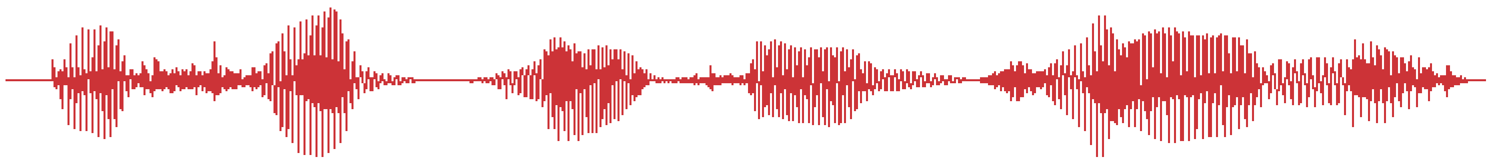
Introduction to Continuous Control Systems

EEME E3601



Week 6

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Linear Differential Equations
Partial Fractions (Recap)

Theorem 3. *If $H(s)$ is a strictly proper ($M \leq N$), rational (ratio of polynomials) function with real coefficients,*

$$
\begin{aligned}
H(s) &= \frac{q_{(M-1)}s^{(M-1)} + q_{(M-2)}s^{(M-2)} + \cdots + q_0}{s^N + p_{(N-1)}s^{(N-1)} + p_{(N-2)}s^{(N-2)} + \cdots + p_0} \\
&= \frac{Q(s)}{(s+s_1)(s+s_2)\cdots(s+s_N)}
\end{aligned}
$$

where $\{s_1, s_2, \cdots, s_N\}$ are distinct roots of $P(s)$, then,

$$H(s) = \frac{R_1}{(s+s_1)} + \frac{R_2}{(s+s_2)} + \cdots + \frac{R_N}{(s+s_N)}$$

where the residues, R_n, are complex numbers given by the following,

$$R_n = [(s+s_n)H(s)]_{s=-s_n}$$

where $n = \{1, 2, \cdots, N\}$
Then, the inverse Laplace Transfor of $H(s)$ is given as,

$$h(t) = \sum_{n=1}^{N} R_n e^{-s_n t}$$

Linear Differential Equations
Partial Fractions Repeated Poles (Recap)

If $H(s)$ is a strictly proper ($M \leq N$), rational (ratio of polynomials) function with real coefficients,

$$\begin{aligned} H(s) &= \frac{q_{(M-1)}s^{(M-1)} + q_{(M-2)}s^{(M-2)} + \cdots + q_0}{s^N + p_{(N-1)}s^{(N-1)} + p_{(N-2)}s^{(N-2)} + \cdots + p_0} \\ &= \frac{Q(s)}{(s+s_1)(s+s_2)\cdots(s+s_i)^r\cdots(s+s_N)} \end{aligned}$$

where the pole s_i is repeated r times, then $H(s)$ may be expanded as follows,

$$H(S) = \frac{R_1}{(s+s_1)} + \frac{R_2}{(s+s_2)} + \cdots + \frac{R_i^{(1)}}{(s+s_i)} + \cdots + \frac{R_i^{(r)}}{(s+s_i)^r} + \cdots + \frac{R_N}{(s+s_N)}$$

where the $R_i^{(1)}$ through $R_i^{(r)}$ associated with the repeated pole, s_i may be computed as follows,

$$R_i^{(r)} = \left[(s+s_i)^r H(s)\right]_{s=-s_i}$$

$$R_i^{(r-2)} = \left[\frac{1}{2!}\frac{d^2}{ds^2}\{(s+s_i)^r H(s)\}\right]_{s=-s_i}$$

$$R_i^{(r-1)} = \left[\frac{d}{ds}\{(s+s_i)^r H(s)\}\right]_{s=-s_i}$$

$$R_i^{(1)} = \left[\frac{1}{(r-1)!}\frac{d^{(r-1)}}{ds^{(r-1)}}\{(s+s_i)^r H(s)\}\right]_{s=-s_i}$$

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Linear Differential Equations
Partial Fractions Complex Conjugate Poles

