Introduction to Continuous Control Systems

EEME E3601

Week 4

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Continuity

Definition 24.12 (Continuity of Functions). *If* $\forall\ \varepsilon > 0\ \exists\delta(\varepsilon) > 0$ *such that* $0 < |t - t_0| < \delta \implies |h(t) - h(t_0)| < \varepsilon$, *then, h(t) is continuous at* $t = t_0$, *or,*

$$\lim_{t \to t_0} h(t) = h(t_0)$$

The function $h(t)$ *is a continuous function if it is continuous for all* t_0.

Used later for defining Analytic Functions

Definition 24.13 (Discontinuous Functions). *A function* $h(t)$ *is discontinuous if for some* t_0,

$$\lim_{t \to t_0} h(t) \neq h(t_0)$$

A function may be discontinuous at a point t_0 *for two main reasons:*

1. $h(t)$ *may not approach any limit as* $t \to t_0$.
2. $h(t)$ *may approach a limit different from* $h(t_0)$.

Let us classify the different points of discontinuity.

Removable Discontinuity

Definition 24.14 (Removable Discontinuity). *When* $\lim_{t \to t_0} h(t)$ *exists, i.e.,*

$$\lim_{t \to t_0} h(t) = \lim_{t \to t_{0+}} h(t) = \lim_{t \to t_{0-}} h(t) \tag{24.35}$$

but is not equal to $h(t_0)$ *or* $h(t_0)$ *is not defined, then* t_0 *is a point of removable discontinuity. ie, It may be deifned such that the discontinuity is removed.*

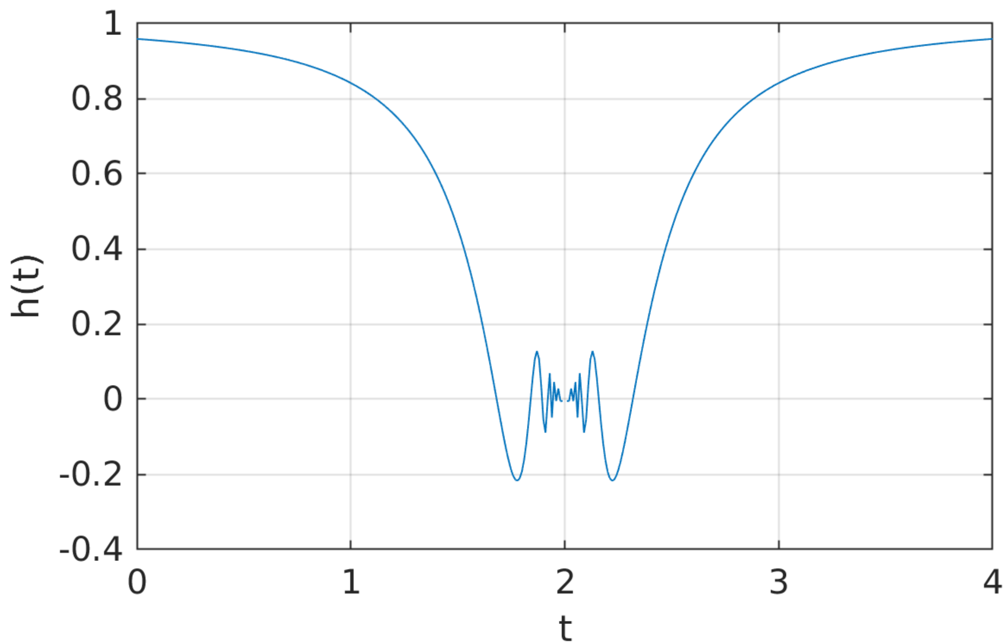
Sample Application: Sampling Theorem allows for a finite number of Removable Discontinuities

Removable Discontinuity (Example)

$$h(t) = (t - t_0)\sin\left(\frac{1}{t - t_0}\right)$$

$h(t_{0^+}) = h(t_{0^-})$, but $h(t_0)$ is not defined.

