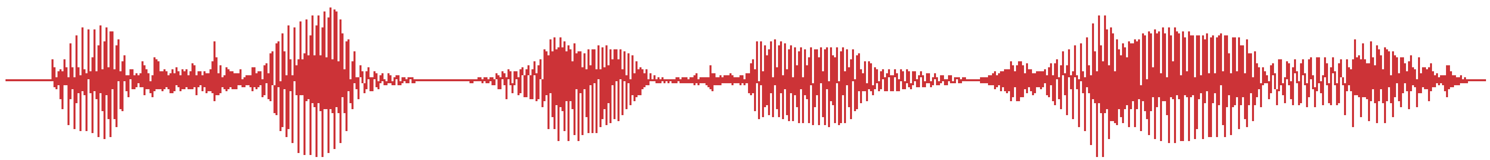
Introduction to Continuous Control Systems

EEME E3601



Week 8

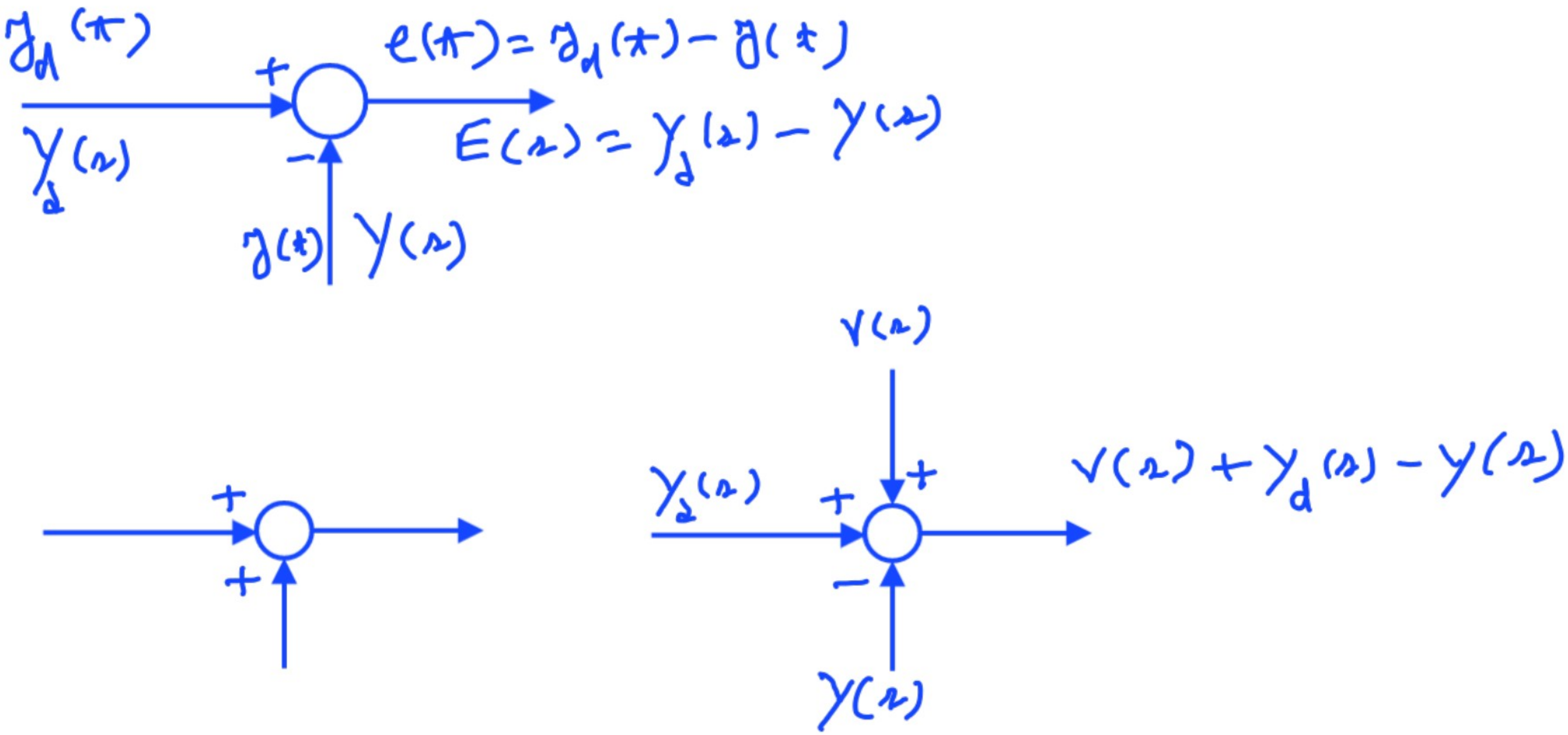
Homayoon Beigi

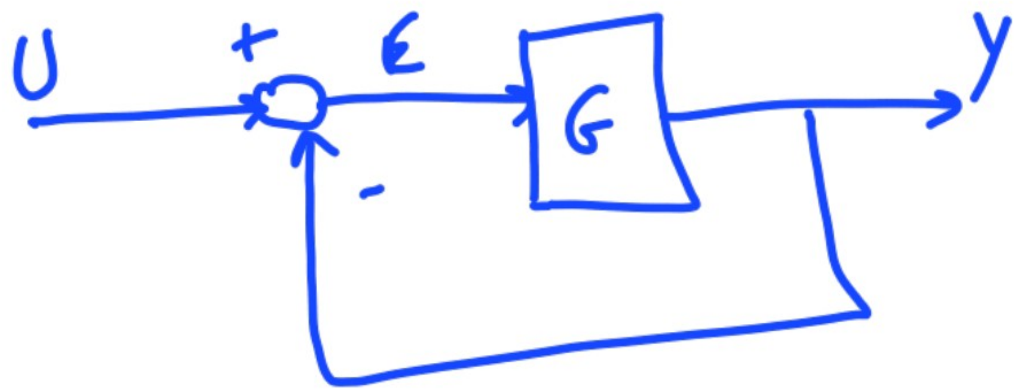
Homayoon.Beigi@columbia.edu
https://www.RecoTechnologies.com/beigi

Mechanical Engineering dept.
&
Electrical Engineering dept.