Let us assume that we have a pole which is complex (includes an imaginary part). In that case, the poles always occur as complex conjugates,

$$\begin{aligned} s_i &= \sigma_i + i\omega_i \\ s_{i+1} &= \sigma_i - i\omega_i = \bar{s}_i \end{aligned}$$

Then the residues associated with these two poles may be written as,

$$\begin{aligned} R_i &= \frac{1}{2}C_i - \frac{1}{2}iS_i \\ R_{i+1} &= \frac{1}{2}C_i + \frac{1}{2}iS_i = \overline{R}_i \end{aligned}$$

The above residues may be alternatively written as,

$$\frac{R_i}{(s+s_i)} + \frac{\overline{R}_i}{(s+\bar{s}_i)} = \frac{C_i(s+\sigma_i) - S_i\omega_i}{(s+\sigma_i)^2 + \omega_i^2}$$

where $C_i = 2\mathscr{R}e(R_i)$ and $S_i = -2\mathscr{I}m(R_i)$ may be computed from

$$2iR_i = S_i + iC_i = \frac{1}{\omega_i}\left[\left((s+\sigma_i)^2 + \omega^2\right) H(s)\right]_{s=-\sigma_i+i\omega_i}$$

The inverse Laplace Transform is then given by,

$$C_i e^{-\sigma_i t}\cos(\omega_i t) + S_i e^{-\sigma_i t}\sin(\omega_i t)$$

which may also be written in terms of the magnitude and phase.

Linear Differential Equations
Laplace Transform Table

Laplace Transform	Time Domain Function
1	Dirac Delta: $\delta(t)$
$\frac{1}{s}$	Unit Step: $u(t)$
$\frac{1}{1-e^{-Ts}}$	$\delta_T \stackrel{\Delta}{=} \sum_{n=0}^{\infty} \delta(t-nT)$
$\frac{1}{s^2}$	Ramp: t
$\frac{1}{s^3}$	$\frac{t^2}{2}$
$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$
$\frac{1}{s+a}$	e^{-at}
$\frac{1}{(s+a)^2}$	te^{-at}
$\frac{a}{s(s+a)}$	$1-e^{-at}$
$\frac{\omega}{(s^2+\omega^2)}$	$\sin(\omega t)$
$\frac{s}{(s^2+\omega^2)}$	$\cos(\omega t)$
$\frac{\omega}{((s+a)^2+\omega^2)}$	$e^{-at}\sin(\omega t)$
$\frac{s+a}{((s+a)^2+\omega^2)}$	$e^{-at}\cos(\omega t)$

Linear Differential Equations Laplace Transform Solution Example

$$\ddot{y}(t) + 7\dot{y}(t) + 12y(t) = 0$$

subject to the following initial conditions

$$\begin{aligned} y(0) &= 4 \\ \dot{y}(0) &= 7 \end{aligned}$$

Take the Laplace Transform of the differential Equation,

$$s^2Y(s) - sy(0) - \dot{y}(0) + 7\left[sY(s) - y(0)\right] + 12Y(s) = 0$$

$$Y(s)\left[s^2 + 7s + 12\right] - 4s - 7 - 28 = 0$$

$$Y(s)\left[s^2 + 7s + 12\right] = 4s + 35$$

$$Y(s) = \frac{4s + 35}{s^2 + 7s + 12}$$

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Linear Differential Equations Laplace Transform Solution Example

$$\ddot{y}(t) + 7\dot{y}(t) + 12y(t) = 0$$

$$\begin{aligned} y(0) &= 4 \\ \dot{y}(0) &= 7 \end{aligned}$$

$$Y(s) = \frac{4s+35}{s^2+7s+12}$$

$$\begin{aligned} s_{1,2} &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{-7 \pm \sqrt{49-48}}{2} \\ &= -3, -4 \end{aligned}$$

$$\begin{aligned} Y(s) &= \frac{4s+35}{s^2+7s+12} \\ &= \frac{4s+35}{(s+3)(s+4)} \\ &= \frac{A}{s+3} + \frac{B}{s+4} \end{aligned}$$

$$\begin{aligned} A &= \frac{4s+35}{s+4}\Big|_{s=-3} \\ &= \frac{-12+35}{1} \\ &= 23 \end{aligned}$$

