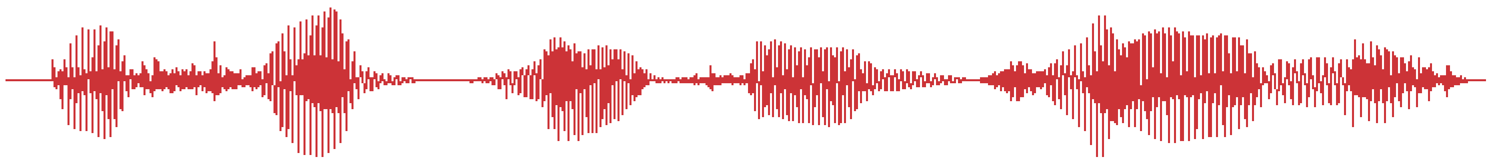
Introduction to Continuous Control Systems

EEME E3601



Week 7

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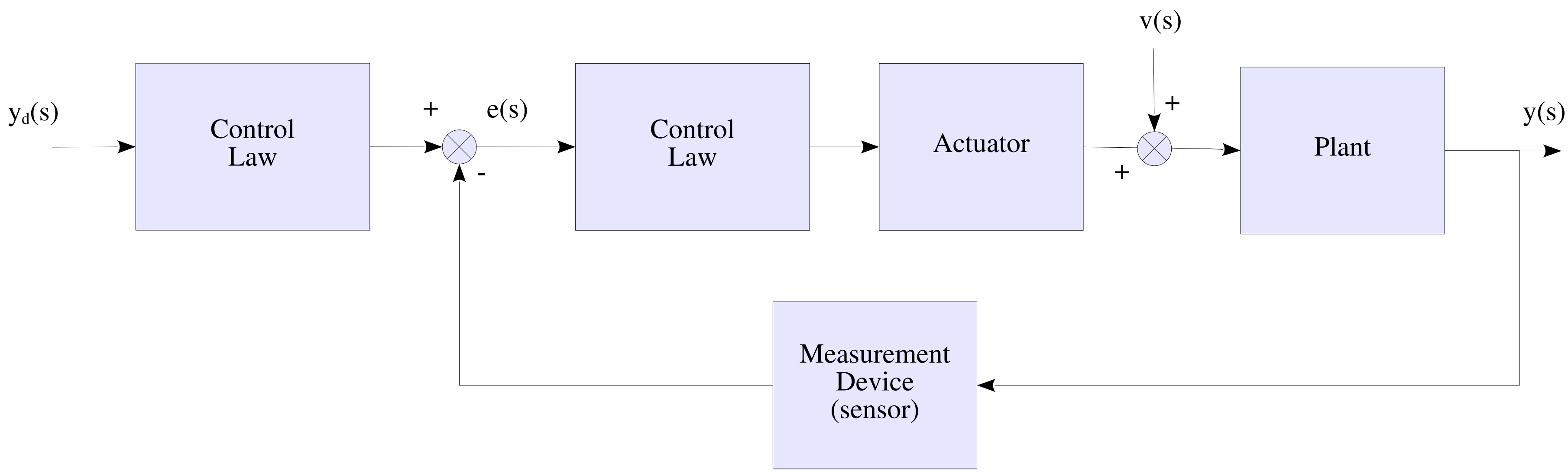
More Complex Feedback Control



Typical Simple Control Laws

$$e(t) = y_d(t) - y(t)$$

P: Proportional Controller

$$m(t) = K\, e(t)$$

D: Derivative Controller

$$m(t) = K \frac{d\, e(t)}{dt}$$

I: Integral Controller

$$m(t) = K \int_0^t e(\tau)\, d\tau$$

PI: Proportional-Integral Controller

$$m(t) = K_1\, e(t) + K_2 \int_0^t e(\tau)\, d\tau$$

PD: Proportional-Derivative Controller

$$m(t) = K_1\, e(t) + K_2 \frac{d\, e(t)}{dt}$$

PID: Proportional-Integral-Derivative Controller

$$m(t) = K_1\, e(t) + K_2 \int_0^t e(\tau)\, d\tau + K_3 \frac{d\, e(t)}{dt}$$

Recall RLC: $$L\frac{dj(t)}{dt} + Rj(t) + \frac{1}{C}\int_0^t j(\tau)d\tau + v(0) = v(t)$$

