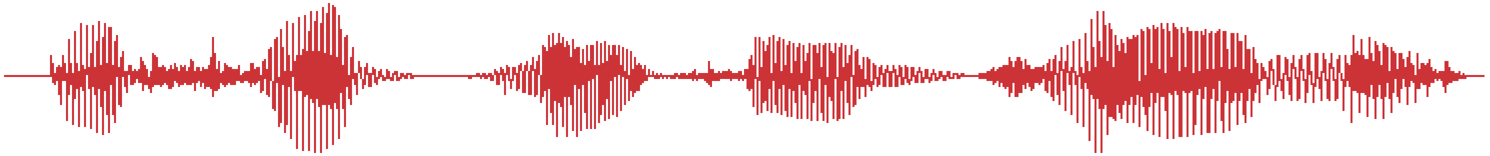
Introduction to Continuous Control Systems

EEME E3601



Week 5

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Zeros and Poles of a Function

Definition 24.44 (Zeros of a Function). *A point s_0 is called a zero of order r of $H(s)$ if*

$$\lim_{s \to s_0} \left[(s - s_0)^{-r} H(s) \right] = M \quad \textit{where } M \neq 0 \wedge M < \infty$$

Definition 24.45 (Isolated Singularities and Poles of a Function). *A point s_0 is called an isolated singularity or an isolated singular point of $H(s)$ if $H(s)$ is not analytic at s_0, but it is analytic in a deleted neighborhood of s_0. s_0 is also called a pole of order r of function $H(s)$ if,*

$$\lim_{s \to s_0} [(s - s_0)^{r} H(s)] = M \quad \textit{where } M \neq 0 \wedge M < \infty$$

Isolated Singularities
(Example)

$$H(s) = \frac{s^2+1}{(s^2+4)(s^2-5s)}$$

is analytic over the entire $\mathbb{C}$ *except at the points where it blows up.* $s = \pm 2i, 0, 5$ *are isolated singularities of* $H(s)$.

Residues

Definition 24.46 (Meromorphic Functions). *A function $H(s)$ which is analytic in a Domain $\mathscr{D} \subset \mathbb{C}$ except at some point of $\mathscr{D}$ where it has poles is said to be meromorphic in $\mathscr{D}$.*

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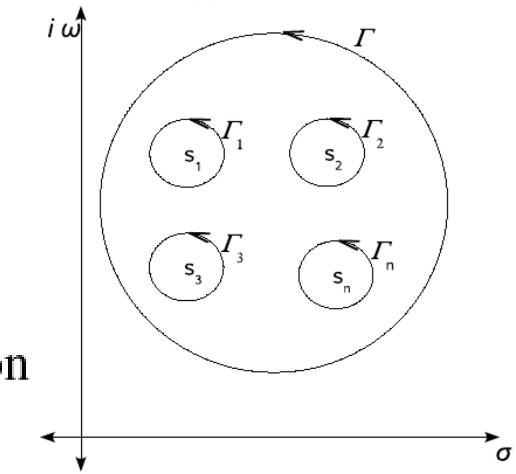
The Cauchy Residue Theorem

Theorem 24.15 (The Cauchy Residue Theorem). *Let Γ be a simple closed contour and $H(s)$ be analytic on Γ and interior to Γ except at a finite number of isolated singular points,* $(s_1, s_2, \cdots, s_n)$

$$\oint_{\Gamma} H(s)ds = 2\pi i \sum_{k=1}^{n} Residue\left[H(s), s_k\right]$$



Proof:

By the Cauchy Integral Theorem for a Multiply Connected Region

$$\oint_{\Gamma} H(s)ds = \sum_{k=1}^{n} \oint_{\Gamma_k} H(s)ds$$

Recall the b_1 coefficient of the Laurent series expansion of $H(s)$ about s_k

$$\oint_{\Gamma_k} H(s)ds = 2\pi i b_1$$

$$= 2\pi i \ \text{Residue}\left[H(s), s_k\right]$$

$$\oint_{\Gamma} H(s)ds = 2\pi i \sum_{k=1}^{n} \text{Residue}\left[H(s), s_k\right]$$