$$\begin{aligned} B &= \frac{4s+35}{s+3}\Big|_{s=-4} \\ &= \frac{-16+35}{-1} \\ &= -19 \end{aligned}$$

$$Y(s) = \frac{23}{s+3} - \frac{19}{s+4}$$

$$\therefore \boxed{y(t) = 23e^{-3t} - 19e^{-4t}}$$

Linear Differential Equations
Laplace Transform Solution
Example

$$\ddot{y}(t) + 4\dot{y}(t) + 13y(t) = u(t) \qquad u(t) \triangleq \begin{cases} 1 & \forall \quad t > 0 \\ 0 & \forall \quad t \leq 0 \end{cases}$$

Subject to the following initial conditions,

$$\begin{aligned} y(0) &= 0 \\ \dot{y}(0) &= 0 \end{aligned}$$

$$s^2Y(s) - s\cancelto{0}{y(0)} - \cancelto{0}{\dot{y}(0)} + \left[4sY(s) - \cancelto{0}{y(0)}\right] + 13Y(s) = \frac{1}{s}$$

$$Y(s)\left[s^2 + 4s + 13\right] = \frac{1}{s}$$

$$Y(s) = \frac{1}{s\left(s^2 + 4s + 13\right)}$$

Linear Differential Equations
Laplace Transform Solution
Example

$$Y(s) = \frac{1}{s\left(s^2 + 4s + 13\right)}$$

$$s_1 = 0$$

$$\begin{aligned} s_{2,3} &= \frac{-h \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-4 \pm 6i}{2} \\ &= -2 \pm 3i \end{aligned}$$

$$Y(s) = \frac{1}{s\left(s - (-2 + 3i)\right)\left(s - (-2 - 3i)\right)}$$

$$\begin{aligned} [(s+2) - 3i]\,[(s+2) + 3i] &= (s+2)^2 - \cancel{3i(s+2)} + \cancel{3i(s+2)} + 9 \\ &= (s+2)^2 + 9 \end{aligned}$$

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Linear Differential Equations
Laplace Transform Solution

$$Y(s) = \frac{A}{s} + \frac{C(s+2)+S(3)}{(s+2)^2+9}$$

$$\begin{aligned}
S+iC &= \frac{1}{\omega_i}\left[\frac{1}{s}\right]_{s=-\sigma+i\omega} \\
&= \frac{1}{3}\left[\frac{1}{s}\right]_{s=-2+3i} \\
&= \frac{1}{3}\left[\frac{1}{-2+3i}\right] \\
&= \frac{1}{3}\frac{1}{-2+3i}\frac{-2-3i}{-2-3i} \\
&= \frac{1}{3}\left[\frac{-2-3i}{4+9}\right] \\
&= \frac{1}{3}\left[\frac{-2-3i}{13}\right] \\
&= -\frac{1}{13}\left[\frac{2}{3}+i\right]
\end{aligned}$$

$$S-iC = -\frac{1}{13}\left[\frac{2}{3}-i\right]$$

Linear Differential Equations
Laplace Transform Solution
Example

$$Y(s) = \frac{A}{s} + \frac{C(s+2)+S(3)}{(s+2)^2+9}$$

$$\begin{aligned} A &= \left[\frac{1}{(s+2-3i)(s+2+3i)}\right]_{s=0} \\ &= \frac{1}{(2-3i)(2+3i)} \\ &= \frac{1}{4+9} \\ &= \frac{1}{13} \end{aligned}$$

$$\begin{aligned} C &= -\frac{1}{13} \\ S &= -\frac{2}{39} \end{aligned}$$

$$\boxed{y(t) = \frac{1}{13} - \frac{1}{13}e^{-2t}\left[\cos(3t) + \frac{2}{3}\sin(3t)\right]}$$

Differential Equations
Rigid Body Dynamics

$$\tau = \mathbf{M}(\boldsymbol{\alpha})\ddot{\boldsymbol{\alpha}} + \mathbf{c}(\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}) + \mathbf{f}(\dot{\boldsymbol{\alpha}}) + \mathbf{g}(\boldsymbol{\alpha}) + \tau_d$$

