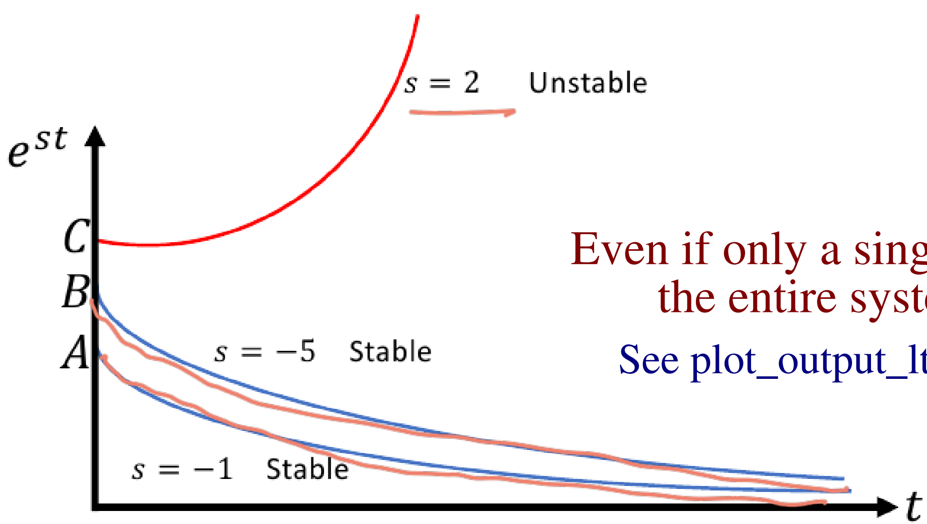
$$H(s) = \frac{1}{s+1}\frac{1}{s+5}\frac{1}{s-2}$$




Partial Fraction Expansion:

$$H(s) = \frac{1}{s+1}\frac{1}{s+5}\frac{1}{s-2} = \underbrace{\frac{A}{s+1}}_{s=-1} + \underbrace{\frac{B}{s+5}}_{s=-5} + \underbrace{\frac{C}{s-2}}_{s=2}$$

Stable Stable Unstable




Even if only a single root is unstable, the entire system is unstable

See plot_output_lti_unstable2.m

# Asymptotic Stability (Testing)

$$H(s) = \frac{1}{s^4 + 9s^3 - 2s^2 + s + 3}$$

Different sign: unstable system

See plot_output_lti_unstable2.m

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## Asymptotic Stability (Routh Criterion)

**Theorem** (Routh Criterion)

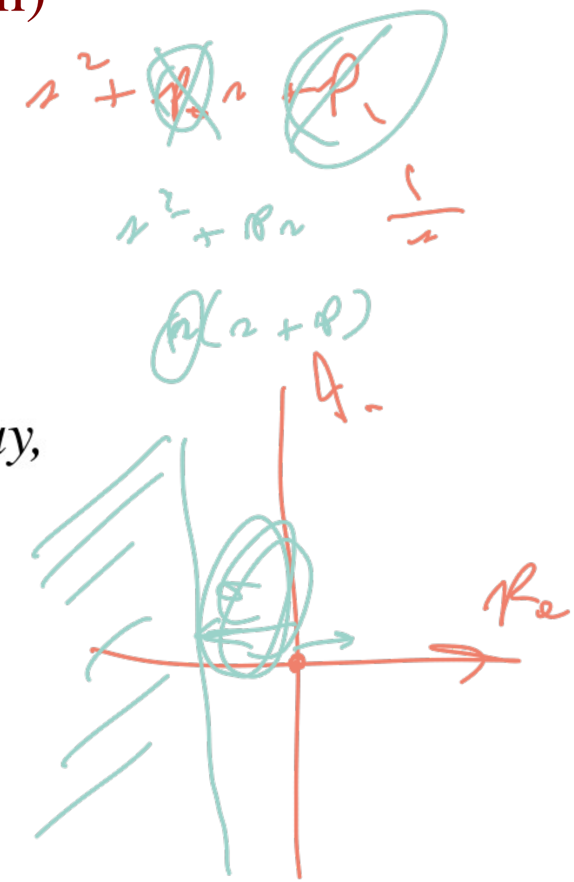
*The number of roots of $P(s)$,*

$$P(s) = s^n + p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \cdots + p_0 = 0$$

*with strictly positive real parts is equal to the number of sign changes in the first column of the Routh array,*

| Row | | | | |
|---|---|---|---|---|
| 1 | $1$ | $p_{n-2}$ | $p_{n-4}$ | $\cdots$ |
| 2 | $p_{n-1}$ | $p_{n-3}$ | $p_{n-5}$ | $\cdots$ |
| 3 | $b_{3,n-1}$ | $b_{3,n-3}$ | $b_{3,n-5}$ | $\cdots$ |
| 4 | $b_{4,n-1}$ | $b_{4,n-3}$ | $b_{4,n-5}$ | $\cdots$ |
| 5 | $b_{5,n-1}$ | $b_{5,n-3}$ | $b_{5,n-5}$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |
| $n+1$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |

Routh Array Table



# Asymptotic Stability (Routh Criterion)

| Row | | | | |
|---|---|---|---|---|
| 1 | 1 | $p_{n-2}$ | $p_{n-4}$ | $\cdots$ |
| 2 | $p_{n-1}$ | $p_{n-3}$ | $p_{n-5}$ | $\cdots$ |
| 3 | $b_{3,n-1}$ | $b_{3,n-3}$ | $b_{3,n-5}$ | $\cdots$ |
| 4 | $b_{4,n-1}$ | $b_{4,n-3}$ | $b_{4,n-5}$ | $\cdots$ |
| 5 | $b_{5,n-1}$ | $b_{5,n-3}$ | $b_{5,n-5}$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |
| $n+1$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |

Routh Array Table

*where* $p_{n-1} \triangleq 0 \;\; \forall \;\; i > n$.

