Introduction to Continuous Control Systems

EEME E3601

Week 9

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Midterm – Nov. 5th
4:10-4:25

Application Laplace Transform and solution of ODEs

State-Space and Exponential Matrix Solutions

Block Diagram Mannipulation

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Linear Differential Equations
A Comparison

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 0$$

Try, $y(t) = Ce^{st}$

$$C\underbrace{\left(s^2 + 3s + 2\right)}_{=0} e^{st} = 0$$

$$s_{1,2} = -1, -2 \qquad \boxed{y(t) = C_1 e^{-t} + C_2 e^{-2t}}$$

Let's use the controllable canonical form,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \cancel{\mathbf{b}u(t)}$$

$$x_1 \triangleq y(t)$$

$$x_2 \triangleq \dot{y}(t)$$

Try, $\mathbf{x}(t) = C\mathbf{v}e^{st}$

Vector

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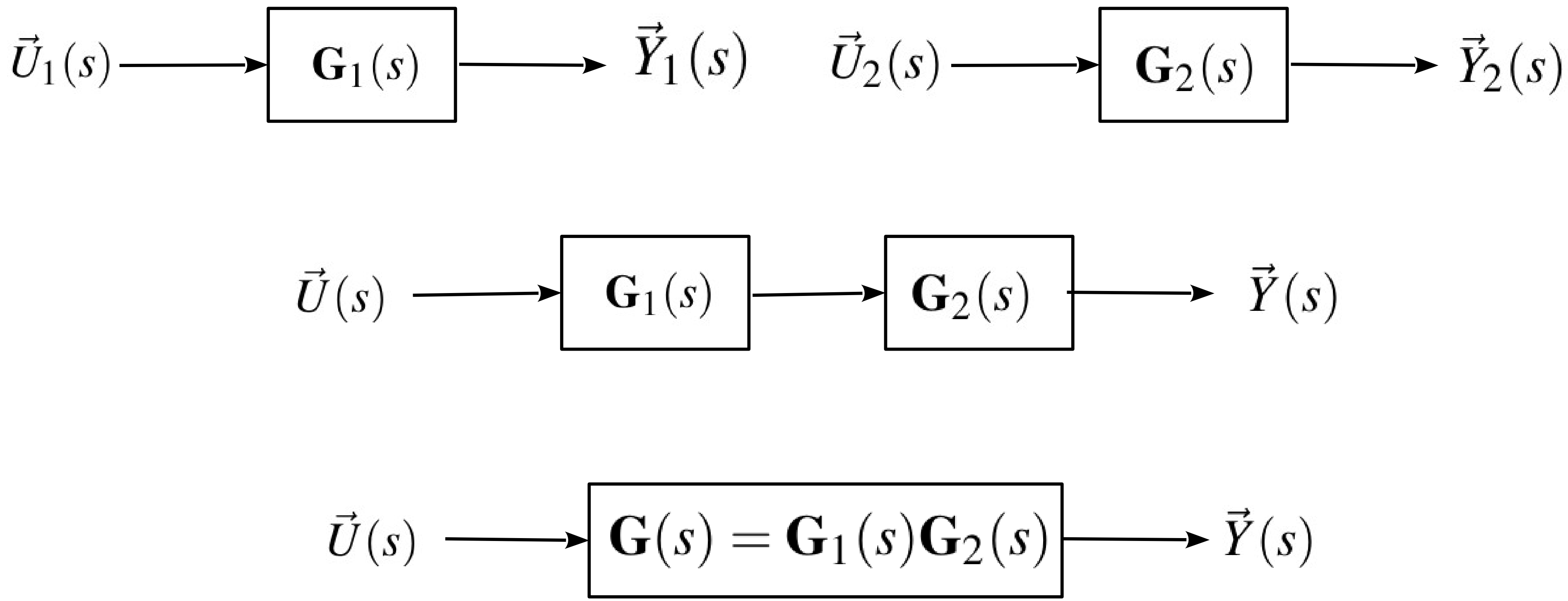
Linear Differential Equations
Solution of Forced Systems



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Linear Differential Equations
A Comparison

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

$$\mathbf{x}(t) = C\mathbf{v}e^{st}$$

$$\cancel{C} s\mathbf{v}\cancel{e^{st}} = \mathbf{A}\cancel{C}\mathbf{v}\cancel{e^{st}}$$

$$\mathbf{A}\mathbf{v} = s\mathbf{v} \quad (Eigenvalue - Eigenvector\ \ Equation)$$

$$(\mathbf{A} - s\mathbf{I})\mathbf{v} = 0$$

Non-trivial Solution when,

$$\left|\mathbf{A} - s\mathbf{I}\right| = 0$$

Linear Differential Equations
A Comparison

$$\left|\mathbf{A} - s\mathbf{I}\right| = 0$$

$$\left| \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} -s & 1 \\ -2 & -3-s \end{vmatrix} = 0$$

$$s(3+s)+2=0$$

$$s^2+3s+2=0 \qquad \textit{Same Charcteristic Polynomial}$$

Linear Differential Equations
A Comparison

Let us pick one Eigenvalue, $s_1 = -2$, and find the associated $\mathbf{v}_1$.