$\tau : \mathscr{R}^1 \mapsto \mathscr{R}^N$	vector of generalized forces applied to the system
$\mathbf{M}(\boldsymbol{\alpha}) : \mathscr{R}^N \mapsto \mathscr{R}^N$	equivalent mass matrix
$\mathbf{c}(\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}) : \mathscr{R}^1 \mapsto \mathscr{R}^N$	vector of generilized forces due to Coriolis and centrifugal effects
$\mathbf{f}(\dot{\boldsymbol{\alpha}}) : \mathscr{R}^1 \mapsto \mathscr{R}^N$	vector of generalized forces due to viscous friction
$\mathbf{g}(\dot{\boldsymbol{\alpha}}) : \mathscr{R}^1 \mapsto \mathscr{R}^N$	vector of generalized gravitational forces
$\tau_d : \mathscr{R}^1 \mapsto \mathscr{R}^N$	vector of disturbances such as friction and other unmodeled foces

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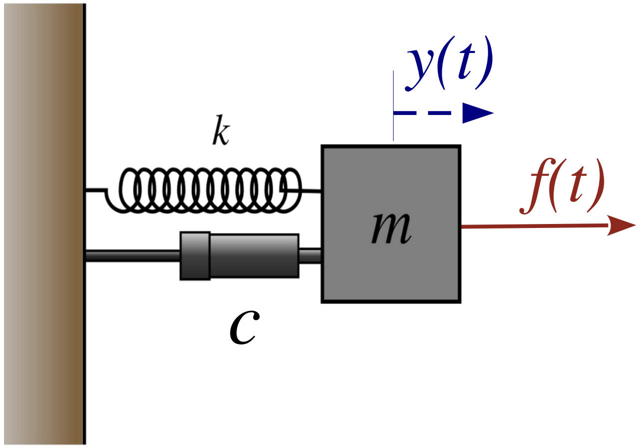

Differential Equations
2nd Order Differential Equation
(Mass-Spring-Dashpot)

$$m\frac{d^2y(t)}{dt^2} + c\frac{dy(t)}{dt} + ky(t) = f(t)$$

Inertial Force — Viscous Damping — Spring Force — Forcing Function

$$m\frac{d^2y(t)}{dt^2} + c\frac{dy(t)}{dt} + ky(t) = 0$$

Homogeneous Equation

$$\frac{d^2y(t)}{dt^2} + \frac{c}{m}\frac{dy(t)}{dt} + \frac{k}{m}y(t) = 0$$

$$\omega_n \stackrel{\Delta}{=} \sqrt{\frac{k}{m}}$$

Natural Angular Frequency

$$\zeta \stackrel{\Delta}{=} \frac{c}{2\sqrt{km}}$$

Damping Ratio

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n\frac{dy(t)}{dt} + \omega_n^2 y(t) = 0$$

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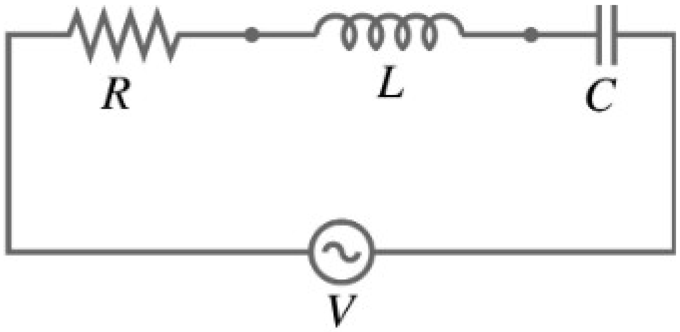
Differential Equations
Integro-Differential Equation
(RLC)



$$L\frac{dj(t)}{dt} + Rj(t) + \frac{1}{C}\int_0^t j(\tau)d\tau + v(0) = v(t)$$

If differentiate both sides of the $L\frac{d^2j(t)}{dt^2} + R\frac{dj(t)}{dt} + \frac{1}{C}j(t) = 0$ to t,

$$L\frac{d^2j(t)}{dt^2} + R\frac{dj(t)}{dt} + \frac{1}{C}j(t) = \frac{dv(t)}{dt}$$