*Also, the coefficients of the* $j^{th}$ *row for* $j \in \{3,4,\cdots,n+1\}$ *are given in terms of the coefficients in the following relations,*

Stable only if all these coefficients are strictly positive

$$b_{3,n-1} = -\frac{1}{p_{n-1}}\begin{vmatrix} 1 & p_{n-2} \\ p_{n-1} & p_{n-3} \end{vmatrix}$$

$$b_{3,n-3} = -\frac{1}{p_{n-1}}\begin{vmatrix} 1 & p_{n-4} \\ p_{n-1} & p_{n-5} \end{vmatrix}$$

$$b_{3,n-5} = -\frac{1}{p_{n-1}}\begin{vmatrix} 1 & p_{n-6} \\ p_{n-1} & p_{n-7} \end{vmatrix}$$

$$b_{4,n-1} = -\frac{1}{b_{3,n-1}}\begin{vmatrix} p_{n-1} & p_{n-3} \\ b_{3,n-1} & b_{3,n-3} \end{vmatrix}$$

$$b_{4,n-3} = -\frac{1}{b_{3,n-1}}\begin{vmatrix} p_{n-1} & p_{n-5} \\ b_{3,n-1} & b_{3,n-5} \end{vmatrix}$$

$$b_{4,n-5} = -\frac{1}{b_{3,n-1}}\begin{vmatrix} p_{n-1} & p_{n-7} \\ b_{3,n-1} & b_{3,n-7} \end{vmatrix}$$

$$b_{5,n-1} = -\frac{1}{b_{4,n-1}}\begin{vmatrix} b_{3,n-1} & b_{3,n-3} \\ b_{4,n-1} & b_{4,n-3} \end{vmatrix}$$

$$b_{5,n-3} = -\frac{1}{b_{4,n-1}}\begin{vmatrix} b_{3,n-1} & b_{3,n-5} \\ b_{4,n-1} & b_{4,n-5} \end{vmatrix}$$

$$b_{5,n-5} = -\frac{1}{b_{4,n-1}}\begin{vmatrix} b_{3,n-1} & b_{3,n-7} \\ b_{4,n-1} & b_{4,n-7} \end{vmatrix}$$

$$\vdots$$

# Asymptotic Stability (Routh Criterion)

*Easy way to remember*

| | | | | | |
|---|---|---|---|---|---|
| 1 | $s^4$ | $a_5$ | $a_3$ | $a_1$ |
| 2 | $s^3$ | $a_4$ | $a_2$ | 0 |
| $-3$ | $s^2$ | $A$ | $B$ | 0 |
| 7 | $s^1$ | $C$ | 0 | |
| 2 | $s^0$ | $D$ | | |

*sign change = 1 root in the RHP*

*sign change = 1 root in the RHP*

*Total of 2 roots in the RHP*

*Negative of determinant*

$$A = \frac{a_4 a_3 - a_5 a_2}{a_4}$$ *scale*

$$B = \frac{a_4 a_1 - a_5 0}{a_4}$$

$$C = \frac{A a_2 - a_4 B}{A}$$

$$D = \frac{CB - A0}{C}$$

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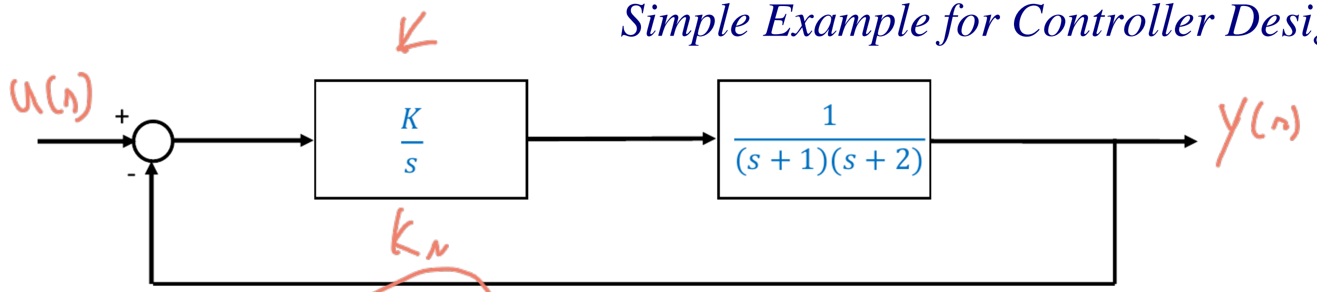
# Asymptotic Stability (Routh Criterion)

*Simple Example for Controller Design*




$$
\begin{aligned}
\frac{Y(s)}{U(s)} &= \frac{C(s)G(s)}{1+C(s)G(s)} \\
&= \frac{\frac{K}{s(s+1)(s+2)}}{1+\frac{K}{s(s+1)(s+2)}} \\
&= \frac{\frac{K}{\cancel{s(s+1)(s+2)}}}{\frac{s(s+1)(s+2)+K}{\cancel{s(s+1)(s+2)}}} \\
&= \frac{K}{s(s+1)(s+2)+K}
\end{aligned}
$$

*Characteristic Equation*

$$s(s+1)(s+2)+K=0 \longrightarrow s^3+3s^2+2s+K=0$$

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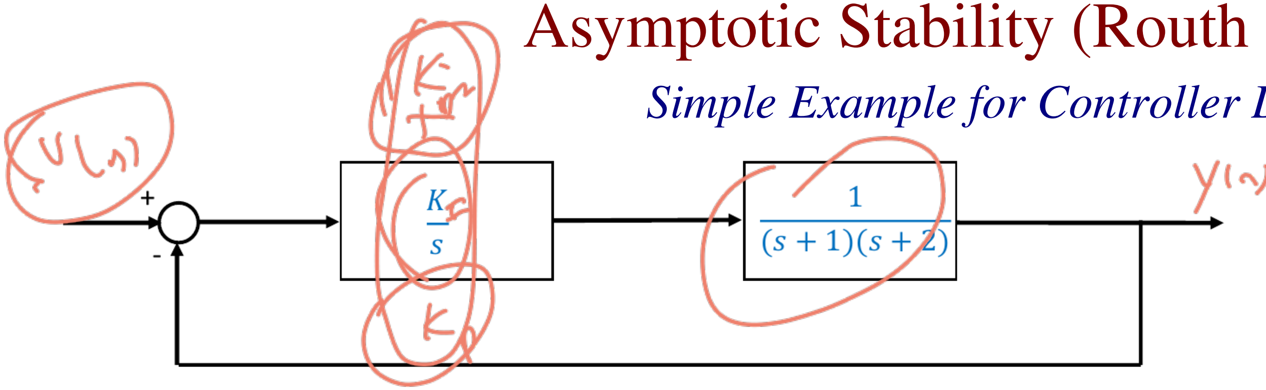
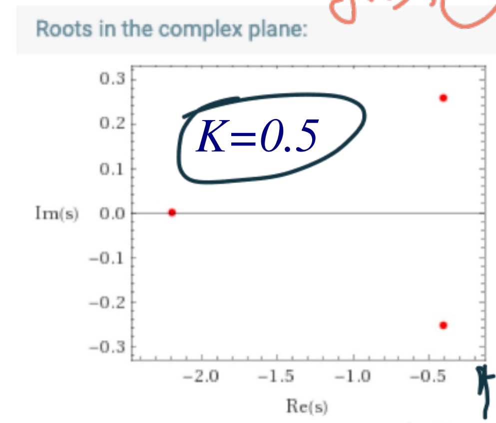
# Asymptotic Stability (Routh Criterion)