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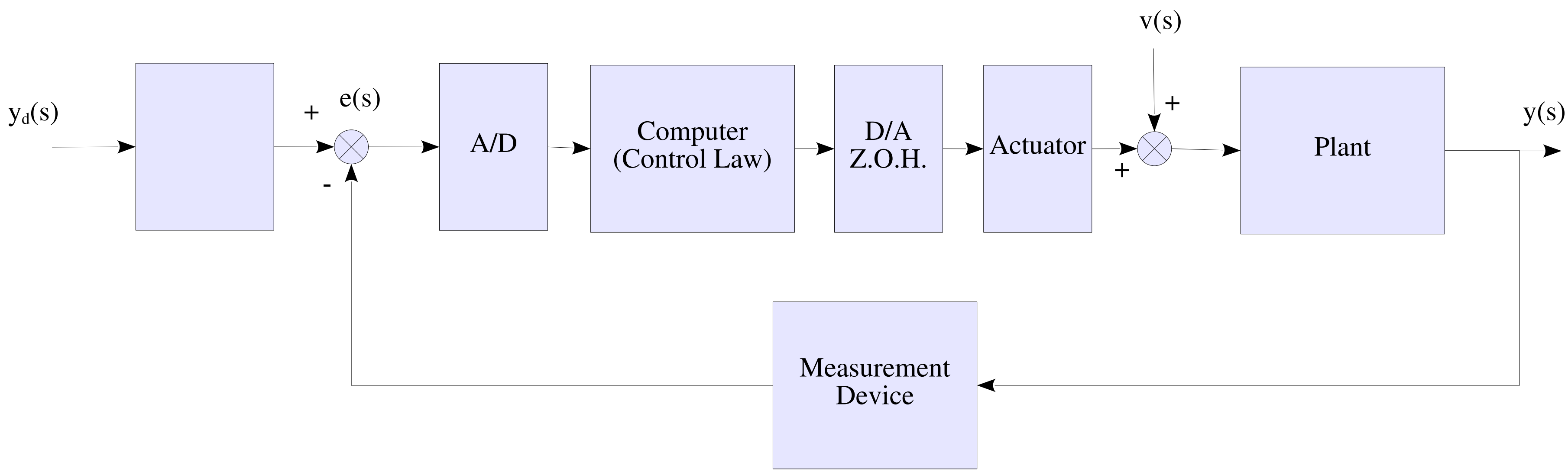
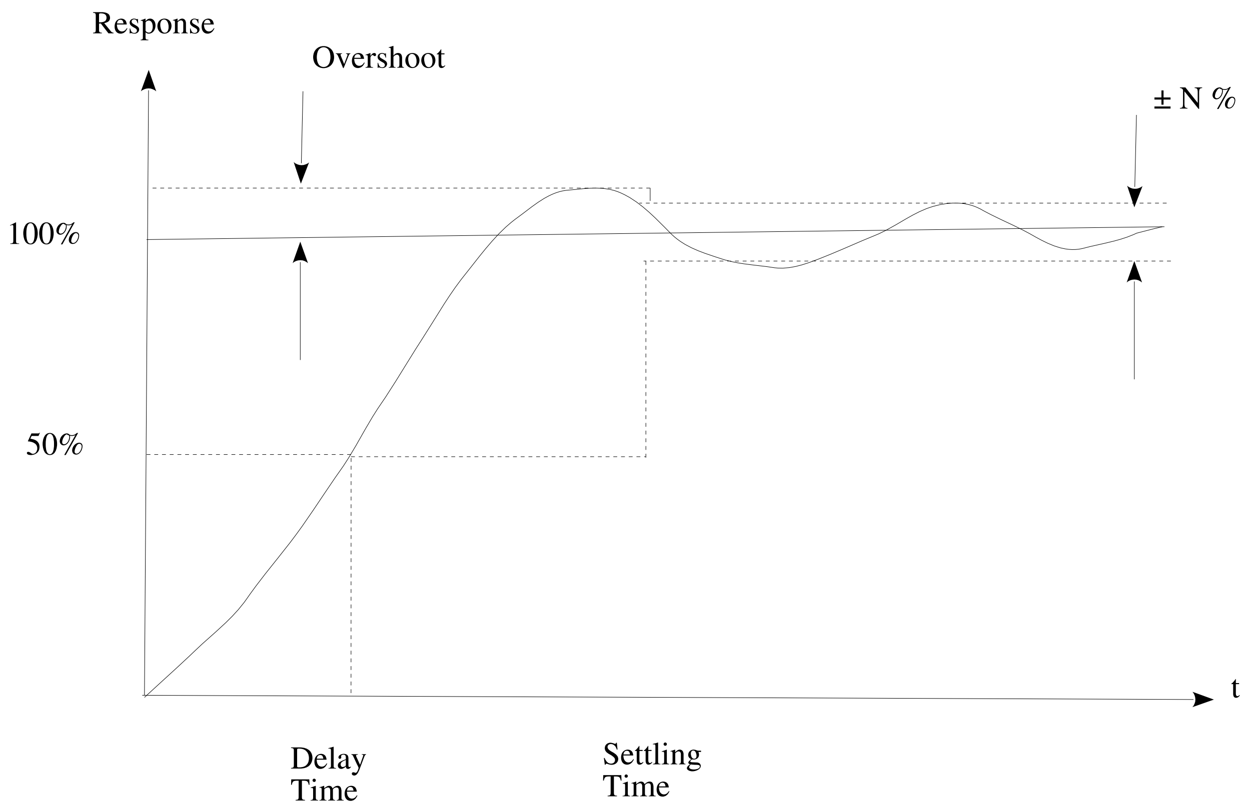
Add Discretization



We will look at this in detail when we speak about discretized systems, difference equations, and the z-Transform

Dynamic System Response

- The Transient Response and the Steady State Response



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Linear Differential Equations

$$\underbrace{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}}_{\dot{\mathbf{x}}(t)} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{\mathbf{x}(t)} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{\mathbf{b}} \underbrace{f(t)}_{u(t)}$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\mathbf{c}^T} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{0}_{d} u(t)$$

Causality

$$\frac{d^2y(t)}{dt^2} + \frac{c}{m}\frac{dy(t)}{dt} + \frac{k}{m}y(t) = 0$$

$$\underbrace{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}}_{\dot{\mathbf{x}}(t)} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{\mathbf{x}(t)} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}}_{\mathbf{b}} \underbrace{v(t)}_{u(t)}$$

$$j(t) = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{\mathbf{c}^T} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{0}_{d} u(t)$$