Differential Equation

$$L\frac{d^2j(t)}{dt^2} + R\frac{dj(t)}{dt} + \frac{1}{C}j(t) = 0$$

Homogeneous Equation

$$\frac{d^2j(t)}{dt^2} + \frac{R}{L}\frac{dj(t)}{dt} + \frac{1}{LC}j(t) = 0$$

Compare to Mass Spring Dashpot

Linear Differential Equations
Causality

$$
\begin{aligned}
\frac{d^{(n)}y(t)}{dt^{(n)}} &+ p_{n-1}\frac{d^{(n-1)}y(t)}{dt^{(n-1)}} + p_{n-2}\frac{d^{(n-2)}y(t)}{dt^{(n-2)}} + \cdots + p_0 y(t) \\
&= q_{n-1}\frac{d^{(n-1)}u(t)}{dt^{(n-1)}} + q_{n-2}\frac{d^{(n-2)}u(t)}{dt^{(n-2)}} + \cdots + q_0 u(t)
\end{aligned}
$$

Sample Plant
(Telescope)

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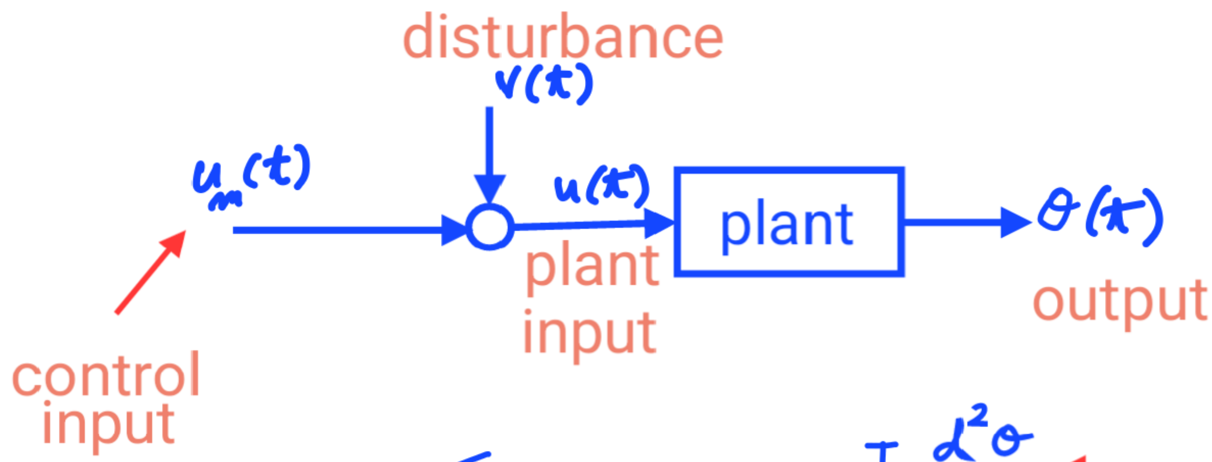
angle