## *Simple Example for Controller Design*

$s^2+3s+2$

$\ddot{y}(t)+3\dot{y}(t)+2y=0$



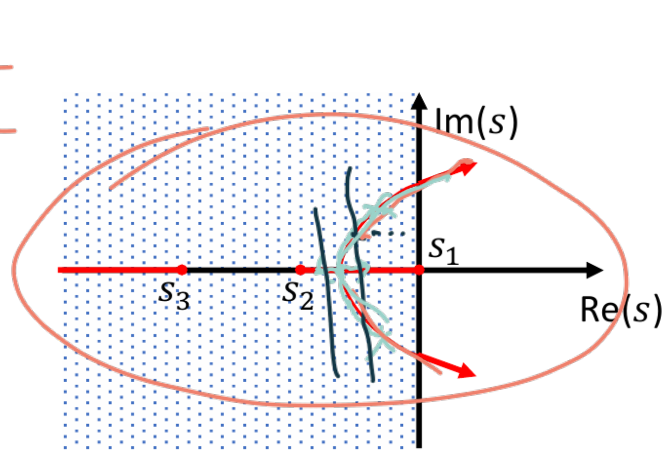

$\omega_n = \sqrt{2}$

$2\zeta\omega_n$

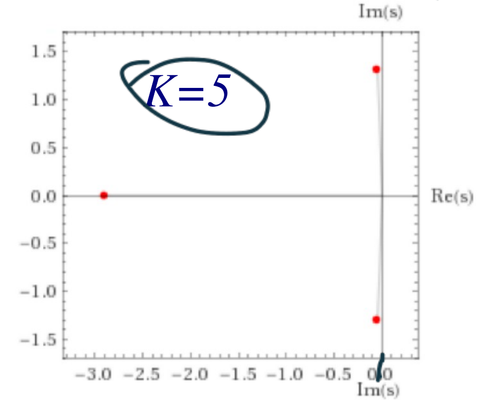
$$s(s+1)(s+2)+K=0 \longrightarrow s^3+3s^2+2s+K=0$$

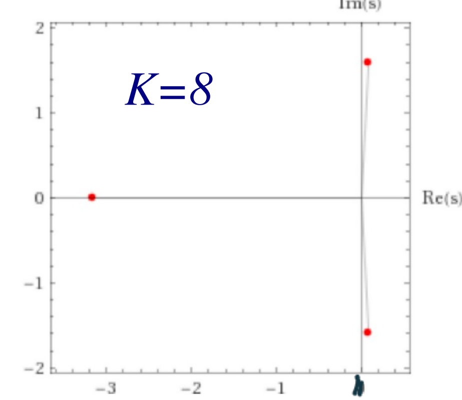
| | | | |
|---|---|---|---|
| $s^3$ | 1 | 2 | 0 |
| $s^2$ | 3 | $K$ | 0 |
| $s^1$ | $\frac{6-K}{3}$ | 0 | 0 |
| $s^0$ | $K$ | | |

ranging $\underbrace{0 < K < 6}_{\text{Range for stability}}$




*Root Locus as 0 < K < 6*



# Asymptotic Stability (Routh Criterion)

| Row | | | | |
|---|---|---|---|---|
| 1 | 1 | $p_{n-2}$ | $p_{n-4}$ | $\cdots$ |
| 2 | $p_{n-1}$ | $p_{n-3}$ | $p_{n-5}$ | $\cdots$ |
| 3 | $b_{3,n-1}$ | $b_{3,n-3}$ | $b_{3,n-5}$ | $\cdots$ |
| 4 | $b_{4,n-1}$ | $b_{4,n-3}$ | $b_{4,n-5}$ | $\cdots$ |
| 5 | $b_{5,n-1}$ | $b_{5,n-3}$ | $b_{5,n-5}$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |
| $n+1$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |

Routh Array Table

two special cases where some intervention is necessary to come to a conclusion

1. **Case:** *There is a 0 in the first column at some row, but there are some nonzero values in the later columns of the same row.*

2. **Case:** *There is a row of all zeros.*

# Asymptotic Stability (Routh Criterion)

1. **Case:** *There is a* $0$ *in the first column at some row, but there are some nonzero values in the later columns of the same row.*

**Procedure:**

- *Assumption: there are no pure imaginary roots that are moved to the right-half plane.*
- *Replace the* $0$ *with a very small number,* $\varepsilon$
- *Continue with the computation of the rest of the coefficients.*
- *Take the limit as* $\varepsilon \to 0$.
- *Count the number of sign changes like the nominal case and the number of sign chages gives the number of roots with real parts in the right half plane.*
- *In case thre is a breakdown in the procedue, factor out all the factors of the form* $(s^2 + \omega^2)$ *and start over.*

## Asymptotic Stability (Routh Criterion)

2. **Case:** *There is a row of all zeros.*

   **Procedure:**

   - *If* $P(s)$ *has no roots at* $s = 0$*, then* $n - j$ *must be even.*
   - *Use the coefficients in row* $j - 1$ *to build the auxiliary polynomial of order* $n - j + 2$*. (ie, This will be a factor of the* $P(s)$*).*
   - *Differentiate the created auxiliary polynomial and enter the coefficients in place of the zeros and continue.*
   - *May need to repeat the procedure more than once.*

# Asymptotic Stability (Routh Criterion)
## Example

Explore the stability of,

$$P(s) = s^4 + 2s^3 + 3s^2 + 4s + 5$$

Compute the Routh array,

| Row | Degree | | | | | |
|---|---|---|---|---|---|---|
| 1 | $s^4$ | 1 | 3 | 5 | 0 | $\cdots$ |
| 2 | $s^3$ | 2 | 4 | 0 | 0 | $\cdots$ |
| 3 | $s^2$ | 1 | 5 | 0 | 0 | $\cdots$ |
| 4 | $s^1$ | $-6$ | 0 | 0 | 0 | $\cdots$ |
| 5 | $s^0$ | 5 | 0 | 0 | 0 | $\cdots$ |