Columbia University, NYC, NY, U.S.A.

comparator



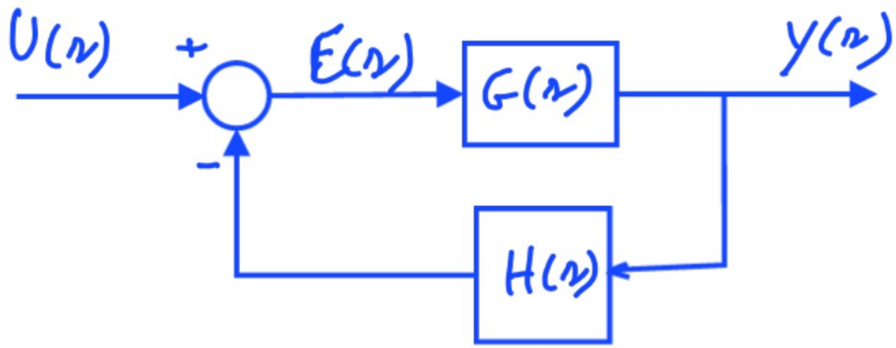


$$y = [U - y]G$$

$$y[1 + G] = GU$$

$$\boxed{y = \frac{G}{1+G}U}$$

homayoon.beigi@columbia.edu



$$E(s) = U(s) - H(s)\,Y(s)$$

$$Y(s) = G(s)\,E(s)$$

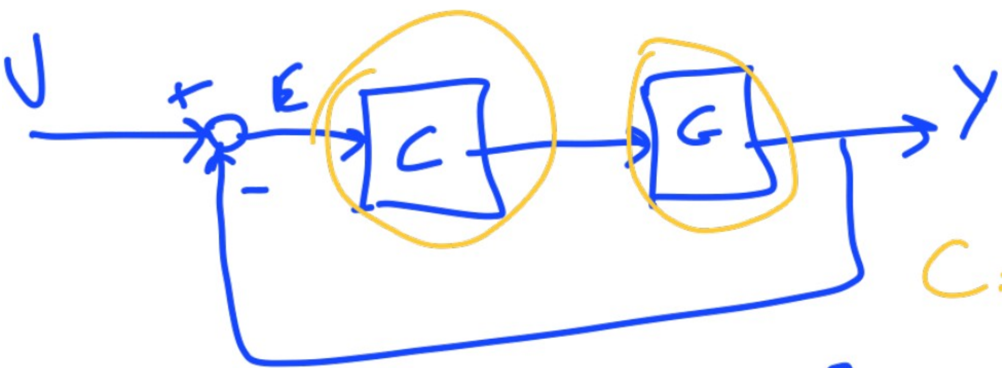
$$Y(s) = G(s)\left[U(s) - H(s)\,Y(s)\right]$$

