Introduction to Continuous Control Systems

EEME E3601

Week 9

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Midterm – Nov. 5th
4:10-4:25

Application Laplace Transform and solution of ODEs

State-Space and Exponential Matrix Solutions

Block Diagram Mannipulation

Linear Differential Equations
A Comparison

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 0$$

Try, $y(t) = Ce^{st}$

$$C\underbrace{(s^2 + 3s + 2)}_{=0}e^{st} = 0$$

$s_{1,2} = -1, -2$

$$y(t) = C_1 e^{-t} + C_2 e^{-2t}$$

Let's use the controllable canonical form,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\cancel{u(t)}$$

$$x_1 \triangleq y(t)$$

$$x_2 \triangleq \dot{y}(t)$$

Try, $\mathbf{x}(t) = C\mathbf{v}e^{st}$

Vector

$\dot{x}(t) = \boxed{A}x$

$A \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$\frac{d}{dt} Ce^{st} = sCe^{st}$

$s_{1,2}$

Ce^{st}

$C_1 e^{s_1 t} + C_2 e^{s_2 t}$

x_1
x_2

$y(0)$
$\dot{y}(0)$

$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$y = C = [1\ 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

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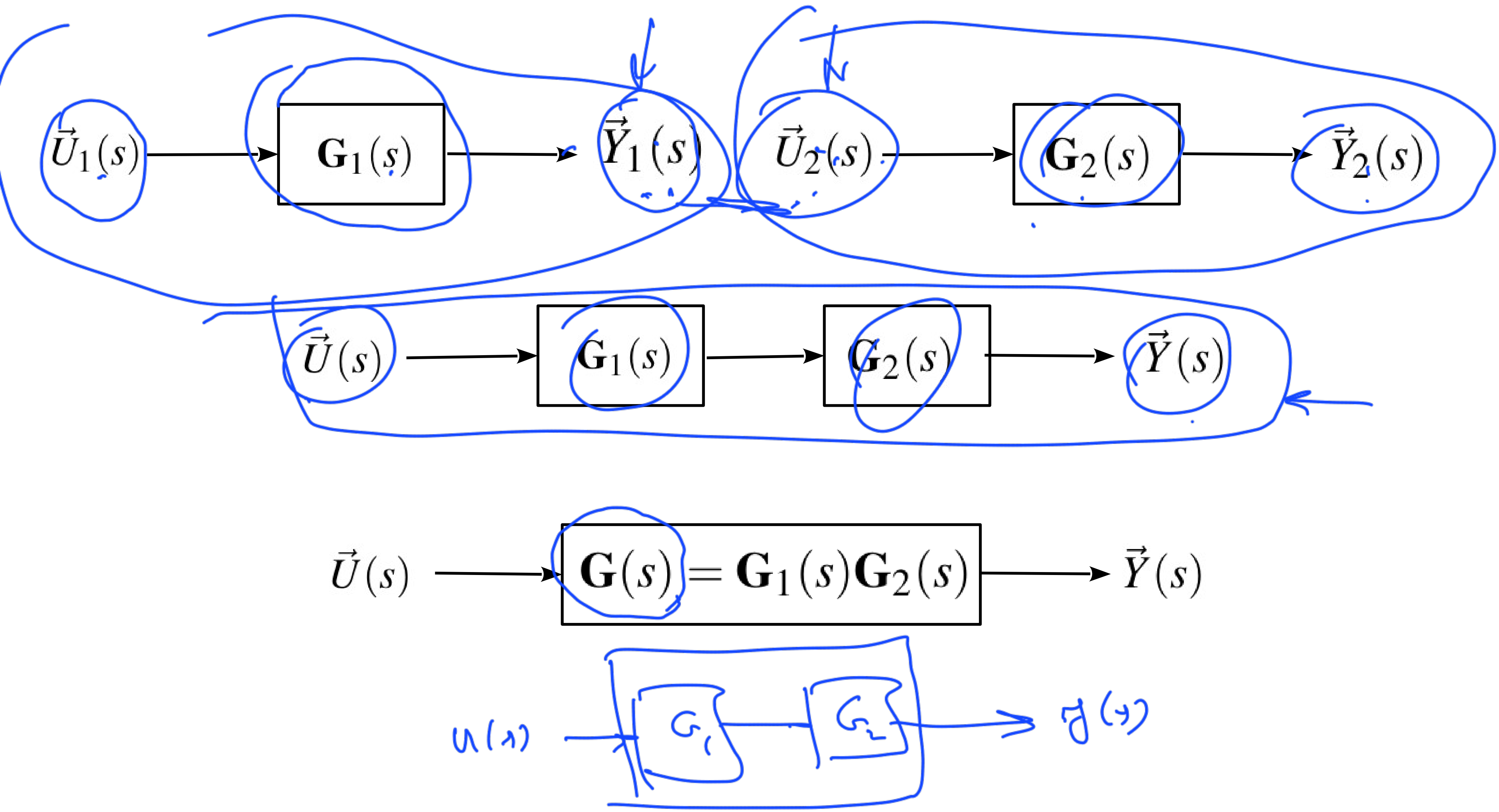
Linear Differential Equations
Solution of Forced Systems