Causality

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t)$$

$$y(t) = \mathbf{c}^T\mathbf{x}(t) + du(t)$$

Set of ***coupled*** 1st order linear differential equations

$$L\frac{dj(t)}{dt} + Rj(t) + \frac{1}{C}\int_0^t j(\tau)d\tau + v(0) = v(t)$$

Linear Differential Equations

Set of ***coupled*** 1st order linear differential equations

$$\boxed{\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t)\\ y(t) &= \mathbf{c}^T\mathbf{x}(t) + du(t)\end{aligned}}$$

Let us consider the homogeneous part of the above equations:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

For simplicity consider the 1-dimensional version with a generic initial condition:

$$\begin{aligned}\dot{x}(t) &= ax(t)\\ x(0) &= x_0\end{aligned}$$

Linear Differential Equations

$$\dot{x}(t) = ax(t) \quad (72)$$
$$x(0) = x_0 \quad (73)$$

Let us assume that the solution is analytic almost everywhere, then we can write it in the form of a Taylor series:

$$x(t) = \sum_{n=0}^{\infty} \alpha_n (t - t_0)^n$$

In the above problem $t_0 = 0$,

$$x(t) = \sum_{n=0}^{\infty} \alpha_n t^n$$

Therefore, using the given first order differential equation,

$$\begin{aligned}\dot{x}(t) &= \sum_{n=1}^{\infty} n\alpha_n t^{n-1} \\ &= a\sum_{n=0}^{\infty} \alpha_n t^n\end{aligned}$$

Linear Differential Equations

$$\dot{x}(t) = \sum_{n=1}^{\infty} n\alpha_n t^{n-1} \tag{76}$$

$$= a\sum_{n=0}^{\infty} \alpha_n t^n \tag{77}$$

Equating the similar power coefficients between Equations 76 and 77,

$$\begin{aligned}
\alpha_1 &= a\alpha_0 \\
\alpha_2 &= a\frac{\alpha_1}{2} = \frac{a^2\alpha_0}{2} \\
\alpha_3 &= a\frac{\alpha_2}{3} = \frac{a^2\alpha_1}{3\times 2} = \frac{a^3\alpha_0}{3!} \\
&\vdots
\end{aligned}$$

Linear Differential Equations

$$x(t) = \sum_{n=0}^{\infty} \alpha_n t^n \qquad (75)$$

$$\begin{aligned}
\alpha_1 &= a\alpha_0 \\
\alpha_2 &= a\frac{\alpha_1}{2} = \frac{a^2\alpha_0}{2} \\
\alpha_3 &= a\frac{\alpha_2}{3} = \frac{a^2\alpha_1}{3\times 2} = \frac{a^3\alpha_0}{3!} \\
&\vdots
\end{aligned}$$

In general,

$$\alpha_n = \frac{a^n\alpha_0}{n!}$$

Writing Equation 75 for the initial condition,

$$x(0) = x_0 = \alpha_0$$

Linear Differential Equations

$$x(t) = \sum_{n=0}^{\infty} \alpha_n t^n$$

$$\alpha_n = \frac{a^n \alpha_0}{n!}$$

$$x(0) = x_0 = \alpha_0$$

$$\therefore x(t) = \underbrace{\left(\sum_{n=0}^{\infty} \frac{a^n t^n}{n!} \right)}_{e^{at}} x_0$$

$$x(t) = e^{at} x_0$$

Linear Differential Equations

$$
\begin{aligned}
\dot{x}(t) &= ax(t) \\
x(0) &= x_0
\end{aligned}
\qquad x(t) = e^{at}x_0
$$

Apply solution to the original linear scalar differential equation:

$$
\begin{aligned}
\dot{x}(t) &= ae^{at}x_0 \\
&= ax(t)
\end{aligned}
\qquad
\begin{aligned}
x(0) &= e^{a0}x_0 \\
&= x(0)
\end{aligned}
$$

Linear Differential Equations

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) \\
\mathbf{x}(0) &= \mathbf{x}_0
\end{aligned}
$$

Let us assume that the solution is analytic almost everywhere, then we can write it in the form of a Taylor series:

$$
\mathbf{x}(t) = \sum_{n=0}^{\infty} \boldsymbol{\alpha}_n (t - t_0)^n
$$

In the above problem $t_0 = 0$,

$$
\mathbf{x}(t) = \sum_{n=0}^{\infty} \boldsymbol{\alpha}_n t^n
$$

Linear Differential Equations

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) \\ \mathbf{x}(0) &= \mathbf{x}_0\end{aligned} \qquad \mathbf{x}(t) = \sum_{n=0}^{\infty} \boldsymbol{\alpha}_n t^n$$

Therefore, using the given first order differential equation,

$$\dot{\mathbf{x}}(t) = \sum_{n=1}^{\infty} n\boldsymbol{\alpha}_n t^{n-1} \tag{89}$$

$$= \mathbf{A}\sum_{n=0}^{\infty} \boldsymbol{\alpha}_n t^n \tag{90}$$

Equating the similar power coefficients between Equations 89 and 90,

$$\begin{aligned}\boldsymbol{\alpha}_1 &= \mathbf{A}\boldsymbol{\alpha}_0 \\ \boldsymbol{\alpha}_2 &= \mathbf{A}\frac{\boldsymbol{\alpha}_1}{2} = \frac{\mathbf{A}^2\boldsymbol{\alpha}_0}{2} \\ \boldsymbol{\alpha}_3 &= \mathbf{A}\frac{\boldsymbol{\alpha}_2}{3} = \frac{\mathbf{A}^2\boldsymbol{\alpha}_1}{3\times 2} = \frac{\mathbf{A}^3\boldsymbol{\alpha}_0}{3!} \\ &\vdots\end{aligned}$$