$$= G(s)U(s) - G(s)H(s)\,Y(s)$$

$$Y(s)\left[1 + G(s)H(s)\right] = G(s)\,U(s)$$

$$Y(s) = \frac{G(s)}{1 + G(s)H(s)}\,U(s)$$

homayoon.beigi@columbia.edu



$$y = CG[U - y]$$

$$y[1 + CG] = CG\,U$$

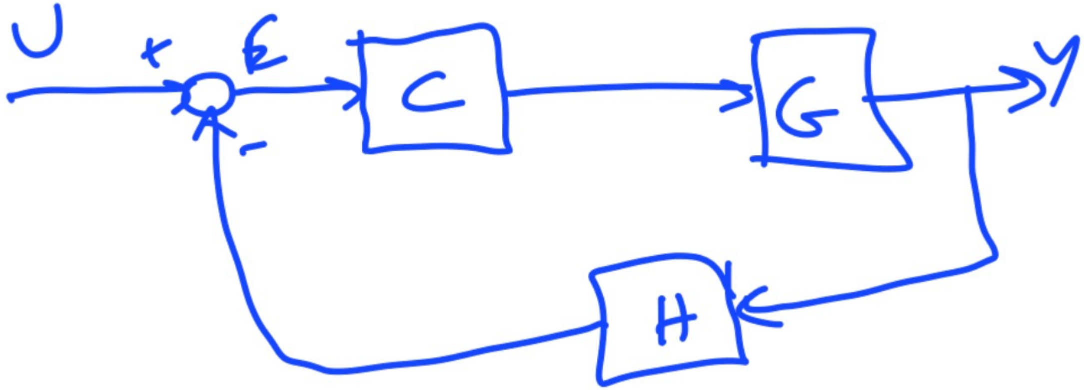
$$\boxed{y = \frac{CG}{1+CG} U}$$

e.g.

$$C = \underset{P}{K_1} + \underset{D}{K_2 s} + \underset{I}{\frac{K_3}{s}} \Leftrightarrow \underset{P}{K_1 e} + \underset{D}{K_2 \frac{de}{dt}} + \underset{I}{K_3 \int_0^t e(\tau)\, d\tau}$$