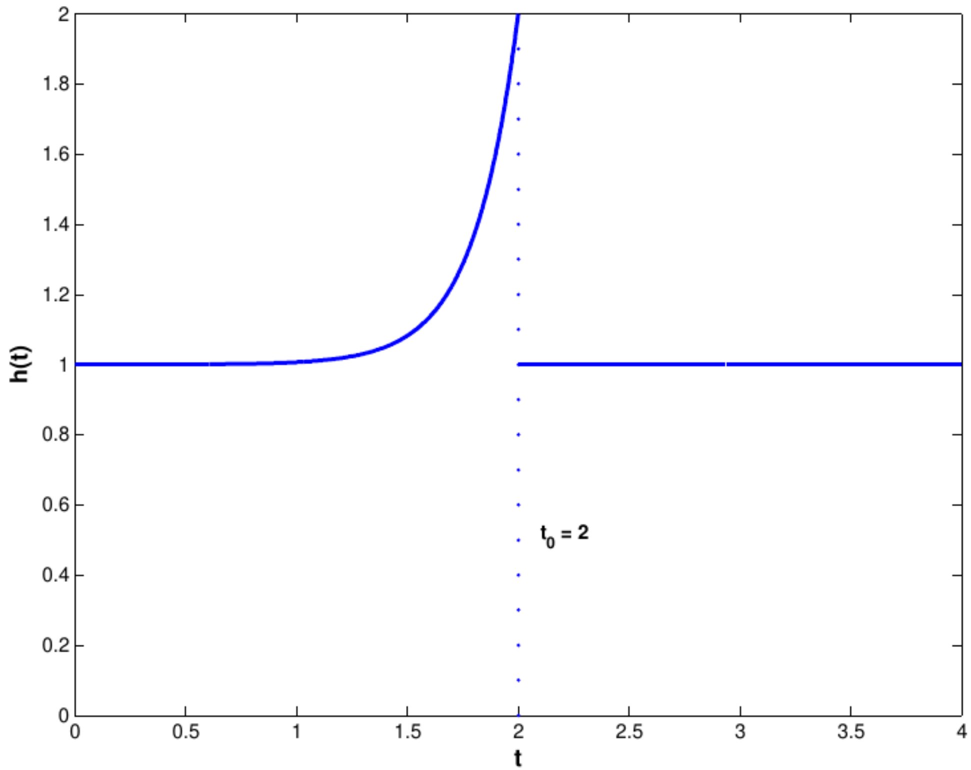
Here, we may postulate that $h(t_0) = 0$ and then $h(t)$ becomes continuous.



Point of ordinary discontinuity at $t = t_0 = 2$

Ordinary Discontinuity (Example)

Definition 24.15 (Ordinary Discontinuity). *In cases where $h(t_{0^+}) \neq h(t_{0^-})$, regardless of the definition of $h(t_0)$, t_0 becomes a point of ordinary discontinuity.*



Point of ordinary discontinuity at $t = t_0 = 2$

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Ordinary Discontinuity (Example)

The following function has an ordinary discontinuity at $t = 0$, $\quad h(t) = \frac{\sin(t)}{|t|}$

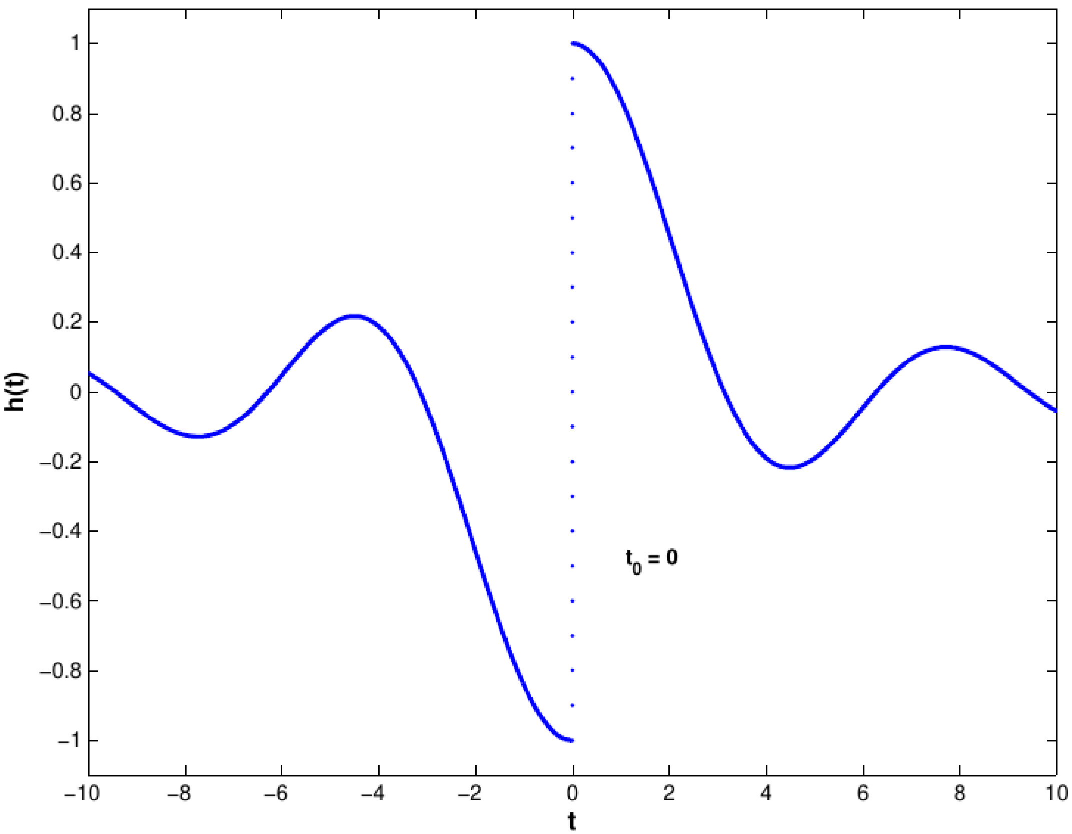
Using l'Hôpital's rule and replacing $|t|$ *with +t and -t respectively.*