Two sign changes

Two roots in the right half place

# Asymptotic Stability (Routh Criterion)
## Example

Explore the stability of,

$$P(s) = s^3 + 7s^2 + 14s + 8$$

Compute the Routh array,

| Row | Degree | | | | |
|---|---|---|---|---|---|
| 1 | $s^3$ | 1 | 14 | 0 | $\cdots$ |
| 2 | $s^2$ | 7 | 8 | 0 | $\cdots$ |
| 3 | $s^1$ | $\frac{90}{7}$ | 0 | 0 | $\cdots$ |
| 4 | $s^0$ | 8 | 0 | 0 | $\cdots$ |

No sign changes
Stable

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## Asymptotic Stability (Routh Criterion)
## Example Case 1

Explore the stability of,

$$P(s) = s^4 + s^3 + 2s^2 + 2s + 1$$

Compute the Routh array,

| Row | Degree | | | | | |
|---|---|---|---|---|---|---|
| 1 | $s^4$ | 1 | 2 | 1 | 0 | $\cdots$ |
| 2 | $s^3$ | 1 | 2 | 0 | 0 | $\cdots$ |
| ~~3~~ | ~~$s^2$~~ | ~~0~~ | ~~1~~ | ~~0~~ | ~~0~~ | ~~$\cdots$~~ |
| 3 | $s^2$ | $\varepsilon$ | 1 | 0 | 0 | $\cdots$ |
| 4 | $s^1$ | $2 - \frac{1}{\varepsilon}$ | 0 | 0 | 0 | $\cdots$ |
| 5 | $s^0$ | 1 | 0 | 0 | 1 | $\cdots$ |

Two sign changes for small $\varepsilon$, $|\varepsilon| < \frac{1}{2}$

Two roots in the right-half plane

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# Asymptotic Stability (Routh Criterion)
## Example Case 2

Explore the stability of, $P(s) = s^5 + 2s^4 + 3s^3 + 6s^2 - 4s - 8$

Since there is a sign change in the coefficients, we know that the polynomial is unstable. However, we would like to know the number of roots in the right-half plane.

| Row | Degree | | | | | |
|---|---|---|---|---|---|---|
| 1 | $s^5$ | 1 | 3 | $-4$ | 0 | $\cdots$ |
| 2 | $s^4$ | 2 | 6 | $-8$ | 0 | $\cdots$ |
| ~~3~~ | $\cancel{s^3}$ | ~~0~~ | ~~0~~ | ~~0~~ | ~~0~~ | $\cancel{\cdots}$ |
| 3 | $s^3$ | 8 | 12 | 0 | 0 | $\cdots$ |
| 4 | $s^2$ | 3 | $-8$ | 0 | 0 | $\cdots$ |
| 5 | $s^1$ | 33.3 | 0 | 0 | 0 | $\cdots$ |
| 6 | $s^0$ | $-8$ | 0 | 0 | 0 | $\cdots$ |

$P_2 = 2s^4 + 6s^2 - 8$

$P_3 = \frac{dP_2}{ds} = 8s^3 + 12s$

One sign change

One roots in the right-half plane

# Asymptotic Stability (Routh Criterion)
## Stability Margin

Have all the poles to the left of -1

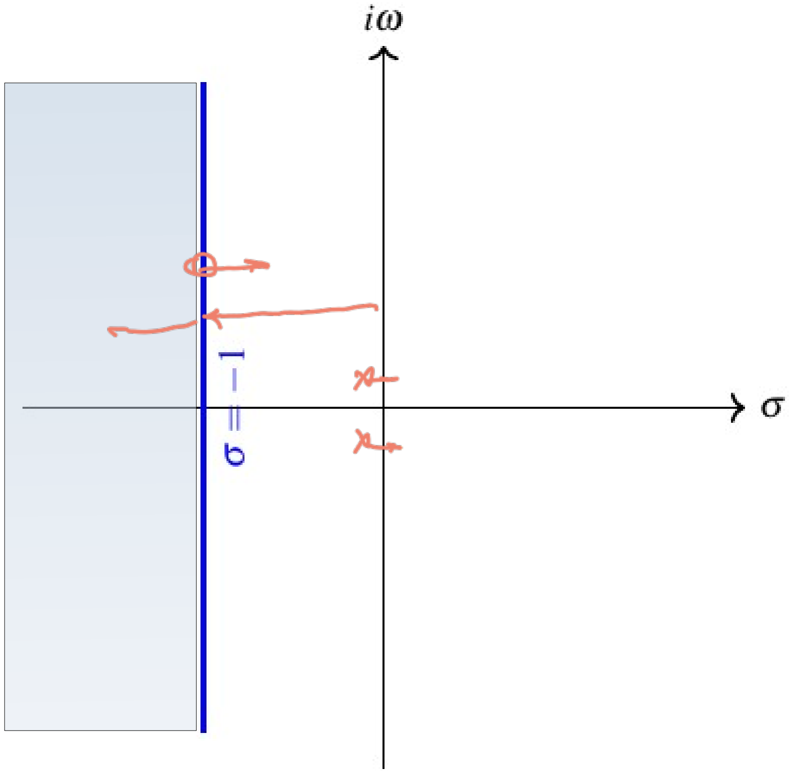
$$\hat{P}(s) = P(s - \sigma_1)$$

If the roots of $P(s)$ are $s_i, i \in \{1, 2, \cdots, n\}$,
then, the roots of $\hat{P}(s)$ would be

$$\hat{s}_i = s_i + \sigma_1, i \in \{1, 2, \cdots, n\}$$

$$\mathscr{Re}\{\hat{s}_i\} < 0 \implies \mathscr{Re}\{s_i\} < -\sigma_1$$

Use the Routh-Hurwitz test on the new polynomial



# Asymptotic Stability (Routh Criterion)
## Stability Margin

*Do all the poles lie to the left of $\sigma = -1$?*

$$P(s) = s^3 + 7s^2 + 14s + 8$$

$$\begin{aligned}
\hat{P}(s) &= P(s-1) \\
&= (s-1)^3 + 7(s-1)^2 + 14(s-1) + 8 \\
&= \underline{s^3 + 4s^2 + 3s + 0}
\end{aligned}$$