$$G = \frac{\omega^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

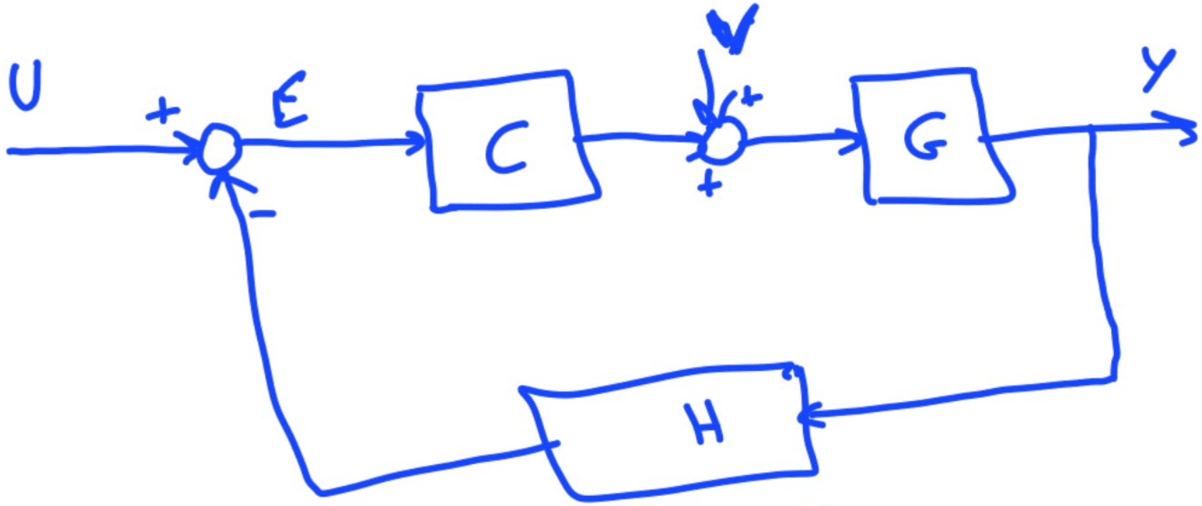
homayoon.beigi@columbia.edu



$$y = CG[U - Hy]$$

$$y[1 + CGH] = CGU$$

$$\boxed{y = \frac{CG}{1+CGH} U}$$

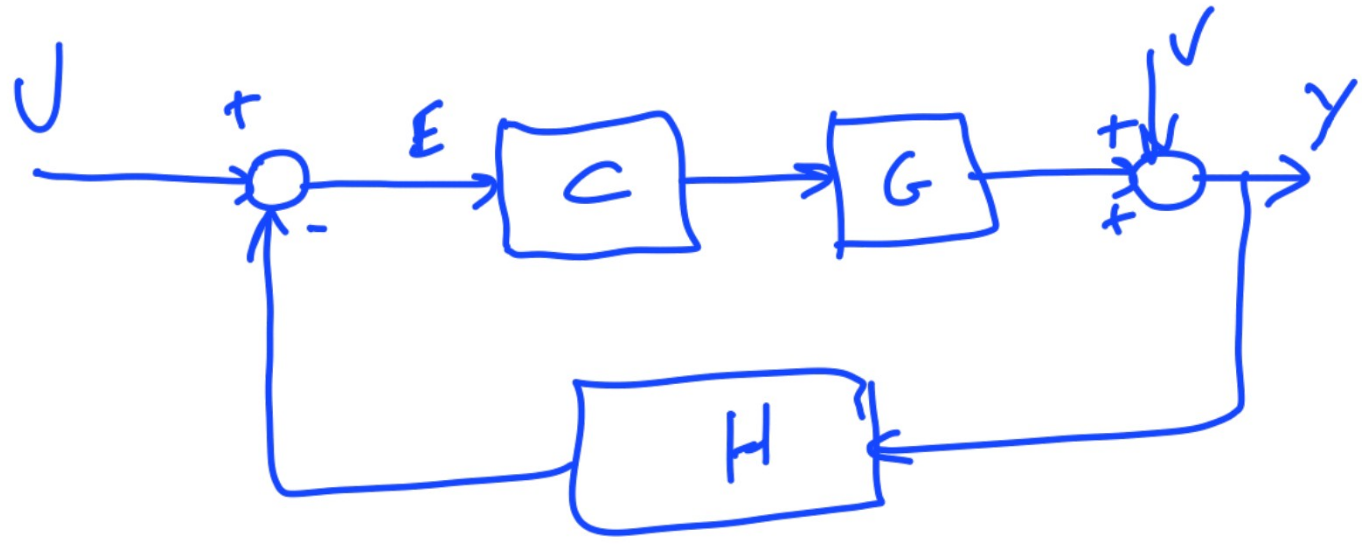


$$Y = \left[\left[U - HY\right]C + V\right]G$$

$$Y\left[1 + CGH\right] = CGU + VG$$

$$\boxed{Y = \frac{CG}{1+CGH}U + \frac{GV}{1+CGH}}$$

homayoon.beigi@columbia.edu



$$y = [U - Hy]\,CG + V$$

$$y\,[1 + HCG] = CGU + V$$

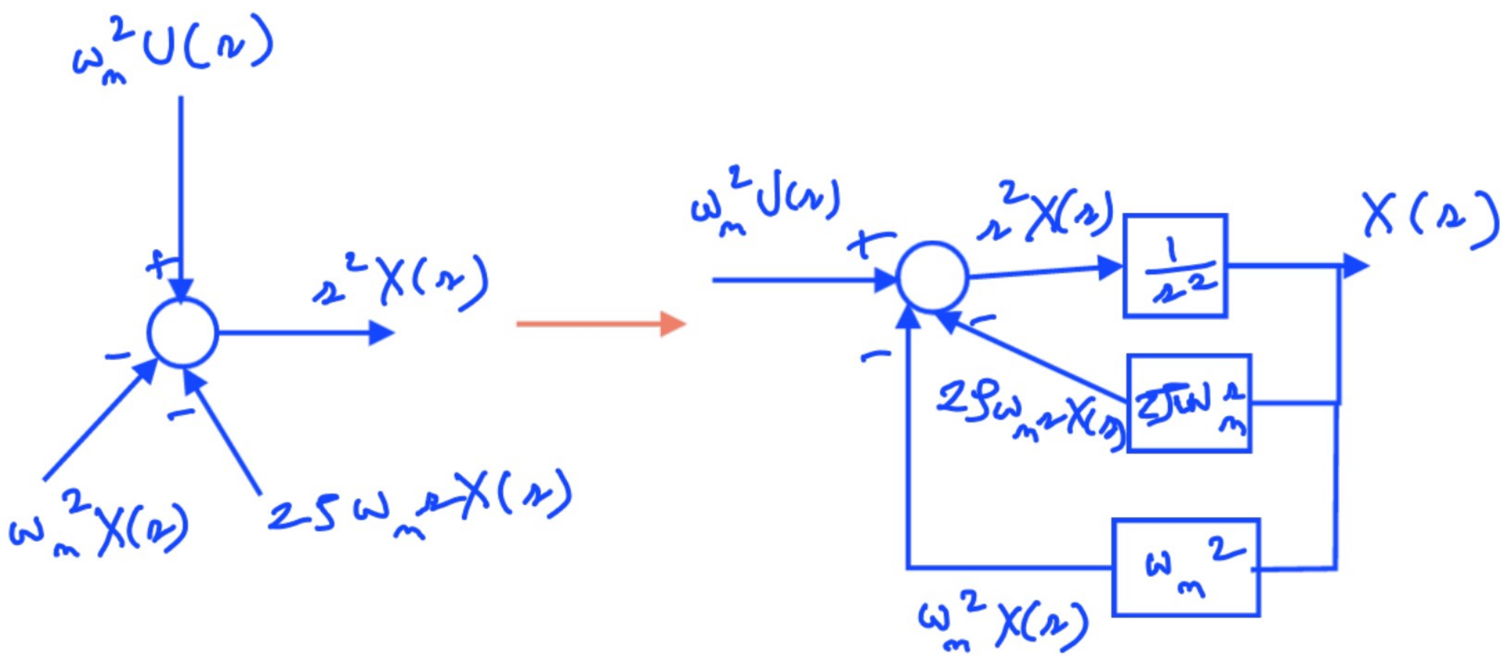
$$\boxed{y = \frac{CG}{1+HCG}U + \frac{V}{1+HCG}}$$