The Residue Evaluation Theorem

Theorem 24.16 (The Residue Evaluation Theorem). *Suppose $H(s)$ is analytic in a neighborhood of $s = s_0$ except at s_0 where it has a pole of order m. Then,*

$$Residue\left[H(s), s_0\right] = \frac{1}{(m-1)!} \lim_{s \to s_0} \frac{d^{(m-1)}}{ds^{(m-1)}} \left[(s-s_0)^m H(s)\right]$$

Proof:

$$l(s) \stackrel{\Delta}{=} (s-s_0)^m H(s)$$

In essence $l(s)$ is analytic everywhere including at $s = s_0$. Now let us write the power series for $l(s)$ about s_0 *(Definition of the Laurent Series, considering the Taylor Series expansion of l(s))*

$$l(s) = \sum_{k=0}^{m-1} b_{m-k}(s-s_0)^k + \sum_{n=0}^{\infty} a_n (s-s_0)^{n+m}$$

Differentiate $l(s)$, $(m-1)$ times and then take its limits as $s \to s_0$,

$$\lim_{s \to s_0} \frac{d^{(m-1)} l(s)}{ds^{(m-1)}} = (m-1)! b_1$$

$$\text{Residue}\left[H(s), s_0\right] = b_1 = \frac{1}{(m-1)!} \lim_{s \to s_0} \frac{d^{(m-1)}}{ds^{(m-1)}} \left[(s-s_0)^m H(s)\right]$$

N.B. For a simple pole, i.e. $m = 1$, $Residue\left[H(s), s_0\right] = \lim_{s \to s_0} \left[(s-s_0) H(s)\right]$

Relation between Functions

Convolution:

The convolution function $Conv(g,h)(t)$ is an indication of the amount of overlap between two functions $g(t)$ and $h(t)$ at t where one function is slid an amount t with respect to the other function. It may be defined in two different ways for Riemann integrable continuous functions

Definition 24.47 (Finite-Domain Convolution)

$$\begin{aligned} Conv(g,h)(t) &= (g * h)(t) \\ &\stackrel{\Delta}{=} \int_0^t g(\tau)h(t-\tau)d\tau \end{aligned}$$

The finite domain definition is popular in control and signal processing where t stands for time. *Often used with Laplace Transforms.*

Relation between Functions

Convolution (Continued):

Definition 24.48 (Infinite-Domain Convolution (Convolution)). *Infinite-Domain Convolution is the more popular definition of Convolution and may be expressed as follows:*

$$\begin{aligned} Conv(g,h)(t) &= (g * h)(t) \\ &\triangleq \int_{-\infty}^{\infty} g(\tau)h(t-\tau)d\tau \end{aligned}$$

Relation between Functions

Convolution (Properties):

Assume that $f(t)$, $g(t)$, and $h(t)$ are general complex-valued functions and that $\gamma = \alpha + i\beta$ is a constant in the complex domain, $\mathbb{C}$.

Property 24.13 (Commutativity of Convolution). *Convolution of functions $g(t)$ and $h(t)$ is commutative, namely,*

$$g * h = h * g$$

Property 24.14 (Associativity of Convolution). *Convolution among functions $f(t)$, $g(t)$ and $h(t)$ is associative, namely,*

$$f * (g * h) = (f * g) * h$$

Property 24.15 (Distributivity of Convolution). *Convolution is distributive, namely,*

$$(f + g) * h = (f * h) + (g * h)$$

Property 24.16 (Scaling Associativity of Convolution). *Convolution is associative with respect to scaling, namely,*

$$\beta\gamma(g * h) = (\beta g) * (\gamma h)$$

Relation between Functions

Correlation:

Another relationship between functions, which is defined in the form of correlation, signifies the location at which the second function has similar features to the first function. It is usually used in cases where a long-term signal is searched for specific features. These features may be represented in a shorter signal which then is slid along the longer signal looking for locations where the similarity peaks. The correlation is defined in mathematical terms as follows,

$$\begin{aligned} Corr(g,h)(t) &= (g \circ h)(t) \\ &\triangleq \int_{-\infty}^{\infty} \overline{g(\tau)} h(t+\tau) d\tau \end{aligned}$$

the independent variable t is called *lag*.