$$(\mathbf{A} - s_1\mathbf{I})\mathbf{v}_1 = \mathbf{0}$$

$$\left(\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}\right)\begin{bmatrix} (\mathbf{v}_1)_{[1]} \\ (\mathbf{v}_1)_{[2]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} (\mathbf{v}_1)_{[1]} \\ (\mathbf{v}_1)_{[2]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 2\,(\mathbf{v}_1)_{[1]} + (\mathbf{v}_1)_{[2]} &= 0 \\ -2\,(\mathbf{v}_1)_{[1]} - (\mathbf{v}_1)_{[2]} &= 0 \end{aligned}$$

The choices of the elements of $\mathbf{v}_1$ are not unique.

Linear Differential Equations
A Comparison

We can pick the following,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Therefore, the following is a solution,

$$\mathbf{x}(t) = C \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t}$$

Linear Differential Equations
A Comparison

We can do the same thing for the second Eigenvector, $\mathbf{v}_2$, Eigenvalue, $s_2 = -1$, and find the associated $\mathbf{v}_2$.

$$(\mathbf{A} - s_2\mathbf{I})\mathbf{v}_2 = \mathbf{0}$$

$$\left(\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)\begin{bmatrix} (\mathbf{v}_2)_{[1]} \\ (\mathbf{v}_2)_{[2]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} (\mathbf{v}_2)_{[1]} \\ (\mathbf{v}_2)_{[2]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 1\,(\mathbf{v}_2)_{[1]} + (\mathbf{v}_2)_{[1]} &= 0 \\ -2\,(\mathbf{v}_2)_{[1]} - 2\,(\mathbf{v}_2)_{[2]} &= 0 \end{aligned}$$

The choices of the elements of $\mathbf{v}_2$ are not unique.

Linear Differential Equations
A Comparison

We can pick the following,

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore, the following is a solution,

$$\mathbf{x}(t) = C \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

$$\mathbf{x}(t) = C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

Let's evaluate the initial conditions,

$$\mathbf{x}(0) = C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

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Linear Differential Equations
A Comparison

$$\mathbf{x}(t) = C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

$$\mathbf{x}(0) = C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{x}(0) = \mathbf{M} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

Eigenvector Matrix

where the columns of $\mathbf{M}$ are $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively.

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \mathbf{M}^{-1}\mathbf{x}(0)$$

$$\mathbf{x}(t) = \underbrace{\mathbf{M} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \mathbf{M}^{-1}}_{e^{\mathbf{A}t}} \mathbf{x}(0)$$

Linear Differential Equations
Controllable Canonical Form (Recap)

$$
\begin{aligned}
\frac{d^{(n)}y(t)}{dt^{(n)}} &+ p_{n-1}\frac{d^{(n-1)}y(t)}{dt^{(n-1)}} + p_{n-2}\frac{d^{(n-2)}y(t)}{dt^{(n-2)}} + \cdots + p_0 y(t) \\
&= q_{n-1}\frac{d^{(n-1)}u(t)}{dt^{(n-1)}} + q_{n-2}\frac{d^{(n-2)}u(t)}{dt^{(n-2)}} + \cdots + q_0 u(t)
\end{aligned}
$$

$$
\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & \dots & . \\ -p_0 & -p_1 & \dots & -p_{n-2} & -p_{n-1} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ . \\ 1 \end{bmatrix} u(t)
$$

$$
y(t) = \begin{bmatrix} q_0 & q_1 & \dots & q_{n-2} & q_{n-1} \end{bmatrix} \mathbf{x}(t)
$$

Observable canonical form

$$\frac{d^3 y}{dt^3} + 2\frac{d^2 y}{dt^2} + 3\frac{dy}{dt} + 4y = u$$

$$\frac{d}{dt}\underbrace{\left(\frac{d}{dt}\underbrace{\left(\frac{dy}{dt} + 2y\right)}_{x_2} + 3y\right)}_{x_1} + 4\underbrace{y}_{x_3} = u$$

$$x_3 = y$$

$$x_2 = \frac{dy}{dt} + 2y \longrightarrow \frac{dx_3}{dt} = x_2 - 2x_3$$

$$x_1 = \frac{dx_2}{dt} + 3y \longrightarrow \frac{dx_2}{dt} = x_1 - 3y$$

$$\frac{dx_1}{dt} = -4x_3 + u$$

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$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -4 \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