homayoon.beigi@columbia.edu

$$X(s)\left[s^2 + 2\zeta\omega_n s + \omega_n^2\right] = \omega_n^2 U(s)$$

$$\downarrow$$

$$\omega_n^2 U(s) - 2\zeta\omega_n s X(s) - \omega_n^2 X(s) = s^2 X(s)$$



Linear Differential Equations

Apply solution to the original linear system of differential equations:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}e^{\mathbf{A}t}\mathbf{x}_0 \\ &= \mathbf{A}\mathbf{x}(t) \end{aligned} \qquad \begin{aligned} \mathbf{x}(0) &= e^{\mathbf{A}0}\mathbf{x}_0 \\ &= \mathbf{x}(0) \end{aligned}$$

Since this system of equations is time-invariant, then, t_0 may be non-zero with the same results. Namley, for $\mathbf{x}(t_0) = \mathbf{x}_0$,

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}_0$$

Transition Matrix Takes the state from t_0 to t

$$e^{\mathbf{A}(t-t_0)} \triangleq \sum_{n=0}^{\infty} \frac{\mathbf{A}^n(t-t_0)^n}{n!}$$

Linear Differential Equations

$$e^{\mathbf{A}t} = I + At + \frac{1}{2!}\mathbf{A}^2 t^2 + \cdots$$

Let us assume that the **A** is diagonalizable through a transformation matrix **M** such that

Eigenvector Matrix

$$\mathbf{A} = \mathbf{M}\mathbf{\Lambda}\mathbf{M}^{-1}$$

where

Eigenvalue Matrix

$$\mathbf{\Lambda} \triangleq \begin{bmatrix} \lambda_1 & 0 & 0 \ \cdots & 0 & \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

Linear Differential Equations

$$e^{\mathbf{A}t} = I + At + \frac{1}{2!}\mathbf{A}^2 t^2 + \cdots \qquad \mathbf{A} = \mathbf{M\Lambda M}^{-1}$$

Substitube for $\mathbf{A}$ in the definition of the Matrix Exponential,

$$e^{\mathbf{A}t} = \underbrace{\mathbf{I}}_{\mathbf{MM}^{-1}} + \mathbf{M\Lambda M}^{-1}t + \frac{1}{2!}\mathbf{M\Lambda}\underbrace{\mathbf{M}^{-1}\mathbf{M}}_{\mathbf{I}}\mathbf{\Lambda M}^{-1}t^2 + \cdots$$

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{M}\left(\mathbf{I} + \mathbf{\Lambda}t + \frac{1}{2!}\mathbf{\Lambda}^2 t^2 + \cdots\right)\mathbf{M}^{-1} \\ &= \mathbf{M}e^{\mathbf{\Lambda}t}\mathbf{M}^{-1} \end{aligned}$$

$$\begin{aligned} e^{\mathbf{A}t} &= e^{\mathbf{M\Lambda M}^{-1}t} \\ &= \mathbf{M}e^{\mathbf{\Lambda}t}\mathbf{M}^{-1} \end{aligned}$$

Linear Differential Equations

$$\boldsymbol{\Sigma}\mathbf{v} = \lambda\mathbf{v}$$

$\mathbf{v} : \mathscr{R}^1 \mapsto \mathscr{R}^D$ Eigenvectors

$\mathbf{x} : \mathscr{R}^1 \mapsto \mathscr{R}^D$

$\boldsymbol{\Sigma} : \mathscr{R}^D \mapsto \mathscr{R}^D$

Eigenvalue associated with Eigenvector **v**

$$(\lambda\mathbf{I} - \boldsymbol{\Sigma})\mathbf{v} = \mathbf{0} \ \forall \ \mathbf{v}$$

Determinant

$$|\lambda\mathbf{I} - \boldsymbol{\Sigma}| = \prod_{i=1}^{D}(\lambda - \lambda_i) = \mathbf{0}$$

Note that in general $\lambda_i \in \mathbb{C}$.

$$\boldsymbol{\Sigma}\mathbf{v}_i = \lambda_i\mathbf{v}_i \ \forall \ i \in \{1, 2, \cdots, D\}$$