For real-valued functions,

$$\begin{aligned} Corr(g,h)(t) &= (g \circ h)(t) \\ &\triangleq \int_{-\infty}^{\infty} g(\tau) h(t+\tau) d\tau \end{aligned}$$

Relation between Functions

Correlation (Continued):

The correlation of a function with itself is called *autocorrelation* and it peaks at the lag $t = 0$, since at lag of zero, the two functions are identical. Correlation is also known as the *sliding inner product* (see Definition 24.49), cross-correlation and the *sliding dot product* of two functions. Like convolution it has interesting connections with the Fourier Transform of functions which will be discussed later in this chapter.

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Laplace Transform

Definition 24.70 (Laplace Transform). *The general Laplace Transform is defined by the following,*

$$H(s) = \mathscr{L}\{h\} \stackrel{\Delta}{=} \int_{-\infty}^{\infty} h(t)e^{-st}dt \quad s \in \mathbb{C} \tag{24.358}$$

Although it is defined for $-\infty < t < \infty$; for speaker recognition, or signal processing in general, we are only interested in functions of time where we may ignore the history of the signal. Assume that a signal started τ seconds prior to the point where our origin of time is defined, then without any loss of generality, we may shift the origin with which we define t to the past by τ seconds so that the function of interest $h(t) = 0 \; \forall \; t < 0$. With this new definition of t, the integral limit t_1 of Equation 24.234 may be set to 0. In fact, to be able to allow for the definition of the *Laplace Transform* of an Impulse function, we will set $t_1 = 0^-$. This issue will be visited later, when we explore the Laplace Transform of the more important functions. Based on this new definition of the origin, we define the, so called, *Unilateral (one-sided) Laplace Transform*.

Laplace Transform

Definition 24.71 (Unilateral (One-Sided) Laplace Transform). *The Unilateral or One-Sided Laplace Transform is defined by the following,*

$$H(s) = \mathscr{L}(h) \triangleq \int_{0^-}^{\infty} h(t)e^{-st}dt \quad s \in \mathbb{C} \tag{24.359}$$

In general, the Laplace Transform of Equation 24.359 does not exist for the whole Complex plane. However, it does exist for functions which grow at most as fast as exponential functions.

Laplace Transform

Theorem 24.25 (Existence and Boundedness of the Unilateral Laplace Transform). *For the Unilateral Laplace Transform to exist, the integral of Equation 24.359 should be bounded, or,*

$$\left|\int_0^\infty h(t)e^{-st}dt\right| < M \quad \text{where } M : M < \infty \tag{24.360}$$

A sufficient condition for this to happen is that the function $h(t)$ would be bounded by an exponential family of functions, namely, that $\exists\ \{k, s_c\}$ such that,

$$|h(t)| \begin{cases} \leq ke^{s_c t} & \forall \quad t \geq 0 \\ = 0 & \forall \quad t < 0 \end{cases} \tag{24.361}$$

Laplace Transform (Existence and Boundedness Proof)