Linear Differential Equations

In general, $\boldsymbol{\alpha}_n = \dfrac{\mathbf{A}^n \boldsymbol{\alpha}_0}{n!}$

$$\mathbf{x}(t) = \sum_{n=0}^{\infty} \boldsymbol{\alpha}_n t^n \qquad (88)$$

Writing Equation 88 for the initial condition,

$$\mathbf{x}(0) = \mathbf{x}_0 = \boldsymbol{\alpha}_0$$

$$\therefore \mathbf{x}(t) = \underbrace{\left(\sum_{n=0}^{\infty} \frac{\mathbf{A}^n t^n}{n!} \right)}_{\triangleq e^{\mathbf{A}t}} \mathbf{x}_0$$

Resembles the exponential function, so we define

$$\mathbf{x}(t) = \underbrace{e^{\mathbf{A}t}\mathbf{x}_0}_{}$$

Be careful of the order multiplication

Linear Differential Equations

Definition (The Matrix Exponential). *Given a constant matrix,* $\mathbf{A} : \mathscr{R}^N \mapsto \mathscr{R}^N$,

$$e^{\mathbf{A}t} \triangleq \sum_{n=0}^{\infty} \frac{\mathbf{A}^n t^n}{n!}$$

is the $\boxed{\textit{Matrix Exponential}}$*, also known as the* $\boxed{\textit{Fundamental Matrix}}$ *or the* $\boxed{\textit{Transition Matrix}}$.

The Matrix Exponential converges absolutely and uniformly for any time interval. This may be shown by the fact that each term of the series is bounded, namely,

$$f_n(t) \triangleq \frac{\mathbf{A}^n t^n}{n!} \qquad |f_n(t)| \le M_n \quad \forall\ t : t_0 \le t \le t_1 \quad \textit{and} \quad n = \{0, 1, 2, \cdots\}$$

Therefore, if $\sum_{n=0}^{\infty} M_n$ converges,

then, $e^{\mathbf{A}t}$ converges absolutely and uniformly in that interval.

$$\boxed{\dot{y}(t) = -\lambda y(t) + u(t)}$$

$\lambda > 0 \Rightarrow$ stable

$y(0) = 0$

$$y(t) = \int_0^t e^{-\lambda(t-\tau)} u(\tau)\, d\tau$$

$$Y(s) = \frac{1}{s+\lambda} U(s)$$

$$y(t) = e^{-\lambda t} \quad t > 0$$

Unit impulse response

$$y(t) = \frac{1}{\lambda} - \frac{1}{\lambda} e^{-\lambda t} \quad t > 0$$

Unit step response

steady state value $\frac{1}{\lambda}$ as $t \to \infty$

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First Order

$$\dot{y}(t) = -\lambda y(t) + u(t)$$

$$sY(s) - \cancelto{0}{y(0)} = -\lambda Y(s) + U(s)$$

impulse response: $u(t) = I(t) \Rightarrow U(s) = 1$

$$sY(s) - 0 = -\lambda Y(s) + 1$$

$$Y(s)(s+\lambda) = 1$$

$$Y(s) = \frac{1}{s+\lambda}$$

$$g(t) = e^{-\lambda t}$$

First Order

$$sY(s) - \cancelto{0}{y(0)} = -\lambda Y(s) + U(s)$$

unit step regsponse: $u(t) = u(t) \Rightarrow U(s) = \frac{1}{s}$

$$sY(s) = -\lambda Y(s) + \frac{1}{s}$$

$$Y(s)(s+\lambda) = \frac{1}{s}$$

$$Y(s) = \frac{1}{s(s+\lambda)} = \frac{A}{s} + \frac{B}{s+\lambda}$$

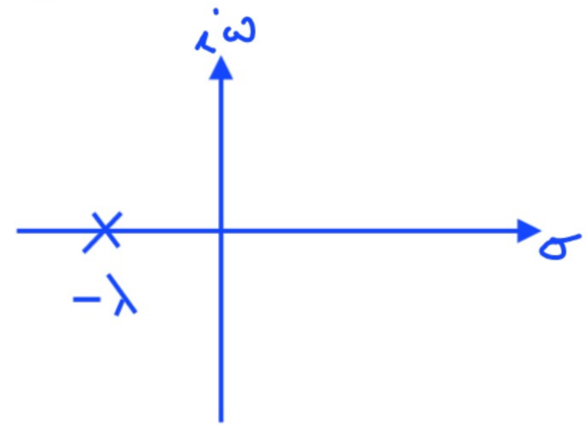
$$A = \left.\frac{1}{s+\lambda}\right|_{s=0} = \frac{1}{\lambda}$$