$\mathbf{V} : \mathscr{R}^D \mapsto \mathscr{R}^D$

Sort Eigenvalues from largest to smallest (largest index is 1)

$$\|\mathbf{v}_i\|_{\mathscr{E}} = 1 \ \forall \ i \in \{1, 2, \cdots, D\}$$

Normalize Eigenvectors by dividing each element by the Euclidean norm of the corresponding vector

$$\boldsymbol{\Sigma}\mathbf{V} = \mathbf{V}\boldsymbol{\Lambda}$$

Matrix form after normalization

$\boldsymbol{\Lambda} : \mathscr{R}^D \mapsto \mathscr{R}^D$

$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_D$

homayoon.beigi@columbia.edu

Linear Differential Equations

$$\boldsymbol{\Sigma}\mathbf{V} = \mathbf{V}\boldsymbol{\Lambda}$$

$$\mathbf{V} : \mathscr{R}^D \mapsto \mathscr{R}^D$$
$$\boldsymbol{\Lambda} : \mathscr{R}^D \mapsto \mathscr{R}^D$$
$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_D$$
$$\|\mathbf{v}_i\|_{\mathscr{E}} = 1 \; \forall \; i \in \{1, 2, \cdots, D\}$$

$$\mathbf{V}^{-1}\boldsymbol{\Sigma}\mathbf{V} = \boldsymbol{\Lambda}$$

Eigenvectors are linearly independent, so the Eigenvector matrix has an inverse†

$$\mathbf{V}^T\mathbf{V} = \mathbf{I}$$

In case V can be made unitary (for example for Symmetric Matrices).

$$\mathbf{V}^{-1}\boldsymbol{\Sigma}\mathbf{V} = \mathbf{V}^T\boldsymbol{\Sigma}\mathbf{V}$$
$$= \boldsymbol{\Lambda}$$

†We are assuming that we have distinct Eigenvalues. If λ_i repeats, then there will be a generalized Eigenvector v_i^r associated with the r^th repeated Eigenvalue, where $\boldsymbol{\Lambda}$ will have a Jordan block form. A repeated Eigenvalue is also known as a multiple Eigenvalue or a degenerate Eigenvalue – see Beigi-2011 for more information.

Transformations
Generalized Eigenvalue Problem

$$\lambda_\alpha \boldsymbol{\Sigma}_\alpha \mathbf{v} = \lambda_\beta \boldsymbol{\Sigma}_\beta \mathbf{v}$$

With special case where $\boldsymbol{\Sigma}_\alpha = \boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}_\beta = \mathbf{I}$

$$\left(\lambda_\beta \boldsymbol{\Sigma}_\beta - \lambda_\alpha \boldsymbol{\Sigma}_\alpha\right) \mathbf{v} = \mathbf{0}$$

$$\boldsymbol{\Lambda}_\alpha \boldsymbol{\Sigma}_\alpha \mathbf{V} = \boldsymbol{\Lambda}_\beta \boldsymbol{\Sigma}_\beta \mathbf{V}$$

$$\lambda_{\alpha_n} \boldsymbol{\Sigma}_\alpha \mathbf{v}_n = \lambda_{\beta_n} \boldsymbol{\Sigma}_\beta \mathbf{v}_n$$

If $\boldsymbol{\Lambda}_\alpha$ has full rank,

$$\boldsymbol{\Lambda} \stackrel{\Delta}{=} \boldsymbol{\Lambda}_\alpha^{-1} \boldsymbol{\Lambda}_\beta$$

$$\lambda_n \stackrel{\Delta}{=} \frac{\lambda_{\beta_n}}{\lambda_{\alpha_n}}$$

$$\boldsymbol{\Sigma}_\alpha \mathbf{V} = \boldsymbol{\Lambda} \boldsymbol{\Sigma}_\beta \mathbf{V}$$

$$\boldsymbol{\Sigma}_\alpha \mathbf{v}_n = \lambda_n \boldsymbol{\Sigma}_\beta \mathbf{v}_n$$

Generalized Eigenvalue and Generalized Eigenvector

homayoon.beigi@columbia.edu

Transformations
Generalized Eigenvalue Problem

If $\boldsymbol{\Lambda}_\alpha$ and $\boldsymbol{\Sigma}_\beta$ have full rank,

$$\boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\Sigma}_\alpha \mathbf{V} = \boldsymbol{\Lambda}\mathbf{V}$$

$$\boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\Sigma}_\alpha \mathbf{v}_n = \lambda_n \mathbf{v}_n$$

The Simple Eigenvalue Problem

However, if $\left|\boldsymbol{\Sigma}_\beta\right| = 0$, then there are p Eigenvectors associated with p Eigenvalues where p is the rank of $\boldsymbol{\Sigma}_\beta$ and the rest of the Eigenvalues will be infinite.

If $|\boldsymbol{\Sigma}_\alpha| \neq 0$,

$$\hat{\boldsymbol{\Lambda}} \stackrel{\Delta}{=} \boldsymbol{\Lambda}_\beta^{-1}\boldsymbol{\Lambda}_\alpha$$

$$\hat{\lambda}_n \stackrel{\Delta}{=} \frac{\lambda_{\alpha_n}}{\lambda_{\beta_n}}$$

In this case, the rest of the Eigenvalues (with index greater than p), will be zero.

$$\hat{\lambda}_n \mathbf{v}_n = \boldsymbol{\Sigma}_\alpha^{-1}\boldsymbol{\Sigma}_\beta \mathbf{v}_n$$

Cayley-Hamilton Theorem

Theorem 4 (Cayley-Hamilton)**.** *If* $\mathbf{A} : \mathscr{R}^n \mapsto \mathscr{R}^n$ *is a matrix with the following characteric equation,*

$$\lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0 = 0$$

then **A** *satisies its own characteristic equation. Namely,*

$$\mathbf{A}^n + \alpha_{n-1}\mathbf{A}^{n-1} + \cdots + \alpha_1\mathbf{A} + \alpha_0\mathbf{I} = 0$$

Spectral Theory

Case 1: Eigenvectors associated with different eigenvalues are always linearly independent.