Proof.
For the Laplace transform to exist, we should show that,

$$\left|\int_0^\infty h(t)e^{-st}dt\right| < M \tag{24.362}$$

We know from Theorem 24.12 that,

$$\begin{aligned}
\left|\int_0^\infty h(t)e^{-st}dt\right| &\le \int_0^\infty \left|h(t)e^{-st}\right| dt && (24.363)\\
&= \int_0^\infty |h(t)|\left|e^{-st}\right| dt \quad \text{(From Theorem 24.1)} && (24.364)\\
&\le \int_0^\infty ke^{s_c t}\left|e^{-st}\right| dt \quad \text{(From Equation 24.361)} && (24.365)\\
&= \int_0^\infty ke^{\sigma_c t}\left|e^{-\sigma t}\right|\left|e^{i\omega_c t}\right|\left|e^{-i\omega t}\right| dt && (24.366)\\
&= \int_0^\infty ke^{\sigma_c t}e^{-\sigma t}dt \quad \left(\text{Since } \left|e^{i\omega_c t}\right| = \left|e^{-i\omega t}\right| = 1\right) && (24.367)\\
&= \int_0^\infty ke^{(\sigma_c-\sigma)t}dt && (24.368)
\end{aligned}$$

Laplace Transform (Existence and Boundedness Proof)

Therefore, if $\exists \{k, s_c = \sigma_c + i\omega_c\}$ such that

$$k \int_0^\infty e^{(\sigma_c - \sigma)t} dt < M \tag{24.369}$$

Consequently, $\exists \{s_c = \sigma_c + i\omega_c\}$ such that

$$\int_0^\infty e^{(\sigma_c - \sigma)t} dt < \tilde{M} \tag{24.370}$$

$$\tag{24.371}$$

where $\tilde{M}$ is another number such that $\tilde{M} < \infty$, or

$$\left| \int_0^\infty h(t) e^{-st} dt \right| < \tilde{M} \tag{24.372}$$

which means that the Laplace Transform exists.

$\square$

Laplace Transform

Unit Step Function,

$$h(t) = \begin{cases} 1 & \forall \ t \geq 0 \\ 0 & \forall \ t < 0 \end{cases}$$

$$H(s) = \int_0^\infty e^{-st} dt = \left[\frac{e^{-st}}{-s} \right]_0^\infty = \frac{1}{s}$$

hence a simple pole at the origin.

Laplace Transform

Unit Ramp Function,

$$h(t) = \begin{cases} t & \forall \ t \geq 0 \\ 0 & \forall \ t < 0 \end{cases}$$

$$H(s) = \int_0^\infty t e^{-st} dt$$

Using Integration by Parts,

$$H(s) = \left[\frac{te^{-st}}{-s} \right]_0^\infty + \frac{1}{s} \int_0^\infty e^{-st} dt = \frac{1}{s^2}$$

Laplace Transform

$$f(t) = e^{-at} \quad t \geq 0$$

$$\begin{aligned} F(s) &= \int_0^\infty e^{-at} e^{-st} dt \\ &= \left[-\frac{e^{-(s+a)t}}{s+a} \right]_0^\infty \\ &= \frac{1}{s+a} \end{aligned}$$

Laplace Transform

Multiplication by a constant

$$\mathscr{L}\{kf(t)\} = kF(s)$$

Sum and Difference

$$\mathscr{L}\{f_1(t) \pm f_2(t)\} = F_1(s) \pm F_2(s)$$

Laplace Transform
Differentiation

$$\mathscr{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - \lim_{t\to 0} f(t)$$
$$= sF(s) - f(0)$$

Show for Homework

$$\mathscr{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s)$$
$$- \lim_{t\to 0}\left[s^{n-1} f(t) + s^{n-2}\frac{df(t)}{dt} + \cdots + \frac{d^{n-1}f(t)}{dt^{n-1}}\right]$$
$$= s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - f^{(n-1)}(0)$$