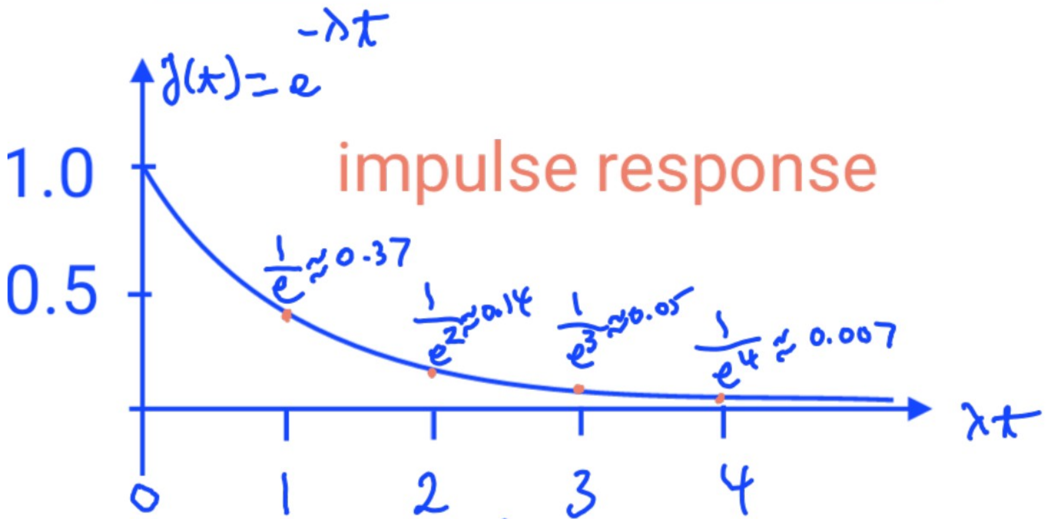
$$B = \left.\frac{1}{s}\right|_{s=-\lambda} = \frac{1}{-\lambda}$$

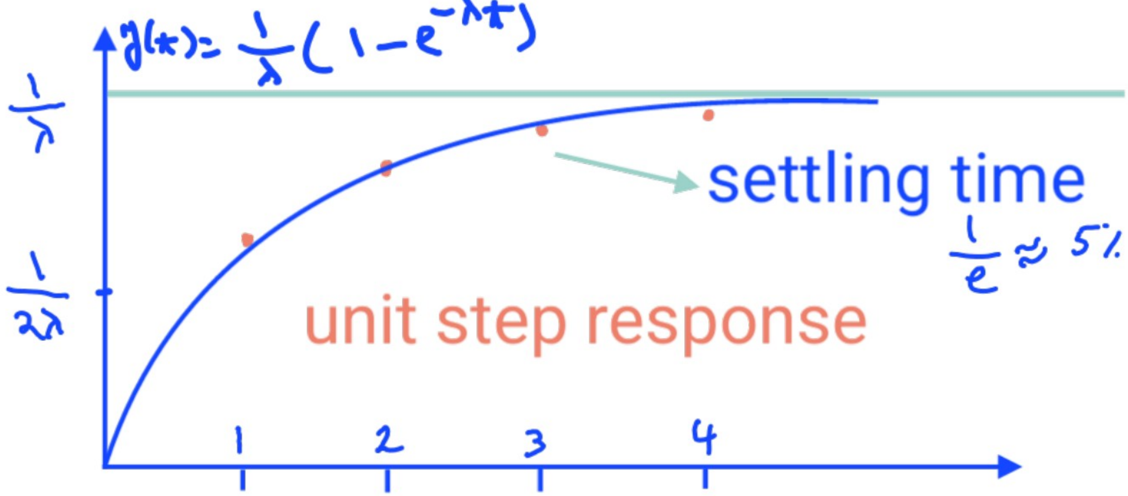
First Order

$$\boxed{\dot{y}(t) = -\lambda y(t) + u(t)}$$

$\lambda > 0 \Rightarrow$ stable







$\frac{1}{\lambda} \triangleq$ time constant

time to drop by $\frac{1}{e}$

Second Order

$$\dot{\vec{x}}(t) = \begin{bmatrix} -\sigma & \omega \\ -\omega & -\sigma \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \vec{x}(t)$$

$s = -\sigma \pm i\omega$ $\quad\quad$ $\sigma > 0 \Rightarrow$ stability

$\vec{x}(0) = \vec{0}$ $\quad\quad$ at rest

$e^{-\sigma t} \sin \omega t$

$$y(t) = \int_0^t \boxed{e^{-\sigma(t-\tau)} \sin(\omega(t-\tau))}\, u(\tau)\, d\tau$$

$\mathcal{L}$

$$Y(s) = H(s)\, U(s) = \frac{\omega}{(s+\sigma)^2 + \omega^2} U(s)$$

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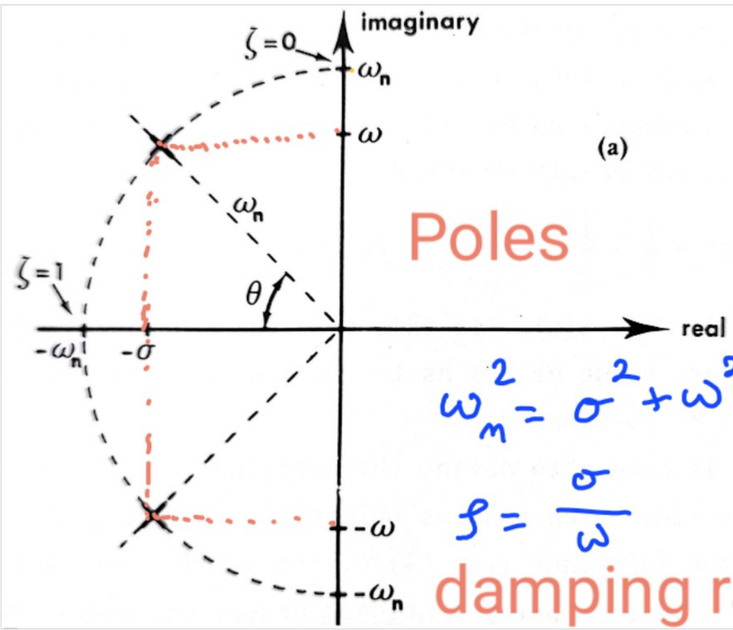
$$\frac{\omega}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$s_{1,2} = \frac{-\cancel{2}\zeta\omega_n \pm i\sqrt{4\omega_n^2 - 4\zeta^2\omega_n^2}}{\cancel{2}}$$