$s(s^2+4s+3)$

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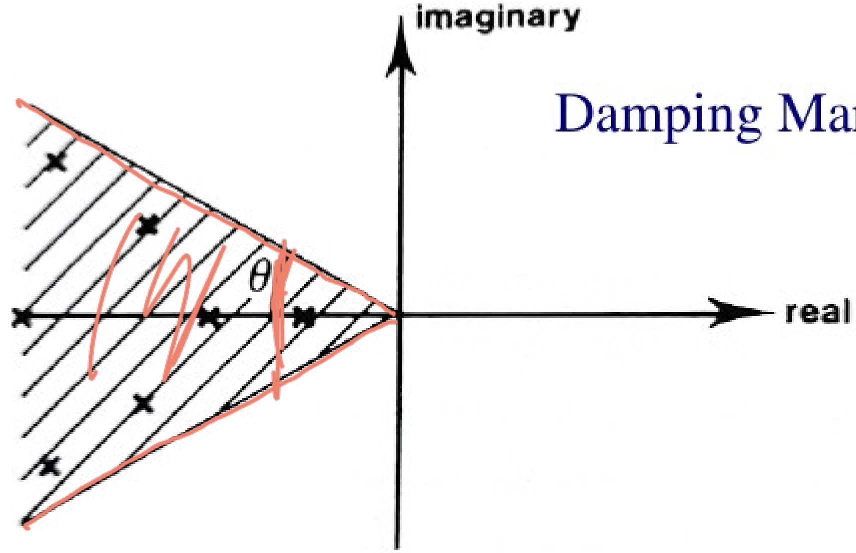
# Asymptotic Stability (Routh Criterion)
## Damping Margin




Damping Margin, $\zeta = \cos(\theta)$

- Testing for a specific damping margin may be done by a coordinate rotation and the generation of a new polynomial in the new coordinate system and testing for stability.

- Another method would be through the use of the Nyquist stability criterion.

# Asymptotic Stability (Lyapunov Stability)

**Definition** (Positive Definite Matrix).

*$\mathbf{Q} : \mathscr{R}^n \mapsto \mathscr{R}^n$ is a positive definite matrix if,*

$$f(\mathbf{x}) = \mathbf{x}^T \mathbf{Q} \mathbf{x} > 0 \;\; \forall \;\; \mathbf{x} \neq \mathbf{0} \tag{322}$$

*or, alternatively, if $\mathbf{Q}$ has all strictly positive Eigenvalues.*

*N.B.*, Equation 322 is the Euclidean norm of $\mathbf{Q}^{\frac{1}{2}}\mathbf{x}$

# Asymptotic Stability (Lyapunov Stability)

Aleksandr Mikhailovich Lyapunov (1857-1918)

- Uses a scalar energy of the system
- Also works for time-variant and nonlinear systems
- We only consider the linear time-invariant case here

# Asymptotic Stability (Lyapunov Stability)

Consider the following homogeneous linear time-invariant system,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

Note that a measure of the energy of the above system may be written as

$$\begin{aligned} \mathscr{E}(\mathbf{x}) &\stackrel{\Delta}{=} \|\mathbf{x}(t)\|^2 \\ &= \mathbf{x}^T\mathbf{x} \end{aligned}$$

If $\mathscr{E}(\mathbf{x}(t)) \to 0 \;\; as \;\; t \to \infty$, then the system is stable.

# Asymptotic Stability (Lyapunov Stability)

In fact, this would work for any quadratic function,

$$v(\mathbf{x}) \stackrel{\Delta}{=} \mathbf{x}^T \mathbf{Q}\mathbf{x}$$

where Q is a symmetric positive definite matrix, in which case,

$$\lambda_{\min} \|\mathbf{x}\|^2 \leq \mathbf{x}^T \mathbf{Q}\mathbf{x} \leq \lambda_{\max} \|\mathbf{x}\|^2$$

Let's impose a slightly stronger condition than requiring $v(\mathbf{x}) \to \mathbf{0}$

## Asymptotic Stability (Lyapunov Stability)

Definition (Quadratic Lyapunov Function).

*The following function,*

$$v(\mathbf{x}) \stackrel{\Delta}{=} \mathbf{x}^T \mathbf{Q} \mathbf{x}$$

*is a quadratic function for the following system,*

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

*if* $\mathbf{Q} > \mathbf{0}$ *and there exists a constant* $k > 0$ *such that,*

$$\dot{v}(\mathbf{x}) \leq -k\|\mathbf{x}\|^2 \quad \forall t$$

*In other words the generalized energy measure,* $v(\mathbf{x})$*, decreases monotonicall as time increases.*

# Asymptotic Stability (Lyapunov Stability)

*Proof.*
Since,

$$\lambda_{\min}||\mathbf{x}||^2 \leq \mathbf{x}^T\mathbf{Q}\mathbf{x} \leq \lambda_{\max}||x||^2$$

$\dot{v}(\mathbf{x})$ may be bounded with respect to the largest Eigenvalue of $A$ as follows,

$$\dot{v}(\mathbf{x}) \leq -k||\mathbf{x}||^2 \leq -\frac{k}{\lambda_{\max}}v(\mathbf{x})$$

## Asymptotic Stability (Lyapunov Stability)

*Proof.*
Since,

$$\lambda_{\min} ||\mathbf{x}||^2 \leq \mathbf{x}^T \mathbf{Q} \mathbf{x} \leq \lambda_{\max} ||x||^2$$

$\dot{v}(\mathbf{x})$ may be bounded with respect to the largest
Eigenvalue of $A$ as follows,

$$\dot{v}(\mathbf{x}) \leq -k||\mathbf{x}||^2 \leq -\frac{k}{\lambda_{\max}} v(\mathbf{x})$$

If we write the above differential equation for the equailty case,

$$\dot{v}(\mathbf{x}) = -\frac{k}{\lambda_{\max}} v(\mathbf{x})$$

then,

$$v(\mathbf{x}) = e^{-(\frac{k}{\lambda_{\max}})t} v(\mathbf{x}_0)$$

## Asymptotic Stability (Lyapunov Stability)

*Proof.* *(continued)*

$$v(\mathbf{x}) = e^{-\left(\frac{k}{\lambda_{\max}}\right)t} v(\mathbf{x}_0)$$

Since $v(\mathbf{x}) \geq 0$, then $v$ may decay even faster than the above solution,

$$v(\mathbf{x}) \leq e^{-\left(\frac{k}{\lambda_{\max}}\right)t} v(\mathbf{x}_0)$$

Using the following inequality again,

$$\dot{v}(\mathbf{x}) \leq -k\|\mathbf{x}\|^2 \leq -\frac{k}{\lambda_{\max}} v(\mathbf{x})$$

then,

$$\begin{aligned} \|x\|^2 &\leq \frac{v(\mathbf{x}(t))}{\lambda_{\min}} \\ &\leq \frac{e^{-\left(\frac{k}{\lambda_{\max}}\right)t} v(\mathbf{x}_0)}{\lambda_{\min}} \xrightarrow[t\to\infty]{} 0 \end{aligned}$$

Therefore the system is asymptotically stable.

