Introduction to Continuous Control Systems

EEME E3601

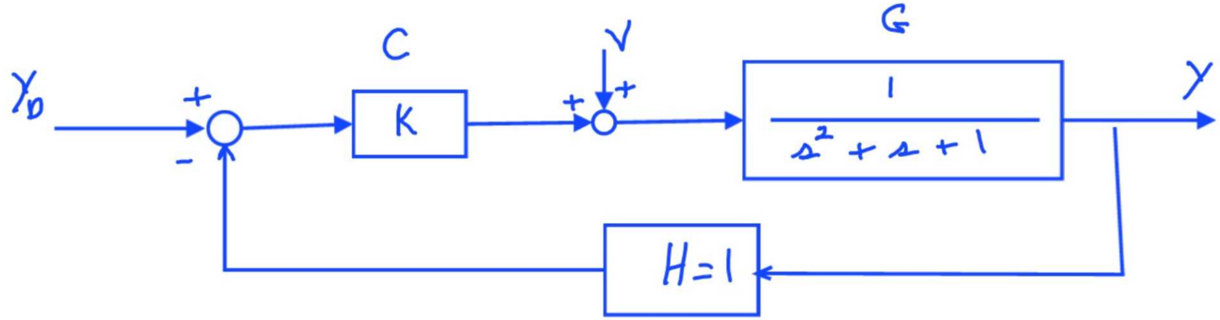
Week 10

Homayoon Beigi
Homayoon.Beigi@columbia.edu
https://www.RecoTechnologies.com/beigi

Mechanical Engineering dept.
&
Electrical Engineering dept.

Columbia University, NYC, NY, U.S.A.

homayoon.beigi@columbia.edu



$$y = \underbrace{\left[\frac{CG}{1+CGH}\right]}_{\displaystyle\frac{K}{s^2+s+(1+K)}} y_D + \underbrace{\left[\frac{G}{1+CGH}\right]}_{\displaystyle\frac{\frac{1}{s^2+s+1}}{1+\frac{K}{s^2+s+1}} = \frac{1}{s^2+s+(1+K)}} V$$

$$\underbrace{\left[s^2 + s + (1+K)\right] y = K y_D + v}$$

$$\boxed{\ddot{y} + \dot{y} + (1+K)y = K y_D + v}$$

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Solution $y(t) = \cancel{y_H(t)}^{\,0} + \cancel{y_{PD}(t)}^{\,y_D} + \cancel{y_{PV}(t)}^{\,0}$

① $y_H = c_1 y_1(t) + c_2 y_2(t)$ function of initial conditions

$s^2 + s + (1+K) = 0 \qquad s_{1,2} = \dfrac{-1 \pm \sqrt{1-4(1+K)}}{2}$

$y_1(t) = e^{s_1 t} \qquad y_2(t) = e^{s_2 t}$

want $y_H \to 0$ as $t \to \infty$ $\forall\, y(0)$

② y_{PD}: particular solution for y_D want $y_{PD} \to y_D$ as $t \to \infty$

③ y_{PV}: particular solution for v want $y_{PV} \to 0$ as $t \to \infty$

$\lim_{t\to\infty} y_H(t) \to 0 \;\; \forall\;\; y(0) \implies$ Asymptotically Stable

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$y(t) = e^{-\alpha_1 t} c_1 + c_2 e^{-\alpha_2 t}$ $\quad y(t) = e^{-\alpha t}(c_1 \cos \omega t + c_2 \sin \omega t)$

→ asymptotically stable: $y_H \to 0$ as $t \to \infty$ $\forall y(0)$ if $u(t)=0$