Laplace Transform
Integration

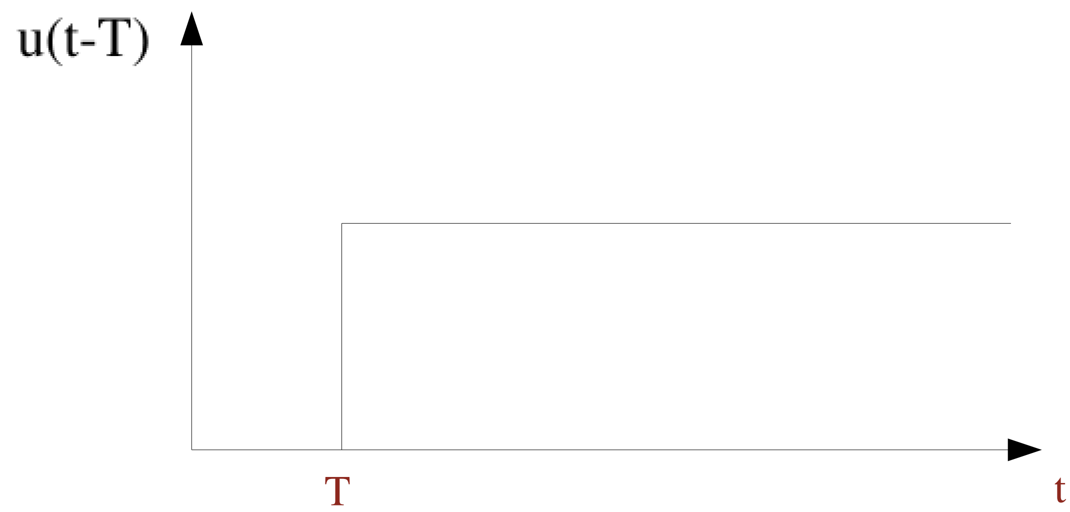
$$\mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

$$\mathscr{L}\left\{\int_0^{t_n}\int_0^{t_{n-1}}\cdots\int_0^{t_1} f(\tau)\, d\tau\, dt_1\, dt_2\cdots\, dt_{n-1}\right\} = \frac{F(s)}{s^n}$$

Laplace Transform
Shift in Time

$$\mathscr{L}\{f(t-T)u(t-T)\} = e^{-Ts}F(s)$$

where u(t-T) denotes the step function which is shifted in time to the right by T seconds.



Laplace Transform
Final Value Theorem

Important

Final Value Theorem for Laplace Transform

If the Laplace transform of $f(t)$ is denoted by $F(s)$ and if $sF(s)$ is analytic on the imaginary axis and in the right half of the s plane, then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Helps Determine the Steady-State Response

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Laplace Transform
Final Value Theorem – Examples

Example 1 (Final Value Theorem).

$$F(s) = \frac{5}{s(s^2+s+2)}$$

$$\begin{aligned}
\lim_{t\to\infty} f(t) &= \lim_{s\to 0} sF(s) \\
&= \lim_{s\to 0} \frac{5}{s^2+s+2} \\
&= \frac{5}{2}
\end{aligned}$$

Laplace Transform
Final Value Theorem – Examples

Example 2 (Final Value Theorem).

$$\begin{aligned} f(t) &= sin(\omega t) \\ F(s) &= \frac{\omega}{s^2+\omega^2} \end{aligned}$$

F(s) has 2 poles on the imaginary axis. Therefore, it is not analytic on the imaginary axis. So we cannot apply the final value theorem, otherwise, $sin(\infty) = 0$ *which is not correct.*

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Laplace Transform
Initial Value Theorem

Important

Initial Value Theorem for Laplace Transform

If the Laplace transform of $f(t)$ is denoted by $F(s)$ and if $sF(s)$ is analytic on the imaginary axis and in the right half of the s plane, then