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## Asymptotic Stability (Lyapunov Stability)

Since

$$v(\mathbf{x}) = \mathbf{x}^T \mathbf{Q} \mathbf{x}$$

using the homogeneous differential equation,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

then

$$\begin{aligned} \dot{v}(\mathbf{x}) &= \frac{d(\mathbf{x}^T \mathbf{Q} \mathbf{x})}{dt} \\ &= \dot{\mathbf{x}}^T \mathbf{Q} \mathbf{x} + \mathbf{x}^T \mathbf{Q} \dot{\mathbf{x}} \\ &= \mathbf{x}^T \mathbf{A}^T \mathbf{Q} \mathbf{x} + \mathbf{x}^T \mathbf{Q} \mathbf{A} \mathbf{x} \\ &= \mathbf{x}^T (\mathbf{A}^T \mathbf{Q} + \mathbf{Q} \mathbf{A}) \mathbf{x} \end{aligned}$$

$\dot{x} = Ax$

$\dot{x}^T = x^T A^T$

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# Asymptotic Stability (Lyapunov Stability)

Let us define $M$ such that,

$$\mathbf{M} \stackrel{\Delta}{=} -(\mathbf{A}^T\mathbf{Q} + \mathbf{Q}\mathbf{A})$$ $< 0$

then, $> 0$

$$\begin{aligned} \dot{v}(\mathbf{x}) &= \mathbf{x}^T(\mathbf{A}^T\mathbf{Q} + \mathbf{Q}\mathbf{A})\mathbf{x} \\ &= -\mathbf{x}^T\mathbf{M}\mathbf{x} \\ &\leq -k\mathbf{x}^T\mathbf{x} \quad \forall t \end{aligned}$$

The above must hold for $t = 0$ and any initial state $\mathbf{x}_0$, which means that $\mathbf{M} > \mathbf{0}$.

$$\mathbf{M} \geq k\mathbf{I} > \mathbf{0}$$

# Asymptotic Stability (Lyapunov Stability)

To determine whether $v$ is a Lyapunov function,

$M = -(A^T Q + Q A)$

- Choose any $\mathbf{Q} > \mathbf{0}$
- Compute $\mathbf{M}$ from its definition, above
- If $\mathbf{M} > \mathbf{0}$, then $v(\mathbf{x}) = \mathbf{x}^T\mathbf{Q}\mathbf{x}$ is a Lyapunov function for the homogeneous system of $\mathbf{A}$.
- If $\mathbf{M} \not> \mathbf{0}$, $v(\mathbf{x})$ is not a Lyapunov function, but there is no conclusion about the stability.

# Asymptotic Stability (Lyapunov Stability)
## Better Alternative

To determine whether $v$ is a Lyapunov function,

- Choose a positive definite matrix, $\mathbf{M} > \mathbf{0}$
- Solve for $\mathbf{Q}$, using this $\mathbf{M}$ in $\mathbf{A}^T\mathbf{Q} + \mathbf{Q}\mathbf{A} = -\mathbf{M}$
- Test for positive definiteness of $\mathbf{Q}$
- This procedure is definitive, so that if $\mathbf{Q} \not> \mathbf{0}$, then the system cannot be stable.

# Asymptotic Stability (Lyapunov Stability) Better Alternative

Theorem (Lyapunov Function Determination).

*Choose* $\mathbf{M} > \mathbf{0}$, *then* $\mathbf{A}$ *is stable if and only if*

$$\mathbf{A}^T\mathbf{Q} + \mathbf{Q}\mathbf{A} = -\mathbf{M}$$

$\dot{x} = Ax$

*has a unique solution,* $\mathbf{Q} > \mathbf{0}$.

*Proof.*

For any $\mathbf{M}$,

$$\mathbf{Q} = \int_0^\infty e^{\mathbf{A}^T t}\mathbf{M}e^{\mathbf{A}t}\,dt \qquad (352)$$

is a solution to the above linear equation. The intergral of 352 must converge for $A$ to be stable. Plugging in for $\mathbf{Q}$,

$$\begin{aligned}
-\mathbf{M} &= \mathbf{A}^T \int_0^\infty e^{\mathbf{A}^T t}\mathbf{M}e^{\mathbf{A}t}dt + \int_0^\infty e^{\mathbf{A}^T t}\mathbf{M}e^{\mathbf{A}t}dt\ \mathbf{A} \\
&= \int_0^\infty \left(\mathbf{A}^T e^{\mathbf{A}^T t}\mathbf{M}e^{\mathbf{A}t} + e^{\mathbf{A}^T t}\mathbf{M}e^{\mathbf{A}t}\mathbf{A}\right) dt \\
&= \left[e^{\mathbf{A}^T t}\mathbf{M}e^{\mathbf{A}t}\right]_0^\infty
\end{aligned}$$

# Asymptotic Stability (Lyapunov Stability) Better Alternative

Theorem (Lyapunov Function Determination).