N.B.

observable controllable

$$A_o = A_c^T$$

$$B_o = C_c^T$$

$$C_o = B_c^T$$

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$$\dddot{y} + 2\ddot{y} + 3\dot{y} + 4y = 6\dot{u} + 5u$$

$$\frac{d}{dt}\left(\frac{d}{dt}\left(\underbrace{\frac{dy}{dt} + 2y}_{x_2}\right) + 3y\right) + 4\underbrace{y}_{x_3} = 6\frac{du}{dt} + 5u$$

$$\frac{d}{dt}\underbrace{\left(\frac{dx_2}{dt} + 3x_3 - 6u\right)}_{x_1} + 4x_3 = 5u$$

$$\left.\begin{aligned} \dot{x}_1 &= -4x_3 + 5u \\ \dot{x}_2 &= x_1 - 3x_3 + 6u \\ \dot{x}_3 &= x_2 - 2x_3 \end{aligned}\right)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 0 & -4 \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix}}^{A_0} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 5 \\ 6 \\ 0 \end{bmatrix}}_{B_0 = C_c^T} u$$

Multi-Input Multi-Output (MIMO)

$$\ddot{y}_1 + 2\dot{y}_1 + y_2 = u_1$$

$$\dot{y}_2 + y_1 = u_2$$

$y_1, \dot{y}_1, y_2$

$x_1 \quad x_2 \quad x_3$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -2x_2 - x_3 + u_1$$

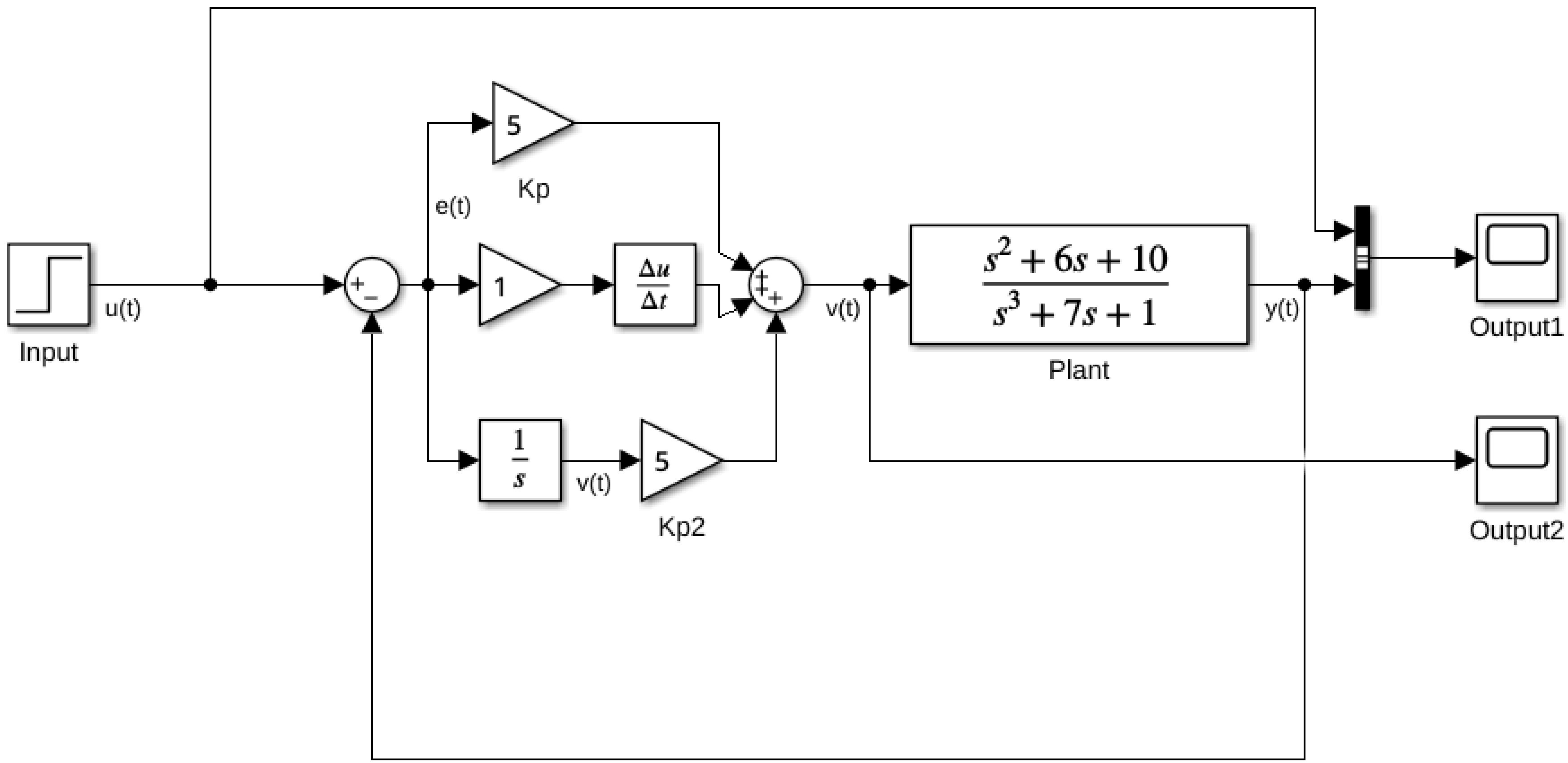
$$\dot{x}_3 = -x_1 + u_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & -1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\boxed{\begin{aligned} \dot{\vec{x}} &= A\vec{x} + B\vec{u} \\ \vec{y} &= C\vec{x} \end{aligned}}$$