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$$

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Laplace Transform
Complex Shifting

$$\mathscr{L}\{e^{\mp at}f(t)\} = F(s \pm a)$$

Laplace Transform
Real Convolution – Complex Shifting

If $f_1(t) = 0, f_2(t) = 0 \quad \forall \; t < 0$

$$
\begin{aligned}
F_1(s)F_2(s) &= \mathscr{L}\left\{\int_0^t f_1(\tau)f_2(t-\tau)d\tau\right\} \\
&= \mathscr{L}\left\{\int_0^t f_2(\tau)f_1(t-\tau)d\tau\right\} \\
&= \mathscr{L}\{f_1(t) * f_2(t)\}
\end{aligned}
$$

$$\therefore \mathscr{L}^{-1}\{F_1(s)F_2(s)\} \neq f_1(t)f_2(t)$$

Laplace Transform (Inversion)

$$h(t) = \frac{1}{2\pi i}\left[\int_{\sigma_c - i\infty}^{\sigma_c + i\infty} H(s)e^{st}ds\right] \tag{24.373}$$

σ_c is chosen so that all the singularities are to the left of the line going parallel to the Imaginary access. Based on the Residue Theorem (Theorem 24.15) , integrating over a semicircle in the left of the s-plane,

$$h(t) = \frac{1}{2\pi i}2\pi i \sum (\text{Residue inside Contour}) \tag{24.374}$$

Laplace Transform (Inversion – Example)

$$H(s) = \frac{1}{(s+a)(s+b)}$$

The Residue at $s = -a$ is,

$$\left[\frac{(s+a)e^{st}}{(s+a)(s+b)}\right]_{s\to -a}$$

and the Residue at $s = -b$ is,

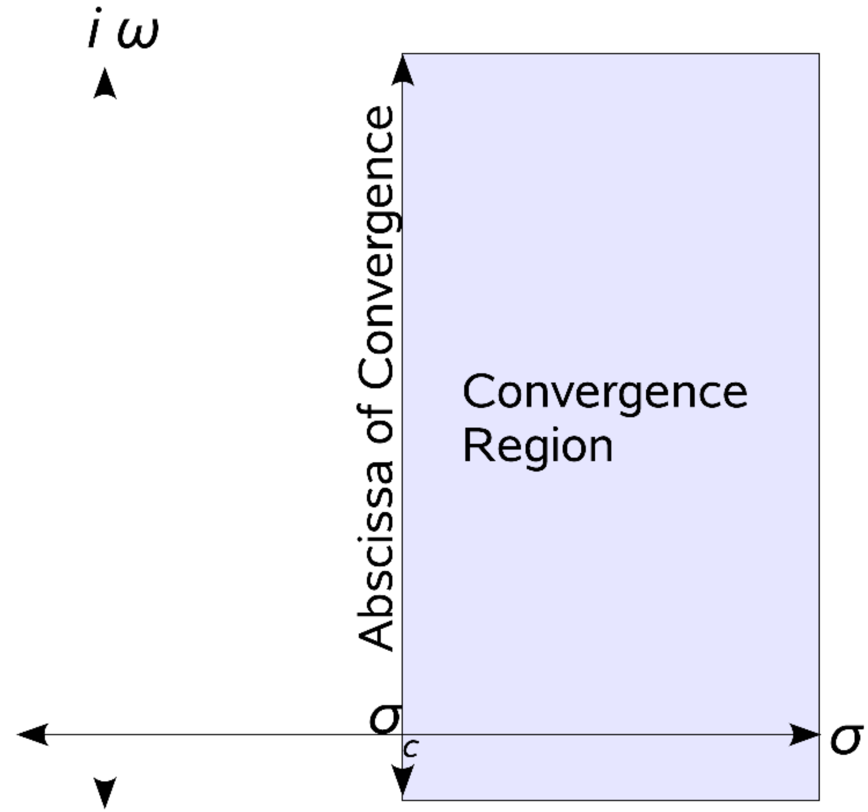
$$\left[\frac{(s+b)e^{st}}{(s+a)(s+b)}\right]_{s\to -b}$$

Therefore,

$$h(t) = \frac{1}{b-a}\left[e^{-at} - e^{-bt}\right]$$

Linear Differential Equations
Laplace Transform Inversion

$$h(t) = \frac{1}{2\pi i}\left[\int_{\sigma_c - i\infty}^{\sigma_c + i\infty} H(s)e^{st}\,ds\right]$$



$$h(t) = \frac{1}{2\pi i} 2\pi i \sum (\text{Residue inside Contour})$$

σ_c is chosen so that all the singularities are to the left of the line going parallel to the Imaginary access. Based on the Residue Theorem (Theorem 24.15) , integrating over a semicircle in the left of the s-plane,