*Proof.* *(Continued)*

The linear map,

$$\mathbf{L}(\mathbf{Q}) \stackrel{\Delta}{=} \mathbf{A}^T\mathbf{Q} + \mathbf{Q}\mathbf{A}$$

is such that $\mathbf{L}(\mathbf{Q}) : \mathscr{R}^{n^2} \mapsto \mathscr{R}^{n^2}$, where, $\mathbf{L}(\mathbf{Q}) = -\mathbf{M}$

has a solution for all $\mathbf{M}$, therefore, $\mathbf{L}(\mathbf{Q})$ has full rank and hence it is invertible. To verify that $\mathbf{Q} > \mathbf{0}$,

$$\begin{aligned}
\mathbf{x}_0^T\mathbf{Q}\mathbf{x}_0 &= \int_0^\infty \mathbf{x}_0^T e^{\mathbf{A}^T t}\mathbf{M}e^{\mathbf{A}t}\,\mathbf{x}_0\, dt \\
&= \int_0^\infty \mathbf{x}^T(t)\mathbf{M}\mathbf{x}(t)\, dt
\end{aligned}$$

where $\mathbf{x}(t)$ can only be $\mathbf{0}$ if $\mathbf{x}_0 = \mathbf{0}$ since $e^{\mathbf{A}t}$ is nonsingular.

$\mathbf{A} > \mathbf{0} \implies \mathbf{x}_0^T\mathbf{Q}\mathbf{x}_0 > 0$ unless $\mathbf{x}_0 = \mathbf{0}$

# Asymptotic Stability (Lyapunov Stability)

## Example

Find a Lyapunov function for

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix} \mathbf{x}(t)$$

*Choose* $\mathbf{M} = \mathbf{I}$ *and solve the following linear equation,*

$$\overset{A^T}{\underbrace{\begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix}}} \overset{Q}{\underbrace{\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}}} + \overset{Q}{\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix}} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}_{\substack{-M \\ -I}}$$

$$\boxed{\begin{aligned} -2Q_{11} - 2Q_{12} &= -1 \\ -3Q_{11} - Q_{12} + Q_{22} &= 0 \\ -6Q_{12} - 4Q_{22} &= -1 \end{aligned}}$$

*which leads to the following slution for* $\mathbf{Q}$*,*

$$\boxed{\mathbf{Q} = \begin{bmatrix} 3 & -3.5 \\ -3.5 & 5.5 \end{bmatrix}}$$

*Therefore the system is stable since* $\mathbf{Q} > \mathbf{0}$.

# Root Locus Plots

$$F(s) = P(s) + KQ(s) = 0$$

$$Q(s) = s^m + q_{m-1}s^{m-1} + q_{m-2}s^{m-2} + \cdots + q_0$$

$$P(s) = s^n + p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \cdots + p_0$$

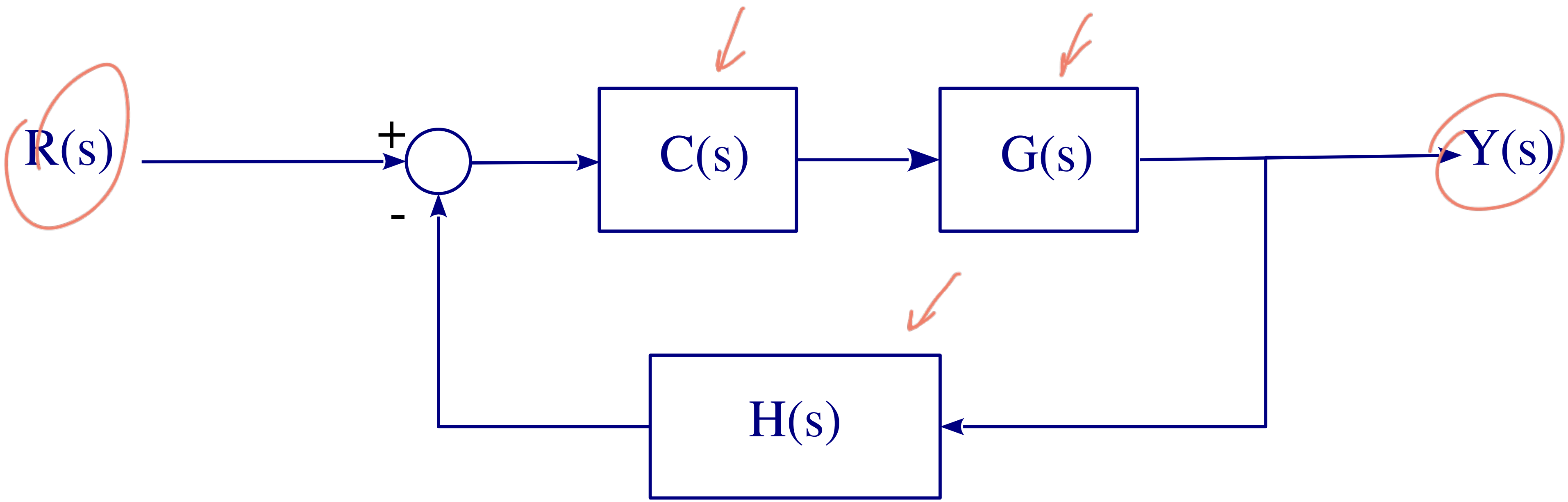
$$1 + K\frac{Q(s)}{P(s)} = 0 \qquad -\infty < K < \infty$$

$P = -KQ$

$1 = -\frac{KQ}{P}$

$1 + \frac{KQ}{P} = 0$

# Root Locus Plots




$$\frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)H(s)}$$

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## Root Locus Plots

Closed-loop transfer function
$$\frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1+C(s)G(s)H(s)}$$

Closed-loop characteristic equation
$$1+C(s)G(s)H(s) = 0$$

Assume C(s)G(s)H(s) has a free parameter, K
$$C(s)G(s)H(s) = \frac{KQ(s)}{P(s)}$$

$$\begin{aligned} 1+C(s)G(s)H(s) &= 1+\frac{KQ(s)}{P(s)} \\ &= \frac{P(s)+KQ(s)}{P(s)} \\ &= 0 \end{aligned}$$

$$F(s) = P(s)+KQ(s) = 0$$

# Root Locus Plots

What if K is not a multiplicative factor in C(s)G(s)H(s)

$$\begin{aligned} 1 + C(s)G(s)H(s) &= s(s+1)(s+2) + s^2 + (3 + 2K)s + 5 \\ &= 0 \end{aligned}$$

Rearrange the equation to have K as a factor

$$s(s+1)(s+2) + s^2 + 3s + 5 = -2Ks \qquad = 1$$

$$1 = \frac{-2Ks}{s(s+1)(s+2) + s^2 + 3s + 5}$$

$$\begin{aligned} 1 + \frac{2Ks}{s(s+1)(s+2) + s^2 + 3s + 5} &= 1 + \frac{KQ(s)}{P(s)} \\ &= 0 \end{aligned}$$