G_1G_2

Linear Differential Equations
A Comparison

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

$$\mathbf{x}(t) = C\mathbf{v}e^{st}$$

$$\cancel{C}s\mathbf{v}\cancel{e^{st}} = \mathbf{A}\cancel{C}\mathbf{v}\cancel{e^{st}}$$

$$\boxed{\mathbf{A}\mathbf{v} = s\mathbf{v}} \quad (Eigenvalue - Eigenvector\ \ Equation)$$

$$(\mathbf{A} - s\mathbf{I})\mathbf{v} = 0$$

Non-trivial Solution when,

$$\boxed{|\mathbf{A} - s\mathbf{I}|} = 0$$

$$x(t) = C\vec{v}e^{st}$$

$$\cancel{C}s\vec{v}e^{st} = A\cancel{C}\vec{v}e^{st}$$

$$\boxed{A\vec{v} = s\vec{v}}$$

$$C\vec{v}e^{st}$$

$$Cs\vec{v}e^{st} = AC\vec{v}e^{st}$$

$$(sI - A)\vec{v} = \vec{0}$$

$$(A - s_1 I)\vec{v}_1 = 0$$

$$C\vec{v}_1 e^{s_1 t} + C v_2 e^{s_2 t}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|A - sI| = \left|\begin{bmatrix} a-s & b \\ c & d-s \end{bmatrix}\right| = 0$$

$$(a-s)(d-s) - cb = 0$$

$$ad - as - ds + s^2 - cb = 0$$

$$\boxed{s^2 - (a+d)s + (ad - cb) = 0}$$

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Linear Differential Equations
A Comparison

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & \cdots & \\ & & & \vdots & \\ -p_0 & -p_1 & \cdots & & -p_{n-1} \end{bmatrix}$$

$$|\mathbf{A} - s\mathbf{I}| = 0$$

$$\left| \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} -s & 1 \\ -2 & -3-s \end{vmatrix} = 0$$

$$s(3+s) + 2 = 0$$

$$\boxed{s^2 + 3s + 2 = 0}$$ *Same Charcteristic Polynomial*

r_1, r_2

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Linear Differential Equations
A Comparison

Let us pick one Eigenvalue, $\boxed{s_1 = -2}$ and find the associated $\mathbf{v}_1$.

$$\boxed{(\mathbf{A} - s_1\mathbf{I})\mathbf{v}_1 = \mathbf{0}}$$

$$\left\{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}\right\}\begin{bmatrix} (\mathbf{v}_1)_{[1]} \\ (\mathbf{v}_1)_{[2]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} (\mathbf{v}_1)_{[1]} \\ (\mathbf{v}_1)_{[2]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 2(\mathbf{v}_1)_{[1]} + (\mathbf{v}_1)_{[2]} &= 0 \\ -2(\mathbf{v}_1)_{[1]} - (\mathbf{v}_1)_{[2]} &= 0 \end{aligned}$$

The choices of the elements of $\mathbf{v}_1$ are not unique.