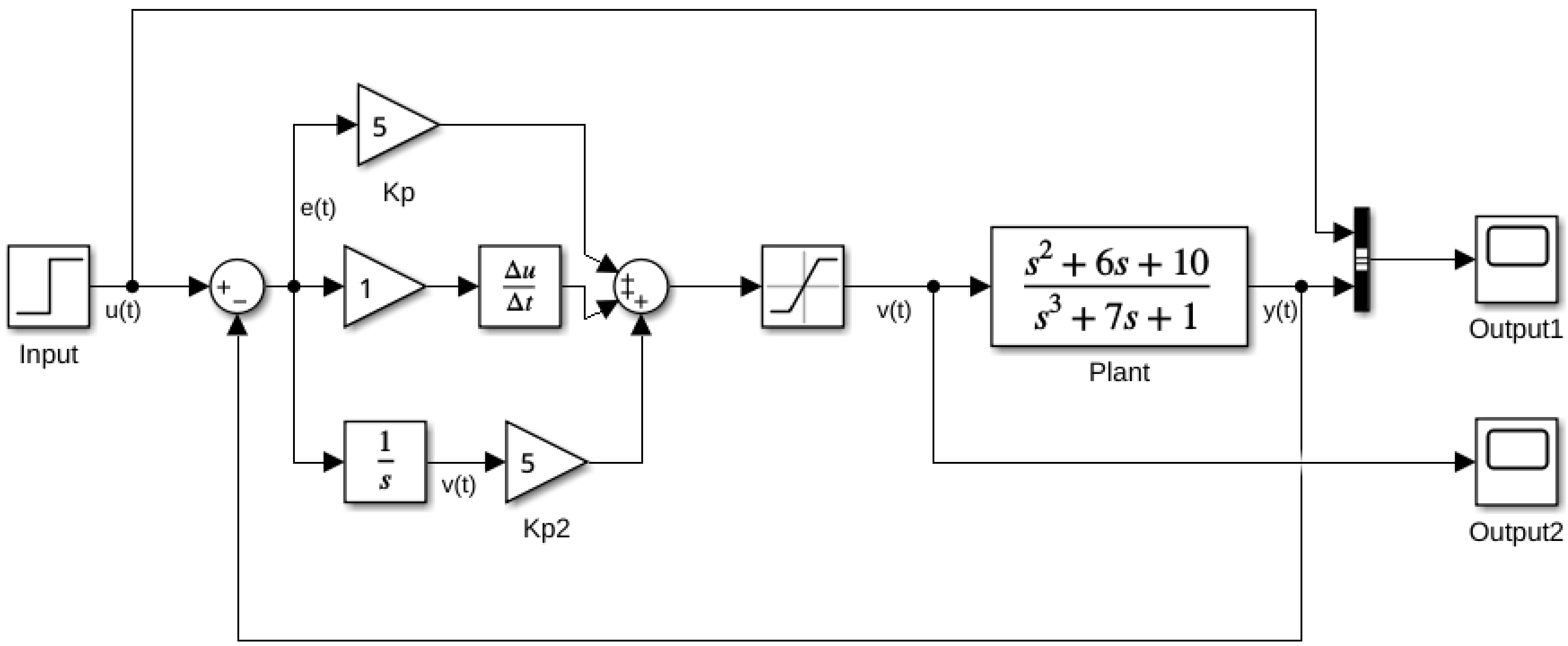
Practical Issues
Matlab and Simulink



Original System

(~/rmeresearch/matlab/simulink/pid_control.slx)

Practical Issues
Matlab and Simulink



Saturated System
(~/rmeresearch/matlab/simulink/pid_control_with_saturation.slx)