$\theta_d(t)$ desired

$\theta(t)$ actual



$$\sum \text{torques} = I \frac{d^2\theta}{dt^2}$$

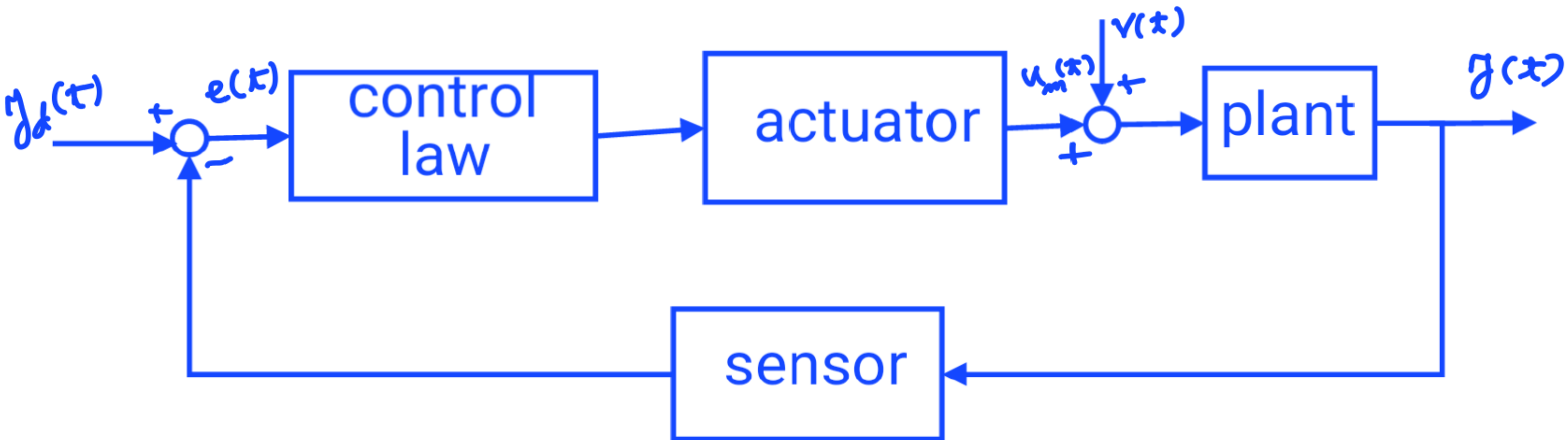
angular accelerations

moment of inertia

$$u(t) - \underbrace{b\frac{d\theta}{dt}}_{\text{damping force}}$$

$$u(t) = u_m(t) + v(t)$$

$$\boxed{I \frac{d^2\theta(t)}{dt^2} + b\frac{d\theta(t)}{dt} = u_m(t) + v(t)}$$



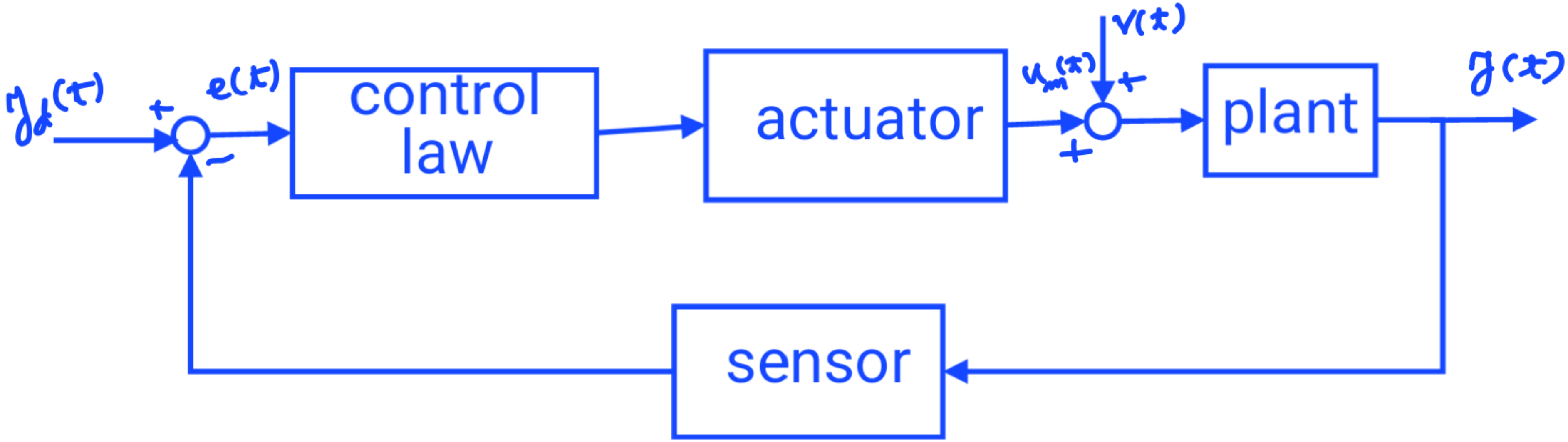
proportional control

$$u_m(t) = K_p e(t)$$

constant $v(t)$ ⟶ constant error

integral control

$$u_m(t) = K_I \int_0^t e(\tau)\, d\tau$$



PI control

$$u_m(t) = K_p e(t) + K_I \int_0^t e(\tau)\, d\tau$$

PD

$$u_m(t) = K_p e(t) + K_d \frac{de(t)}{dt}$$

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P:

$$I \frac{d^2\theta(t)}{dt^2} + b \frac{d\theta(t)}{dt} = K_p e(t) + r(t)$$

$$= K_p \left(\theta_d(t) - \theta(t)\right) + r(t)$$

$$I \frac{d^2\theta(t)}{dt^2} + b \frac{d\theta(t)}{dt} + K_p\theta(t) = K_p\theta_d(t) + r(t)$$

PD:

$$I \frac{d^2\theta(t)}{dt^2} + b \frac{d\theta(t)}{dt} = K_p e(t) + K_D \frac{de(t)}{dt} + r(t)$$

$$I \frac{d^2\theta(t)}{dt^2} + b \frac{d\theta(t)}{dt} + K_p\theta(t) = K_p\theta_d(t) + \underbrace{K_D \frac{d\theta(t)}{dt}}_{\text{assuming constant } \theta_d(t)} + r(t)$$

PID:

$$u_m(t) = K_p e(t) + K_I \int_0^t e(\tau)\, d\tau + K_D \frac{de(t)}{dt}$$

lead compensator

lag compensator

filters

pole- zero cancellation

etc.

Differential Equations
State-Space Representation

$$
\begin{aligned}
x_1(t) &\stackrel{\Delta}{=} y(t) \\
x_2(t) &\stackrel{\Delta}{=} \frac{dy(t)}{dt} = \frac{dx_1(t)}{dt}
\end{aligned}
$$

Recall

$$\frac{d^2y(t)}{dt^2} + \frac{c}{m}\frac{dy(t)}{dt} + \frac{k}{m}y(t) = 0$$

$$\therefore \frac{dx_2(t)}{dt} = -\frac{k}{m}x_1(t) - \frac{c}{m}x_2(t) + f(t)$$

$$
\begin{aligned}
\frac{dx_1(t)}{dt} &= x_2(t) \\
\frac{dx_2(t)}{dt} &= -\frac{k}{m}x_1(t) - \frac{c}{m}x_2(t) + \frac{1}{m}f(t)
\end{aligned}
$$

$$\underbrace{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}}_{\dot{\mathbf{x}}(t)} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{\mathbf{x}(t)} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{\mathbf{b}} \underbrace{f(t)}_{u(t)}$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\mathbf{c}^T} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{0}_{d} u(t)$$

Causality

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \\
y(t) &= \mathbf{c}^T\mathbf{x}(t) + du(t)
\end{aligned}
$$

Differential Equations
State-Space Representation

Recall

$$L\frac{dj(t)}{dt} + Rj(t) + \frac{1}{C}\int_0^t j(\tau)d\tau + v(0) = v(t)$$

$$x_1(t) \triangleq \int_0^t j(\tau)d\tau$$

$$x_2(t) \triangleq \frac{dx_1(t)}{dt} = j(t)$$

$$\therefore \frac{dx_2(t)}{dt} = -\frac{1}{LC}x_1(t) - \frac{R}{L}x_2(t) + \frac{1}{L}v(t)$$

$$\frac{dx_1(t)}{dt} = x_2(t)$$

$$\frac{dx_2(t)}{dt} = -\frac{1}{LC}x_1(t) - \frac{R}{L}x_2(t) + \frac{1}{L}v(t)$$

$$\underbrace{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}}_{\dot{\mathbf{x}}(t)} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{\mathbf{x}(t)} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}}_{\mathbf{b}} \underbrace{v(t)}_{u(t)}$$

$$j(t) = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{\mathbf{c}^T} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{0}_{d} u(t)$$