→ unstable: at least one solution where $y_{H_i} \to \infty$ as $t \to \infty$

At least one of the homogeneous solutions

→ Lyapunov stable: no solution → ∞

some solution may not go to zero

→ Marginal stability: some solution does not decay

(For all practical purposes not considered to be stable)

but it does not blow up either

the rest do the same or decay

→ Bounded Input- Bounded Output (BIBO) stability:

if $\vec{x}(0) = \vec{0}$

$$\|\vec{u}(t)\| \le k_1 < \infty \quad \forall\, 0 \le t < \infty \implies \|\vec{y}(t)\| \le k_2 < \infty \quad \forall\, 0 \le t < \infty$$

BIBO ⟹ all poles are strictly in the left half plane

Asymptotic Stability

$y(t) \rightarrow 0 \;\; as \;\; t \rightarrow \infty \;\; \forall \;\; y(0) \;\; \text{if} \;\; u(t) = 0$

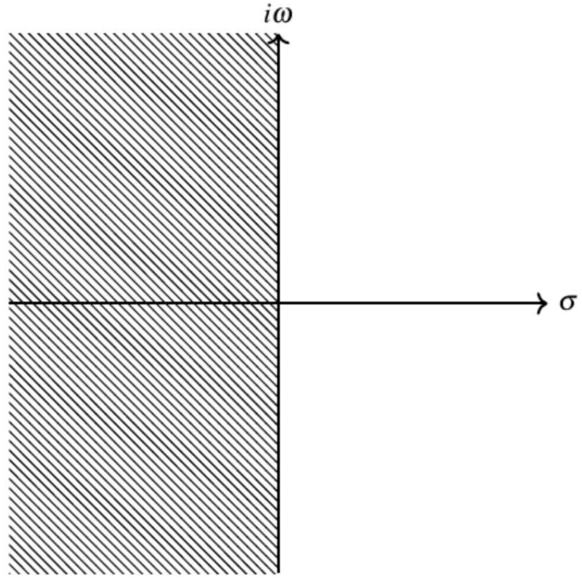
Theorem (Asymptotic Stability)

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{aligned} \tag{267}$$

The system of Equation 267 is asymptotically stable if and only if all of its poles lie strictly in the left half-plane,

$$\mathscr{Re}\{\lambda_i\} < 0 \quad i \in \{1, 2, \cdots, n\}$$

where λ_i are the eigenvalues of **A**.



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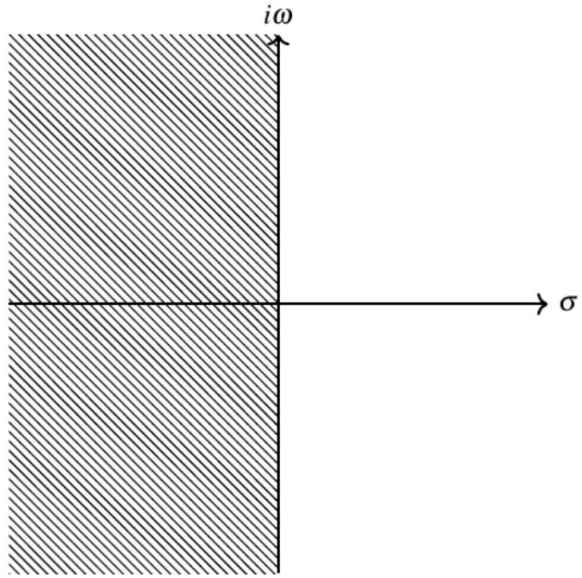
Asymptotic Stability



$$y(t) \to 0 \;\; as \;\; t \to \infty \;\; \forall \;\; y(0) \;\; \text{if} \;\; u(t) = 0$$

Theorem (Asymptotic Stability)

Equivalently, the system described by Equation 270,

$$\begin{aligned} \frac{d^{(n)}y(t)}{dt^{(n)}} \quad &+ \quad p_{n-1}\frac{d^{(n-1)}y(t)}{dt^{(n-1)}} + p_{n-2}\frac{d^{(n-2)}y(t)}{dt^{(n-2)}} + \cdots + p_0 y(t) \\ &= \quad q_{n-1}\frac{d^{(n-1)}u(t)}{dt^{(n-1)}} + q_{n-2}\frac{d^{(n-2)}u(t)}{dt^{(n-2)}} + \cdots + q_0 u(t) \end{aligned} \tag{270}$$

is asymptotically stable if and only if all of its poles, roots of the characteristic polynomial,

$$P(s) = s^n + p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \cdots + p_0 \tag{271}$$

are strictly in the left half plane.