Case 2: If for every eigenvalue λ that is repeated (eg, repeats k-times), the $rank(\mathbf{A} - \lambda\mathbf{I})$ is n (ie, a set of k independent eigenvectors can be found for that eigenvalue, then the matrix can be diagonalized.

Case 3: If $\mathbf{A}$ has repeated eigenvalues for which the condition in case 2 does not hold, then $\mathbf{A}$ cannot be diagonalized and the best we can do is to transform it to a Jordan canonical form which makes use of generalized eigenvectors.

When we do not have enough independent eigenvctors for a repeated eigenvalue, we generate a chain of generalized eigenvectors from each true eigenvector.

homayoon.beigi@columbia.edu

Spectral Theory

The first generated eigenvector satisfies:

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{v}^{(i,1)} = \mathbf{v}^{(i)}$$

The second generated eigenvector satisfies:

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{v}^{(i,2)} = \mathbf{v}^{(i,1)}$$

Note that,

$$\begin{aligned} (\mathbf{A} - \lambda_i \mathbf{I})^2 \mathbf{v}^{(i,1)} &= \mathbf{0} \\ (\mathbf{A} - \lambda_i \mathbf{I})^3 \mathbf{v}^{(i,2)} &= \mathbf{0} \\ &\vdots \end{aligned}$$

Spectral Theory

Consider the case where λ_1 appears 3 times, and having only 1 true eigenvector. All other Eigenvalues are distinct.

$$
\begin{aligned}
\mathbf{A}\mathbf{v}^{(1)} &= \lambda_1 \mathbf{v}^{(1)} \\
\mathbf{A}\mathbf{v}^{(1,1)} &= \lambda_1 \mathbf{v}^{(1,1)} + \mathbf{v}^{(1)} \\
\mathbf{A}\mathbf{v}^{(1,2)} &= \lambda_1 \mathbf{v}^{(1,2)} + \mathbf{v}^{(1,1)} \\
\mathbf{A}\mathbf{v}^{(4)} &= \lambda_4 \mathbf{v}^{(4)} \\
&\vdots \\
\mathbf{A}\mathbf{v}^{(n)} &= \lambda_n \mathbf{v}^{(n)}
\end{aligned}
$$

Spectral Theory

$$\left[\mathbf{A}\mathbf{v}^{(1)}|\mathbf{A}\mathbf{v}^{(1,1)}|\mathbf{A}\mathbf{v}^{(1,2)}|\mathbf{A}\mathbf{v}^{(4)}|\cdots|\mathbf{A}\mathbf{v}^{(n)}\right] = \left[\lambda_1\mathbf{v}^{(1)}|\lambda_1\mathbf{v}^{(1,1)}+\mathbf{v}^{(1)}|\lambda_1\mathbf{v}^{(1,2)}+\mathbf{v}^{(1,1)}|\lambda_4\mathbf{v}^{(4)}|\cdots|\lambda_n\mathbf{v}^{(n)}\right]$$

$$\mathbf{A}\left[\mathbf{v}^{(1)}|\mathbf{v}^{(1,1)}|\mathbf{v}^{(1,2)}|\mathbf{v}^{(4)}|\cdots|\mathbf{v}^{(n)}\right] = \underbrace{\left[\mathbf{v}^{(1)}|\mathbf{v}^{(1,1)}|\mathbf{v}^{(1,2)}|\mathbf{v}^{(4)}|\cdots|\mathbf{v}^{(n)}\right]}_{\mathbf{M}}\underbrace{\begin{bmatrix} \lambda_1 & 1 & 0 \ \cdots & 0 & 0 \\ 0 & \lambda_1 & 1 & \cdots & 0 \\ 0 & 0 & \lambda_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix}}_{\mathbf{J}}$$

$$\boxed{\mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \mathbf{J}}$$

$$\therefore e^{\mathbf{A}t} = \mathbf{M}e^{\mathbf{J}t}\mathbf{M}^{-1}$$

Transformations
Generalized Eigenvalue Problem

Repeated Eigenvalues result in a Jordan block for as follows,

$$\begin{bmatrix} \lambda_1 & 1 & 0 \cdots & 0 & 0 \\ 0 & \lambda_1 & 1 & \cdots & 0 \\ 0 & 0 & \lambda_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