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Linear Differential Equations
A Comparison

We can pick the following,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Therefore, the following is a solution,

$$\mathbf{x}(t) = C \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t}$$

$$= C\, v_1\, e^{\lambda_1 t}$$

Linear Differential Equations
A Comparison

We can do the same thing for the second Eigenvector, $\mathbf{v}_2$,

Eigenvalue, $s_2 = -1$, and find the associated $\mathbf{v}_2$.

$$(\mathbf{A} - s_2\mathbf{I})\mathbf{v}_2 = \mathbf{0}$$

$$\left\{ \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\} \begin{bmatrix} (\mathbf{v}_2)_{[1]} \\ (\mathbf{v}_2)_{[2]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} (\mathbf{v}_2)_{[1]} \\ (\mathbf{v}_2)_{[2]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \boxed{1\,(\mathbf{v}_2)_{[1]} + (\mathbf{v}_2)_{[1]}} &= 0 \\ \boxed{-2\,(\mathbf{v}_2)_{[1]} - 2\,(\mathbf{v}_2)_{[2]}} &= 0 \end{aligned}$$

The choices of the elements of $\mathbf{v}_2$ are not unique.

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Linear Differential Equations
A Comparison

We can pick the following,

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore, the following is a solution,

$$\mathbf{x}(t) = C \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

$$\boxed{\mathbf{x}(t) = C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}}$$

$$\left[\vec{v}_1 \,|\, \vec{v}_2 \right]$$

Let's evaluate the initial conditions,

$$\mathbf{x}(0) = C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$\vec{x}(0) = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Linear Differential Equations
A Comparison

$$\mathbf{x}(t) = C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

$$\vec{x}(t) = e^{A(t-t_0)} \vec{x}(t_0)$$

$$\mathbf{x}(0) = C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\boxed{\mathbf{x}(0) = \mathbf{M} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}}$$

Eigenvector Matrix

where the columns of $\mathbf{M}$ are $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively.

$$\boxed{\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \mathbf{M}^{-1}\mathbf{x}(0)}$$

$$x(t) = C\vec{v}e^{\lambda t}$$

$$\mathbf{x}(t) = \underbrace{\mathbf{M} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \mathbf{M}^{-1}}_{e^{\mathbf{A}t}} \mathbf{x}(0)$$

$$\vec{x}(t) = e^{At}\vec{x}(0)$$

Linear Differential Equations
Controllable Canonical Form (Recap)

$$\frac{d^{(n)}y(t)}{dt^{(n)}} + p_{n-1}\frac{d^{(n-1)}y(t)}{dt^{(n-1)}} + p_{n-2}\frac{d^{(n-2)}y(t)}{dt^{(n-2)}} + \cdots + p_0 y(t)$$

$$= q_{n-1}\frac{d^{(n-1)}u(t)}{dt^{(n-1)}} + q_{n-2}\frac{d^{(n-2)}u(t)}{dt^{(n-2)}} + \cdots + q_0 u(t)$$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & \dots & . \\ -p_0 & -p_1 & \dots & -p_{n-2} & -p_{n-1} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ . \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} q_0 & q_1 & \dots & q_{n-2} & q_{n-1} \end{bmatrix} \mathbf{x}(t)$$

Observable canonical form

$$\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 4y = u$$

$$\frac{d}{dt}\underbrace{\left(\frac{d}{dt}\underbrace{\left(\frac{dy}{dt} + 2y\right)}_{x_2} + 3y\right)}_{x_1} + 4\underbrace{y}_{x_3} = u$$

$$x_3 = y$$

$$x_2 = \frac{dy}{dt} + 2y \rightarrow \frac{dx_3}{dt} = x_2 - 2x_3$$

$$x_1 = \frac{dx_2}{dt} + 3y \rightarrow \frac{dx_2}{dt} = x_1 - 3y$$

$$\frac{dx_1}{dt} = -4x_3 + u$$

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$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -4 \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

N.B.

observable controllable

$$A_o = A_c^T$$
$$B_o = C_c^T$$
$$C_o = B_c^T$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{bmatrix}$$

$$\dddot{y} + 2\ddot{y} + 3\dot{y} + 4y = 6\dot{u} + 5u$$

$$\frac{d}{dt}\left(\frac{d}{dt}\left(\underbrace{\frac{dy}{dt} + 2y}_{x_2}\right) + 3y\right) + 4\underbrace{y}_{x_3} = 6\frac{du}{dt} + 5u$$

$$\begin{bmatrix}0\\0\\1\end{bmatrix} \qquad \begin{bmatrix}0 & 0 & 1\end{bmatrix}$$

$$\frac{d}{dt}\left(\underbrace{\frac{dx_2}{dt} + 3x_3 - 6u}_{x_1}\right) + 4x_3 = 5u$$

$$\dot{x}_1 = -4x_3 + 5u$$

$$\dot{x}_2 = x_1 - 3x_3 + 6u$$

$$\dot{x}_3 = x_2 - 2x_3$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \overbrace{\begin{bmatrix}0 & 0 & -4\\ 1 & 0 & -3\\ 0 & 1 & -2\end{bmatrix}}^{A_0}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} + \underbrace{\begin{bmatrix}5\\6\\0\end{bmatrix}}_{B_0 = C_c^T} u$$