Bounded-Input Bounded-Output (BIBO) Stability

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t)\end{aligned}$$

$$\|\mathbf{u}(t)\| \leq K_1 < \infty \ \forall \ 0 \leq t < \infty \implies \|\mathbf{y}(t)\| \leq K_2 < \infty \ \forall \ 0 \leq t < \infty$$

$$\begin{aligned}\frac{d^{(n)}y(t)}{dt^{(n)}} &+ p_{n-1}\frac{d^{(n-1)}y(t)}{dt^{(n-1)}} + p_{n-2}\frac{d^{(n-2)}y(t)}{dt^{(n-2)}} + \cdots + p_0 y(t)\\ &= q_{n-1}\frac{d^{(n-1)}u(t)}{dt^{(n-1)}} + q_{n-2}\frac{d^{(n-2)}u(t)}{dt^{(n-2)}} + \cdots + q_0 u(t)\end{aligned}$$

If all the initial conditions are zero, $y(0) = \dot{y}(0) = \ddot{y}(0) = \cdots = y^{(n-1)}(0) = 0$

and,

$$|u(t)| \leq K_1 < \infty \ \forall \ 0 \leq t < \infty \implies |y(t)| \leq K_2 < \infty \ \forall \ 0 \leq t < \infty$$

Asymptotic Stability (Testing)

Lemma (Positive Coefficients for Stability). Necessary – But not Sufficient

$$P(s) = s^n + p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \cdots + p_0 = 0 \tag{275}$$

If $P(s)$ is stable, the all of its coefficients, p_i, including the coefficient for s^n (usually 1), must have the same sign,

$$p_i > 0 \quad \forall \quad i \in \{0, 1, \cdots, n-1, n\}$$

assuming that $P(s)$ may be multiplied by -1 to make $p_{n-1} > 0$, if needed.

Asymptotic Stability (Testing)

Lemma (Positive Coefficients for Stability). Necessary – But not Sufficient

$$P(s) = s^n + p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \cdots + p_0 = 0 \qquad (275)$$

If $P(s)$ is stable, the all of its coefficients, p_i, including the coefficient for s^n (usually 1), must have the same sign,

$$p_i > 0 \quad \forall \; i \in \{0, 1, \cdots, n-1, n\}$$

assuming that $P(s)$ may be multiplied by -1 to make $p_{n-1} > 0$, if needed.

Proof.

Euqation 275 may be rewritten in terms of a set of complex conjugate and real roots,

$$P(s) = \left[(s-\sigma_1)^2 + \omega_1^2\right] \cdots \left[(s-\sigma_{m-1})^2 + \omega_{m-1}^2\right] (s - \lambda_{m+1}) \cdots (s - \lambda_n) \qquad (277)$$

with $\sigma_i \pm i\omega_i$, $i \in \{1, 3, \cdots, m-1\}$, and λ_j, $j \in \{m+1, m+2, \cdots, n\}$ as its roots.

$$\sigma_i \quad < \quad 0 \;\; i \in \{1, 3, \cdots, m-1\} \qquad\qquad \lambda_j \quad < \quad 0 \;\; j \in \{m+1, m+2, \cdots, n\}$$

all $\{-\sigma_i, \omega_i^2, -\lambda_j\}$ must be positive

multipying the terms in 277 would cause all the coefficients of 275 to be positive. □

homayoon.beigi@columbia.edu

Asymptotic Stability (Testing)

Theorem (Hurwitz Criterion) Necessary and Sufficient

$P(s)$ is stable if and only if the Hurwitz determinants are all positive.