$$= -\underbrace{\zeta\omega_n}_{\sigma} \pm i\,\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega}$$

$$\frac{\omega_n\sqrt{1-\zeta^2}}{(s+\sigma)^2 + \omega_n^2(1-\zeta^2)}$$

Second Order

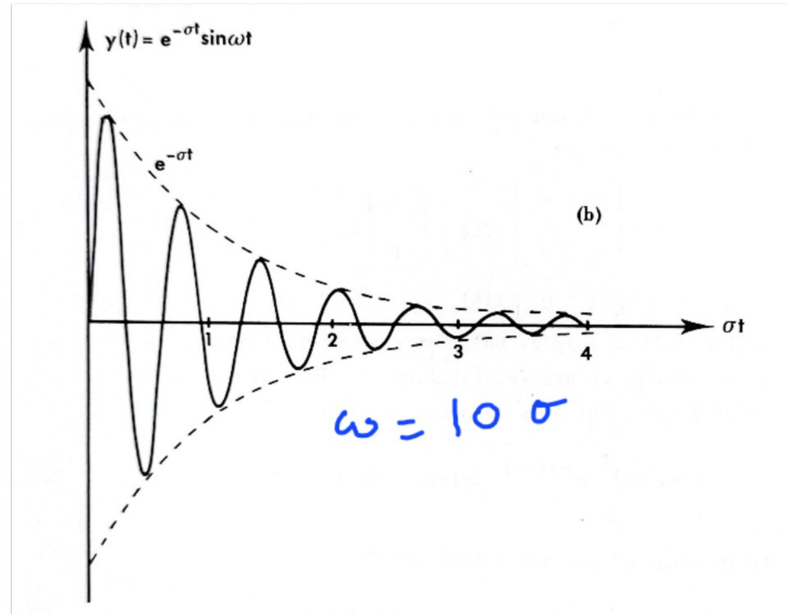


Poles

$$\omega_n^2 = \sigma^2 + \omega^2$$

$$\zeta = \frac{\sigma}{\omega}$$

damping ratio



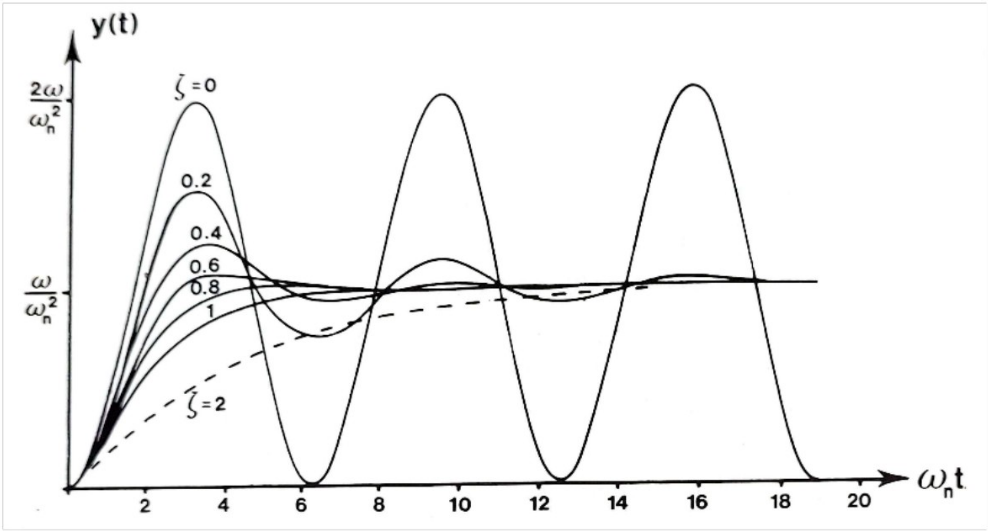
$$\omega = 10\,\sigma$$

$y(t) = e^{-\sigma t}\sin(\omega t) \quad t > 0$ impulse response $\quad \frac{1}{\sigma} \rightarrow$ time constant

$$H(s) = \frac{\omega}{(s+\sigma)^2 + \omega^2} = \frac{\omega}{s^2 + 2\zeta\omega_n + \omega_n^2} \qquad \zeta = \cos\theta$$

$$\omega = \omega_n\sqrt{1-\zeta^2} \qquad \sigma = \omega_n \zeta \qquad \theta = \tan^{-1}\left[\frac{\omega}{-\sigma}\right]$$