Multi-Input Multi-Output (MIMO)

$$\ddot{y}_1 + 2\dot{y}_1 + y_2 = u_1$$

$$\dot{y}_2 + y_1 = u_2$$

$$\begin{array}{ccc} y_1, & \dot{y}_1, & y_2 \\ | & | & | \\ x_1 & x_2 & x_3 \end{array}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -2x_2 - x_3 + u_1$$

$$\dot{x}_3 = -x_1 + u_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & -1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\boxed{\begin{aligned} \dot{\vec{x}} &= A\vec{x} + B\vec{u} \\ \vec{y} &= C\vec{x} \end{aligned}}$$

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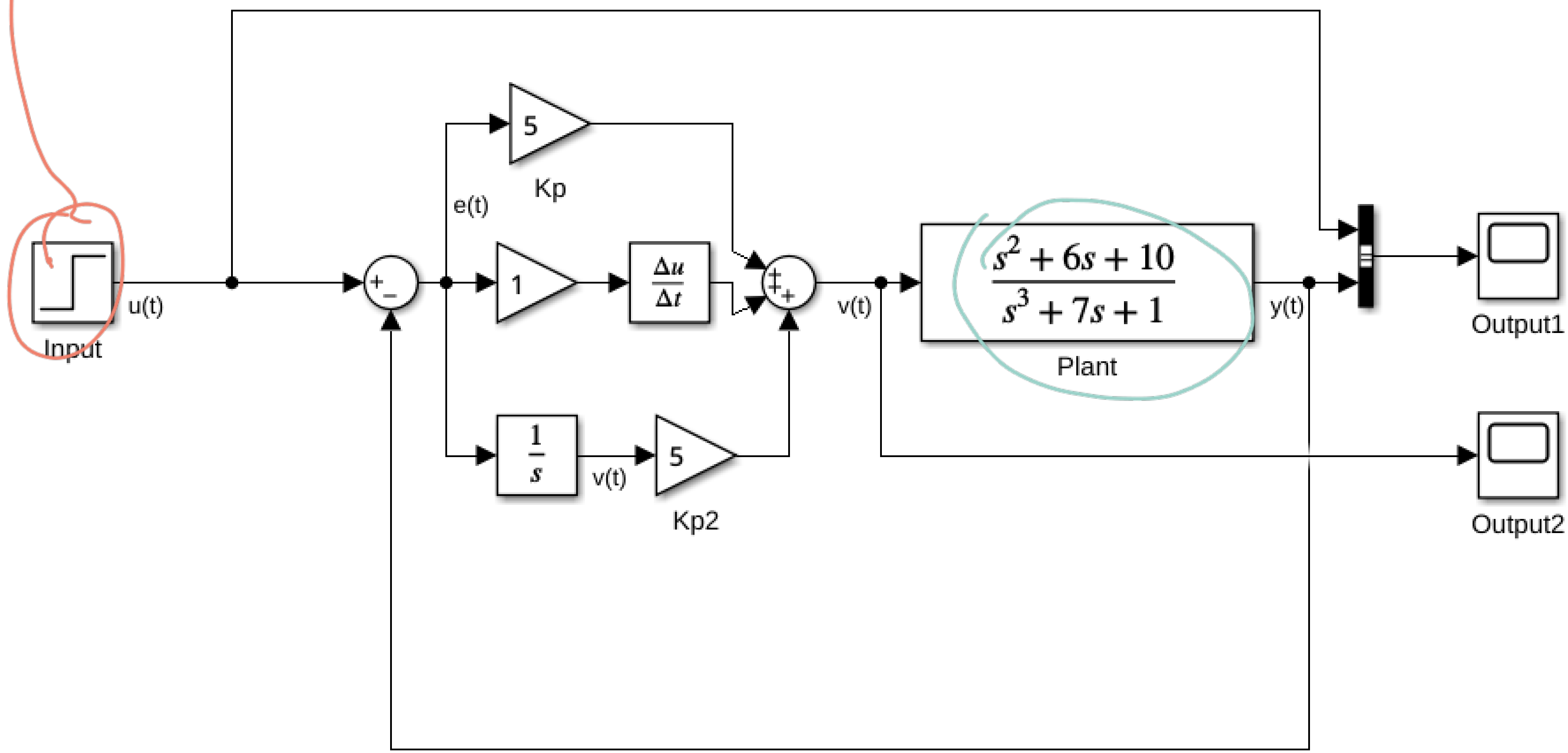
Practical Issues
Matlab and Simulink