Causality

$$\longrightarrow$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t)$$

$$y(t) = \mathbf{c}^T\mathbf{x}(t) + du(t)$$

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Linear Differential Equations
Causality

$$\frac{d^{(n)}y(t)}{dt^{(n)}} + p_{n-1}\frac{d^{(n-1)}y(t)}{dt^{(n-1)}} + p_{n-2}\frac{d^{(n-2)}y(t)}{dt^{(n-2)}} + \cdots + p_0 y(t)$$
$$= q_{n-1}\frac{d^{(n-1)}u(t)}{dt^{(n-1)}} + q_{n-2}\frac{d^{(n-2)}u(t)}{dt^{(n-2)}} + \cdots + q_0 u(t)$$

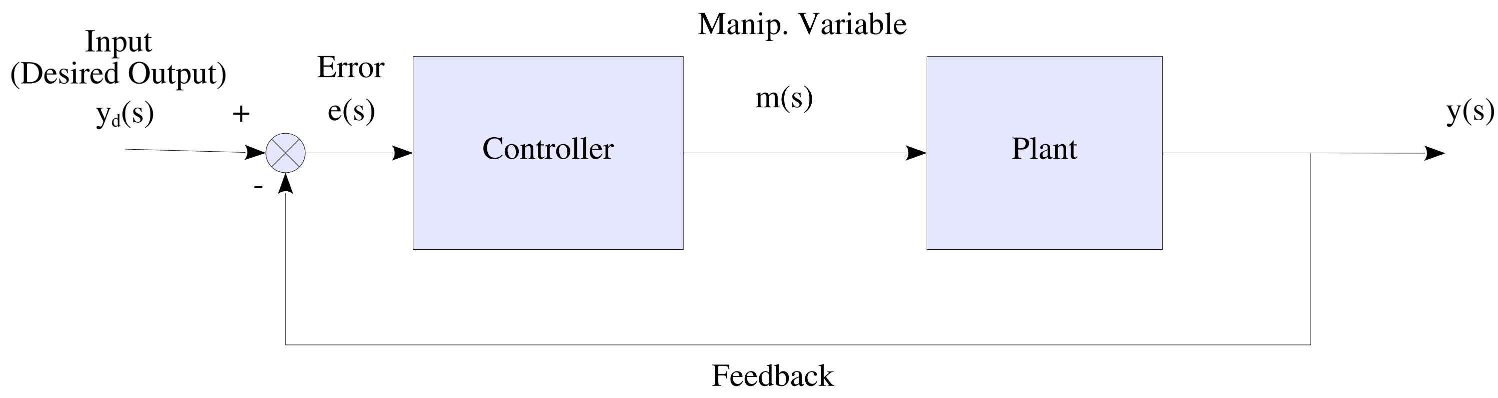
Controllable Canonical Form

Show for Homework

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & \dots & . \\ -p_0 & -p_1 & \dots & -p_{n-2} & -p_{n-1} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ . \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} q_0 & q_1 & \dots & q_{n-2} & q_{n-1} \end{bmatrix} \mathbf{x}(t)$$

Feedback Control



Output: Variable to be controlled

Input (Command): What we want the output to be

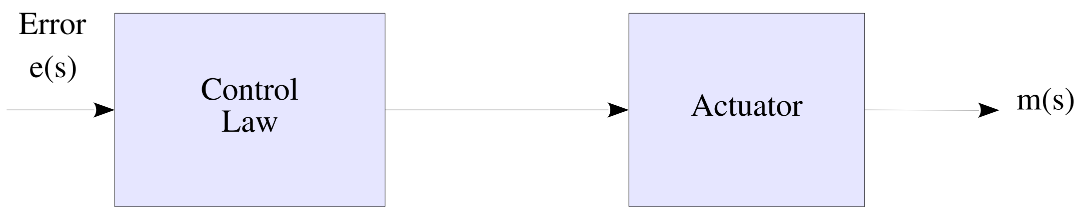
Plant: System that needs to be controlled

- Measure the output
- Compare it to the desired output
- Manipulated Variable: The physical variable we adjust to make the plant produce the desired output

The solution to the differential equation
Representing the plant

Controller

- Looks at the error & decides how to change the manipulated variable



Compensators & Filters

- Frequency
  - Bandpass Filter
  - Notch Filter
  - Lowpass Filter
  - Highpass Filter
- Phase
  - Phase-Lead Compensator
  - Phase-Lag Compensator
  - Phase-Lead-Lag Compensator