Linear Differential Equations
Laplace Transform Inversion – Example

$$H(s) = \frac{1}{(s+a)(s+b)}$$

The Residue at $s = -a$ is, $\left[\frac{(s+a)e^{st}}{(s+a)(s+b)}\right]_{s \to -a}$

and the Residue at $s = -b$ is, $\left[\frac{(s+b)e^{st}}{(s+a)(s+b)}\right]_{s \to -b}$

$$h(t) = \frac{1}{b-a}\left[e^{-at} - e^{-bt}\right]$$

Linear Differential Equations
Partial Fractions

Theorem 3. *If $H(s)$ is a strictly proper ($M \leq N$), rational (ratio of polynomials) function with real coefficients,*

$$
\begin{aligned}
H(s) &= \frac{q_{(M-1)}s^{(M-1)} + q_{(M-2)}s^{(M-2)} + \cdots + q_0}{s^N + p_{(N-1)}s^{(N-1)} + p_{(N-2)}s^{(N-2)} + \cdots + p_0} \\
&= \frac{Q(s)}{(s+s_1)(s+s_2)\cdots(s+s_N)}
\end{aligned}
$$

where $\{s_1, s_2, \cdots, s_N\}$ are distinct roots of $P(s)$, then,

$$H(s) = \frac{R_1}{(s+s_1)} + \frac{R_2}{(s+s_2)} + \cdots + \frac{R_N}{(s+s_N)}$$

where the residues, R_n, are complex numbers given by the following,

$$R_n = [(s+s_n)H(s)]_{s=-s_n}$$

where $n = \{1, 2, \cdots, N\}$
Then, the inverse Laplace Transfor of $H(s)$ is given as,

$$h(t) = \sum_{n=1}^{N} R_n e^{-s_n t}$$

Linear Differential Equations
Partial Fractions Repeated Poles

If $H(s)$ is a strictly proper ($M \leq N$), rational (ratio of polynomials) function with real coefficients,

$$\begin{aligned} H(s) &= \frac{q_{(M-1)}s^{(M-1)} + q_{(M-2)}s^{(M-2)} + \cdots + q_0}{s^N + p_{(N-1)}s^{(N-1)} + p_{(N-2)}s^{(N-2)} + \cdots + p_0} \\ &= \frac{Q(s)}{(s+s_1)(s+s_2)\cdots(s+s_i)^r\cdots(s+s_N)} \end{aligned}$$

where the pole s_i is repeated r times, then $H(s)$ may be expanded as follows,

$$H(S) = \frac{R_1}{(s+s_1)} + \frac{R_2}{(s+s_2)} + \cdots + \frac{R_i^{(1)}}{(s+s_i)} + \cdots + \frac{R_i^{(r)}}{(s+s_i)^r} + \cdots + \frac{R_N}{(s+s_N)}$$

where the $R_i^{(1)}$ through $R_i^{(r)}$ associated with the repeated pole, s_i may be computed as follows,

$$R_i^{(r)} = \left[(s+s_i)^r H(s)\right]_{s=-s_i}$$

$$R_i^{(r-1)} = \left[\frac{d}{ds}\{(s+s_i)^r H(s)\}\right]_{s=-s_i}$$

$$R_i^{(r-2)} = \left[\frac{1}{2!}\frac{d^2}{ds^2}\{(s+s_i)^r H(s)\}\right]_{s=-s_i}$$

$$R_i^{(1)} = \left[\frac{1}{(r-1)!}\frac{d^{(r-1)}}{ds^{(r-1)}}\{(s+s_i)^r H(s)\}\right]_{s=-s_i}$$