Second Order



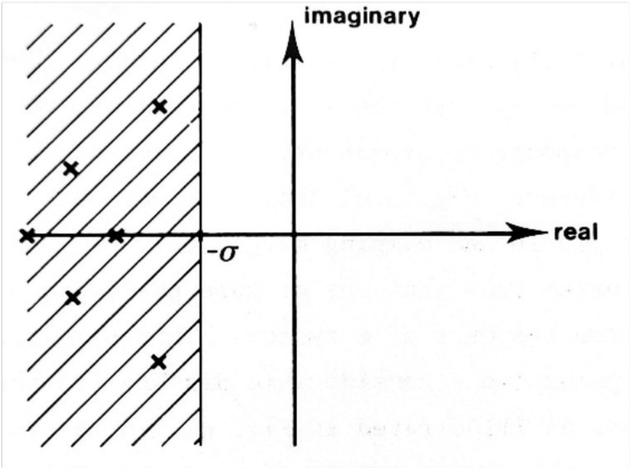
Uinit step response

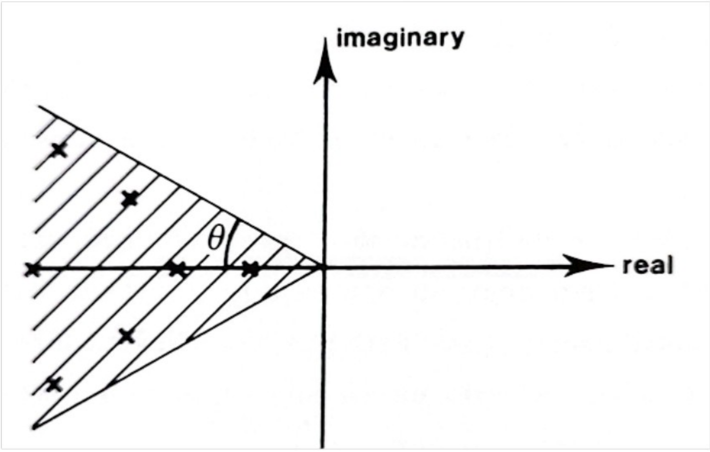
$\zeta = 0 \Rightarrow$ undamped

$0 < \zeta < 1 \Rightarrow$ underdamped

$\zeta = 1 \Rightarrow$ critically damped

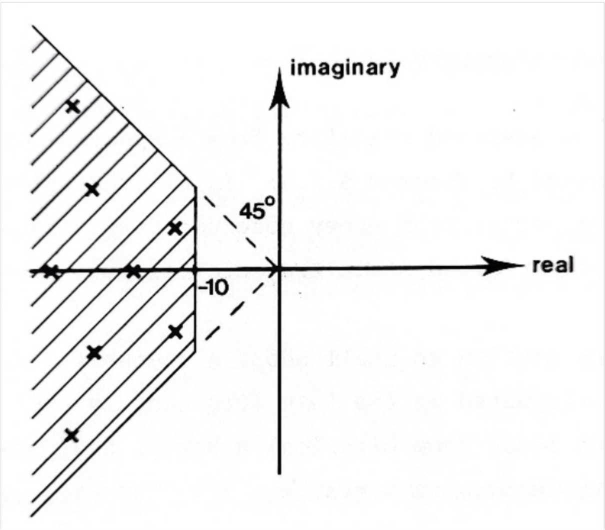
$\zeta > 1 \Rightarrow$ overdamped





stability margin σ

damping margin $\theta = \cos^{-1}\zeta$



stability margin 10 damping margin $0.7 = \cos(45^\circ)$

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$$\frac{1}{\omega_n^2}\ddot{x} + \frac{2\zeta}{\omega_n}\dot{x} + x = 0 \qquad \frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1 = 0$$

$$s_{1,2} = \frac{-\frac{\cancel{2}\zeta}{\omega_{\cancel{n}}} \pm i\sqrt{-\frac{\cancel{4}\zeta^2}{\cancel{\omega_n^2}}+\frac{\cancel{4}}{\cancel{\omega_n^2}}}}{\frac{\cancel{2}}{\omega_n^{\cancel{2}}}}$$

$$= -\overbrace{\zeta\omega_n}^{\sigma} \pm i\overbrace{\omega_n\sqrt{1-\zeta^2}}^{\omega}$$

$$y(t) = e^{-\zeta\omega_n t}\left(C_1\cos\omega_n\sqrt{1-\zeta^2}\,t + C_2\sin\omega_n\sqrt{1-\zeta^2}\,t\right)$$

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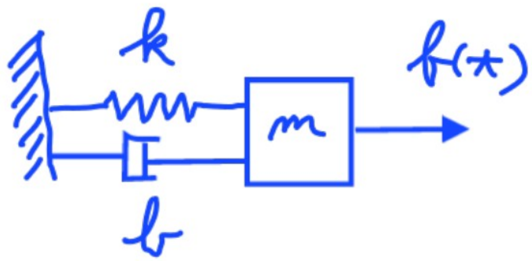
$$m\ddot{x} + b\dot{x} + kx = f(t)$$