Jordan Block Form
(Different Blocks)

$$\exp \begin{bmatrix} \lambda_1 & 0 & 0 \;\cdots & 0 & \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix} t = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \;\cdots & 0 & \\ 0 & e^{\lambda_2 t} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & e^{\lambda_{n-1} t} & 0 \\ 0 & 0 & 0 & 0 & e^{\lambda_n t} \end{bmatrix}$$

Jordan Block Form (Different Blocks)

$$\exp\begin{bmatrix} 0 & 1 & 0 \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \ldots & \ldots & \ldots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} t = \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n-2}}{(n-2)!} \\ \ldots & \ldots & \ldots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Jordan Block Form
(Different Blocks)

$$\exp \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix} t = e^{\lambda t} \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n-2}}{(n-2)!} \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Jordan Block Form (Different Blocks)

$$\exp\begin{bmatrix} \mathbf{A}_1 & 0 & 0 \cdots & 0 & \\ 0 & \mathbf{A}_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \mathbf{A}_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{A}_n \end{bmatrix} t = \begin{bmatrix} e^{\mathbf{A}_1 t} & 0 & 0 \cdots & 0 & \\ 0 & e^{\mathbf{A}_2 t} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & e^{\mathbf{A}_{n-1} t} & 0 \\ 0 & 0 & 0 & 0 & e^{\mathbf{A}_n t} \end{bmatrix}$$

homayoon.beigi@columbia.edu

Jordan Block Form
(Different Blocks)

$$\exp\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} t = \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}$$

$$\exp\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} t = e^{\sigma t}\begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}$$

Linear Differential Equations Ignoring Input

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$s\vec{X}(s) - \mathbf{x}(0) = \mathbf{A}\vec{X}(s) + \cancel{\mathbf{B}\vec{U}(s)}$$

$$(s\mathbf{I} - \mathbf{A})\vec{X}(s) = \mathbf{x}(0)$$

$$\vec{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0)$$

Compare to the solution in time, $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$

This means that, $\boxed{\mathscr{L}\{e^{\mathbf{A}t}\} = (s\mathbf{I} - \mathbf{A})^{-1}}$

Linear Differential Equations Including Input

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t)\end{aligned}$$

Since there is an input, we can set,

$$\mathbf{x}(0) = \mathbf{0}$$

$$s\vec{X}(s) = \mathbf{A}\vec{X}(s) + \mathbf{B}\vec{U}(s) \rightarrow (s\mathbf{I} - \mathbf{A})\vec{X}(s) = \mathbf{B}\vec{U}(s)$$

$$\begin{aligned}\vec{X}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\vec{U}(s)\\ \vec{Y}(s) &= \left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\right]\vec{U}(s)\end{aligned}$$

If the output depends on the input directly,

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)\end{aligned} \quad \longleftrightarrow \quad \begin{aligned}\vec{X}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\vec{U}(s)\\ \vec{Y}(s) &= \underbrace{\left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right]}_{\mathbf{G}(s)}\vec{U}(s)\end{aligned}$$

Linear Differential Equations
Solution of Forced Systems

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

$$\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t)$$

Multiply the above equation by e^{-At} from the left,

$$e^{-\mathbf{A}t}\left[\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t)\right] = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t) \qquad (192)$$

The left hand side of the above is a perfect differential,

$$e^{-\mathbf{A}t}\left[\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t)\right] = \frac{d}{dt}\left[e^{-\mathbf{A}t}\mathbf{x}(t)\right] \qquad (193)$$

Combining Equations 192 and 193,

$$\frac{d}{dt}\left[e^{-\mathbf{A}t}\mathbf{x}(t)\right] = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t) \qquad (194)$$

Linear Differential Equations
Solution of Forced Systems

$$\frac{d}{dt}\left[e^{-\mathbf{A}t}\mathbf{x}(t)\right] = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t) \tag{194}$$

Integrating both sides of Equation 194,

$$\int_{t_0}^{t} \frac{d}{d\tau}\left[e^{-\mathbf{A}\tau}\mathbf{x}(\tau)\right] d\tau = \int_{t_0}^{t} e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau$$

$$e^{-\mathbf{A}t}\mathbf{x}(t) - e^{-\mathbf{A}t_0}\mathbf{x}(t_0) = \int_{t_0}^{t} e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{196}$$

Multiplying Equation 196 from the left by $e^{\mathbf{A}t}$,

$$\mathbf{x}(t) - e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) = \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

$$\boxed{\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau}$$

Linear Differential Equations
Solution of Forced Systems

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

$$\begin{aligned}\vec{X}(s) &= (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\vec{U}(s)\\ \vec{Y}(s) &= \underbrace{\left[\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\right]}_{\text{Transfer Function Matrix}}\vec{U}(s)\end{aligned}$$

Resolvent Matrix: $\mathscr{L}\{e^{\mathbf{A}t}\} = (s\mathbf{I}-\mathbf{A})^{-1}$

Homework 7

Please see Courseworks