$s^{(n)} + P_{n-1} s^{(n-1)} + P_{n-2} s^{(n-2)} + P_{n-3} s^{(n-3)}$

$$D_1 = |p_{n-1}| = p_{n-1}$$

$$D_2 = \begin{vmatrix} p_{n-1} & p_{n-3} \\ 1 & p_{n-2} \end{vmatrix}$$

$$D_3 = \begin{vmatrix} p_{n-1} & p_{n-3} & p_{n-5} \\ 1 & p_{n-2} & p_{n-4} \\ 0 & p_{n-1} & p_{n-3} \end{vmatrix}$$

$$\cdots$$

$$D_n = \begin{vmatrix} p_{n-1} & p_{n-3} & p_{n-5} & \cdots & 0 & 0 \\ 1 & p_{n-2} & p_{n-4} & \cdots & 0 & 0 \\ 0 & p_{n-1} & p_{n-3} & \cdots & 0 & 0 \\ 0 & 1 & p_{n-2} & \cdots & 0 & 0 \\ 0 & 0 & p_{n-1} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & p_0 & 0 \\ 0 & 0 & 0 & \cdots & p_1 & 0 \\ 0 & 0 & 0 & \cdots & p_2 & p_0 \end{vmatrix}$$

$$\boxed{D_i > 0 \quad i \in \{1, 2, \cdots, n\}}$$

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Asymptotic Stability (Testing)

Theorem (Liénard-Chipart). Necessary and Sufficient

Assuming that the coefficients of $P(s)$ are all positive,

$$p_i > 0 \quad i \in \{0, 1, 2, \cdots, n-1\}$$

then, $P(s)$ is stable if and only if,

$$D_i > 0 \quad i \in \{2, 4, \cdots, n-1\} \quad (when\ n\ is\ odd)$$

or

$$D_i > 0 \quad i \in \{3, 5, \cdots, n-1\} \quad (when\ n\ is\ even)$$

Asymptotic Stability (Testing)

Example

Consider the following system,

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & -4 & -3 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & -6 & 4 & 0 \end{bmatrix} \mathbf{x}(t)$$

with the characteristic polynomial, $P(s) = s^4 + 2s^3 + 3s^2 + 4s + 5$

Since $n = 4$ *is even, we require,* $D_i > 0 \quad i \in \{3, 5, \cdots, n-1\}$

Therefore, we need to evaluate only one determinant,

$$D_3 = \begin{vmatrix} 2 & 4 & 0 \\ 1 & 3 & 5 \\ 0 & 2 & 4 \end{vmatrix} = -12$$

See plot_output_lti_statespace.m

Since D_3 *is negative, according to the combination of Hurwitz Theorem and Liénard-Chipart Theorem, this system is unstable.*

homayoon.beigi@columbia.edu

Asymptotic Stability (Testing)

Example

Consider the following system,

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & -4 & -3 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & -6 & 4 & 0 \end{bmatrix} \mathbf{x}(t)$$

Note that using the Hurwitz theorem only, we would have also had to compute,

$$D_1 = |2| = 2 \qquad D_2 = \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 2$$

$$D_4 = \begin{vmatrix} 2 & 4 & 0 & 0 \\ 1 & 3 & 5 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 3 & 5 \end{vmatrix} = -60$$

Asymptotic Stability (Testing)

Example

Consider the following system,

$$\dddot{y}(t) + 7\ddot{y}(t) + 14\dot{y}(t) + 8y(t) = \dot{u}(t) + 3u(t)$$

with the following characteristic polynomial,

$$P(s) = s^3 + 7s^2 + 14s + 8$$

See plot_output_lti.m

All the coefficients are positive, so we need to check,

$$D_2 = \begin{bmatrix} 7 & 8 \\ 1 & 14 \end{bmatrix} = 90$$

Since D_2 is positive, the system is stable.