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# Root Locus Plots

$$
\begin{aligned}
1 + \frac{2Ks}{s(s+1)(s+2) + s^2 + 3s + 5} &= 1 + \frac{KQ(s)}{P(s)} \\
&= 0
\end{aligned}
$$

$$
Q(s) = 2s
$$

$$
\begin{aligned}
P(s) &= s(s+1)(s+2) + s^2 + 3s + 5 \\
&= (s^2 + s)(s+2) + s^2 + 3s + 5 \\
&= s^3 + 2s^2 + s^2 + 2s + s^2 + 3s + 5 \\
&= s^3 + 4s^2 + 5s + 5
\end{aligned}
$$

$$
\frac{Q(s)}{P(s)} = \frac{2s}{s^3 + 4s^2 + 5s + 5}
$$

# Root Locus Plots

$$\frac{Q(s)}{P(s)} = \frac{2s}{s^3 + 4s^2 + 5s + 5}$$

We can define new transfer functions that do not contain K,

$$C(s)G(s)H(s) = KC_1(s)G_1(s)H_1(s)$$

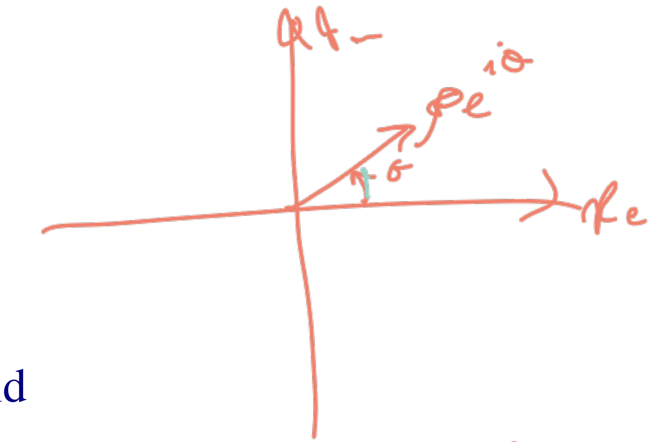
Such that,

$$C_1(s)G_1(s)H_1(s) = -\frac{1}{K}$$

Therefore we can say the following about the magnitude of the above equation,

$$|C_1(s)G_1(s)H_1(s)| = \frac{1}{|K|} \text{ where } -\infty < K < \infty$$

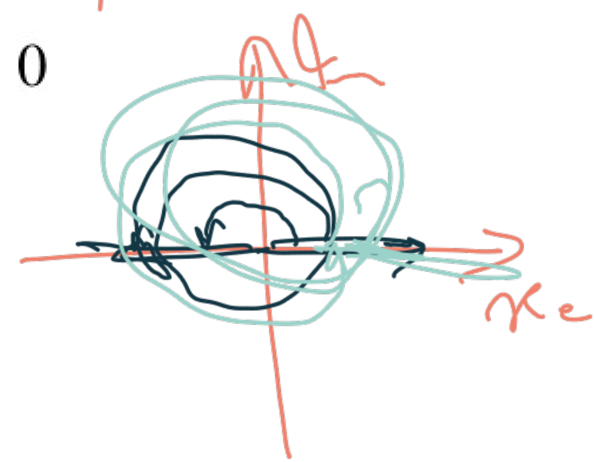
## Root Locus Plots




$$C_1(s)G_1(s)H_1(s) = -\frac{1}{K}$$

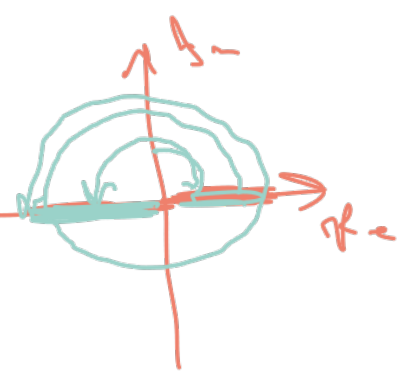
Also, the following phase angle conditions also hold

$$\begin{aligned}
\angle C_1(s)G_1(s)H_1(s) &= (2l+1)\pi \quad \text{where} \quad K \geq 0 \\
&= \text{Odd multiples of } \pi
\end{aligned}$$




$$\begin{aligned}
\angle C_1(s)G_1(s)H_1(s) &= 2l\pi \quad \text{where} \quad K \leq 0 \\
&= \text{Even multiples of } \pi
\end{aligned}$$

where $l \in \{0, \pm 1, \pm 2, \cdots\}$



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# Root Locus Plots

$$
\begin{aligned}
C(s)G(s)H(s) &= KC_1(s)G_1(s)H_1(s) \\
&= \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}
\end{aligned}
$$

$$
\begin{aligned}
|C_1(s)G_1(s)H_1(s)| &= \frac{\prod_{j=1}^{m}|s+z_j|}{\prod_{k=1}^{n}|s+p_k|} \\
&= \frac{1}{|K|}
\end{aligned}
$$

$\rho e^{i\theta}$

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# Root Locus Plots

$$
\begin{aligned}
C(s)G(s)H(s) &= KC_1(s)G_1(s)H_1(s) \\
&= \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}
\end{aligned}
$$

Use to draw Root Loci

$$
\left\{
\begin{aligned}
\angle C_1(s)G_1(s)H_1(s) &= \sum_{j=1}^{m} \angle(s+z_j) - \sum_{k=1}^{n} \angle(s+p_k) \\
&= (2l+1)\pi \quad \text{where} \quad K \geq 0 \\
\angle C_1(s)G_1(s)H_1(s) &= \sum_{j=1}^{m} \angle(s+z_j) - \sum_{k=1}^{n} \angle(s+p_k) \\
&= 2l\pi \quad \text{where} \quad K \leq 0
\end{aligned}
\right.
$$

where $l \in \{0, \pm 1, \pm 2, \cdots\}$

# Root Locus Plots

$$
\begin{aligned}
|C_1(s)G_1(s)H_1(s)| &= \frac{\prod_{j=1}^{m} |s+z_j|}{\prod_{k=1}^{n} |s+p_k|} \\
&= \frac{1}{|K|}
\end{aligned}
$$

After the Root Locus is created, use the following to compute the value of K

$$
|K| = \frac{\prod_{k=1}^{n} |s+p_k|}{\prod_{j=1}^{m} |s+z_j|}
$$