$\dddot{y}(t)+7\dot{y}(t)+y(t) = \ddot{u}(t)+6\dot{u}(t)+10u$

$Y(s)(s^3+7s+1) = U(s)(s^2+6s+10)$



Original System

(~/rmeresearch/matlab/simulink/pid_control.slx)

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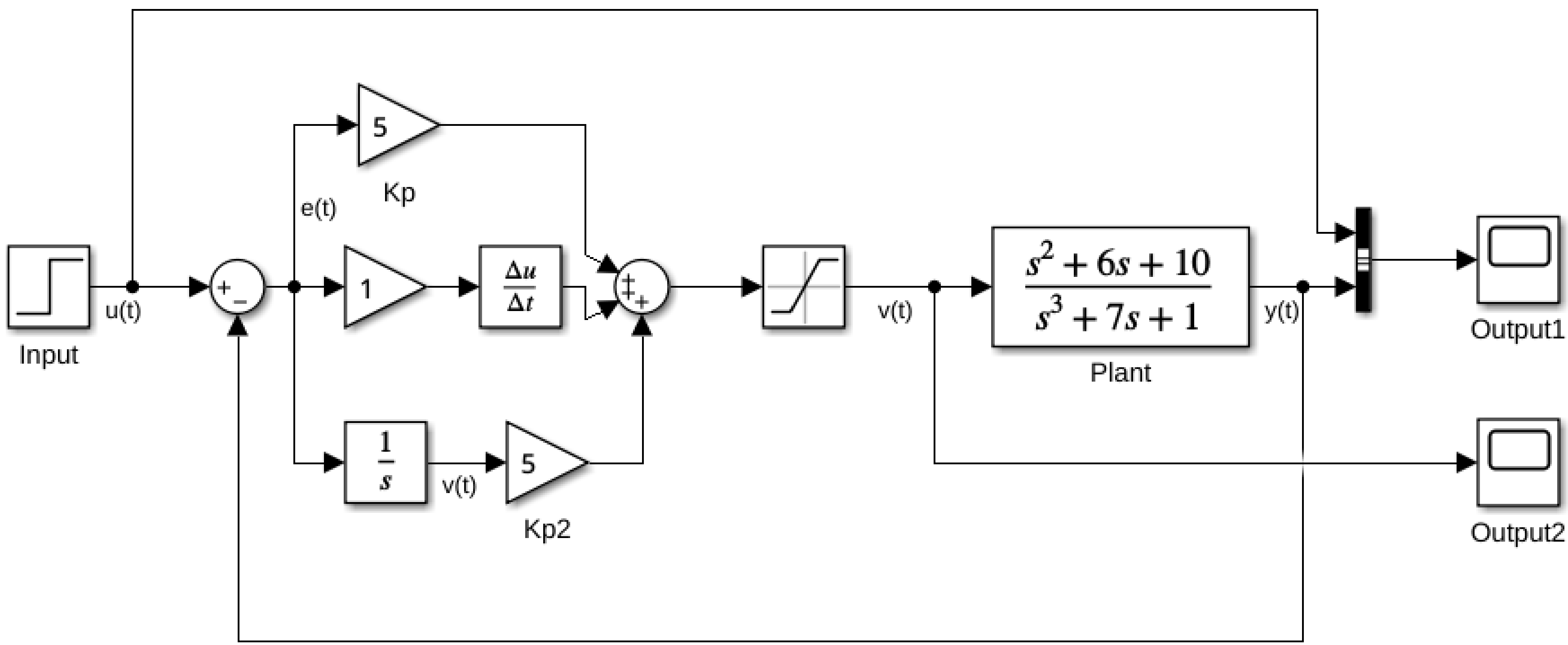
Practical Issues
Matlab and Simulink



Saturated System
(~/rmeresearch/matlab/simulink/pid_control_with_saturation.slx)