$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}f(t)$$

where $r(t) \triangleq \frac{f(t)}{k}$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x + \frac{k}{m}r(t)$$

same units

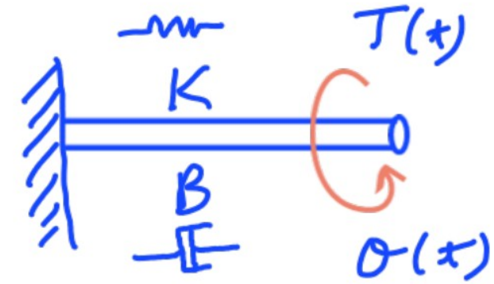
$$\frac{b}{m} = 2\zeta\omega_n$$

$$\frac{k}{m} = \omega_n^2$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 r(t)$$

$$\boxed{\frac{X(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}}$$

Torque



$$I\,\ddot{\theta}(t) + B\,\dot{\theta}(t) + K\,\theta(t) = T(t)$$

$$\ddot{\theta} + \frac{B}{I}\dot{\theta} + \frac{K}{I}\theta = \frac{1}{I}T$$

$$r(t) \triangleq \frac{T}{K}$$

$$\ddot{\theta} + \frac{B}{I}\dot{\theta} + \frac{K}{I}\theta = \frac{K}{I}r$$

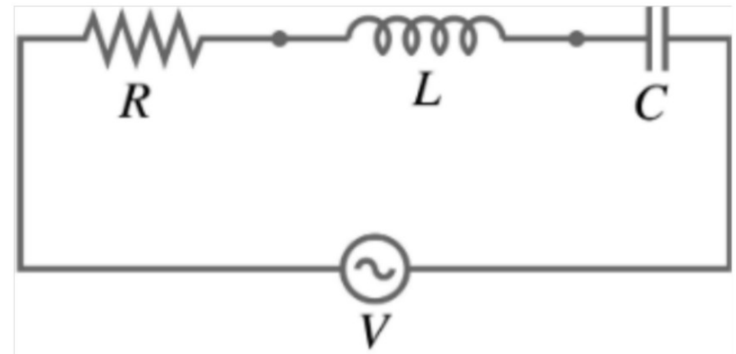
$$\omega_m^2 \triangleq \frac{K}{I}$$

$$2\zeta\omega_m \triangleq \frac{B}{I}$$

$$\ddot{\theta} + 2\zeta\omega_m\dot{\theta} + \omega_m^2\theta = \omega_m^2 r$$

$$\boxed{\frac{\Theta(s)}{R(s)} = \frac{\omega_m^2}{s^2 + 2\zeta\omega_m + \omega_m^2}}$$

RLC Circuit



$$\ddot{j} + \frac{R}{L}\dot{j} + \frac{1}{LC} j = f(t)$$

$$\ddot{j} + 2\zeta\omega_n \dot{j} + \omega_n^2 j = \omega_n^2 r$$

$$\omega_n^2 \triangleq \frac{1}{LC}$$

$$2\zeta\omega_n \triangleq \frac{R}{L}$$

$$r \triangleq \frac{f(t)}{\omega_n^2}$$

$$\boxed{\frac{J(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

example

$$H(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \Longrightarrow U(s) = \frac{1}{s} \text{ or } u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$= \frac{K_0}{s} + \frac{K_{-\sigma+i\omega}}{s+\sigma-i\omega} + \frac{K_{-\sigma-i\omega}}{s+\sigma+i\omega}$$

recall $\sigma = \zeta\omega_n$

$\omega = \omega_n\sqrt{1-\zeta^2}$

residues

$$K_0 = sH(s)\Big|_{s=0} = 1$$

residues (cont.)

$$K_{-\sigma+i\omega} = (s+\sigma-i\omega)\,H(s)\Big|_{s=-\sigma+i\omega}$$

$$= \frac{\omega_n^2}{2i\omega(-\sigma+i\omega)} = \frac{\omega_n}{2\omega}\, e^{-i\left(\theta+\frac{\pi}{2}\right)}$$

recall $\theta = \tan^{-1}\left[\frac{\omega}{-\sigma}\right]$

$$K_{-\sigma-i\omega} = (s+\sigma+i\omega)\,H(s)\Big|_{s=-\sigma-i\omega}$$

$$= \frac{\omega_n^2}{-2i\omega(-\sigma-i\omega)} = \frac{\omega_n}{2\omega}\, e^{i\left(\theta+\frac{\pi}{2}\right)}$$

Inverse Laplace

$$H(s) = \frac{1}{s} + \frac{\omega_n}{2\omega}\left[\frac{e^{-i(\theta+\frac{\pi}{2})}}{s+\sigma-i\omega} + \frac{e^{i(\theta+\frac{\pi}{2})}}{s+\sigma+i\omega}\right]$$

$$h(t) = 1 + \frac{\omega_n}{2\omega}\left[e^{-i(\theta+\frac{\pi}{2})}e^{(-\sigma+i\omega)t} + e^{i(\theta+\frac{\pi}{2})}e^{(-\sigma-i\omega)t}\right]$$

$$= 1 + \frac{\omega_n}{2\omega}e^{-\alpha t}\sin(\omega t-\theta)$$

$$= 1 + \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t-\theta\right)$$

where

$$\theta = \tan^{-1}\left[\frac{\omega}{-\sigma}\right]$$

Linear Differential Equations

Apply solution to the original linear system of differential equations:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}e^{\mathbf{A}t}\mathbf{x}_0 \\ &= \mathbf{A}\mathbf{x}(t)\end{aligned} \qquad \begin{aligned}\mathbf{x}(0) &= e^{\mathbf{A}0}\mathbf{x}_0 \\ &= \mathbf{x}(0)\end{aligned}$$

Since this system of equations is time-invariant, then, t_0 may be non-zero with the same results. Namley, for $\mathbf{x}(t_0) = \mathbf{x}_0$,

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}_0$$

Transition Matrix Takes the state from t_0 to t

$$e^{\mathbf{A}(t-t_0)} \triangleq \sum_{n=0}^{\infty} \frac{\mathbf{A}^n(t-t_0)^n}{n!}$$

Homework 6

See Assignments on Courseworks