$$\lim_{t\to 0^+} \frac{\sin(t)}{|t|} = \lim_{t\to 0^+} \frac{\sin(t)}{t} = 1$$

$$\lim_{t\to 0^-} \frac{\sin(t)}{|t|} = \lim_{t\to 0^-} \frac{\sin(t)}{-t} = -1$$

$$\therefore \quad \lim_{t\to 0^+} \frac{\sin(t)}{|t|} \neq \lim_{t\to 0^-} \frac{\sin(t)}{|t|}$$

Namely, the limit does not exist.



Point of ordinary discontinuity at $t = t_0 = 0$ $\left(h(t) = \frac{\sin(t)}{|t|}\right)$

Infinite Discontinuity

Definition 24.15 (A Point of Infinite Discontinuity). *The following are the different cases of infinite discontinuities,*

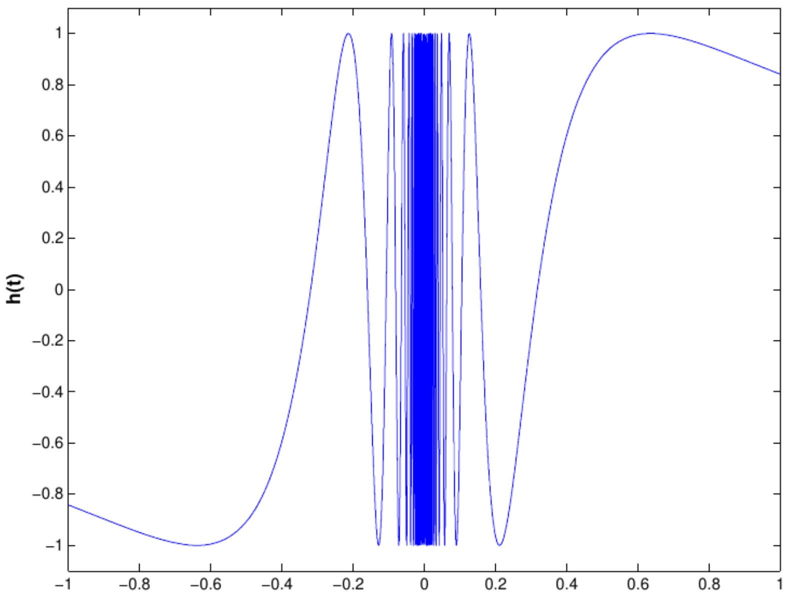
1. $h(t_{0^+}) = h(t_{0^-}) = \pm\infty$
2. $h(t_{0^+}) = +\infty$ *and* $h(t_{0^-}) = -\infty$
3. $h(t_{0^+}) = \pm\infty$ *and* $h(t_{0^-})$ *exists*
4. $h(t_{0^+})$ *exists and* $h(t_{0^-}) = \pm\infty$

Point of Oscillatory Discontinuity

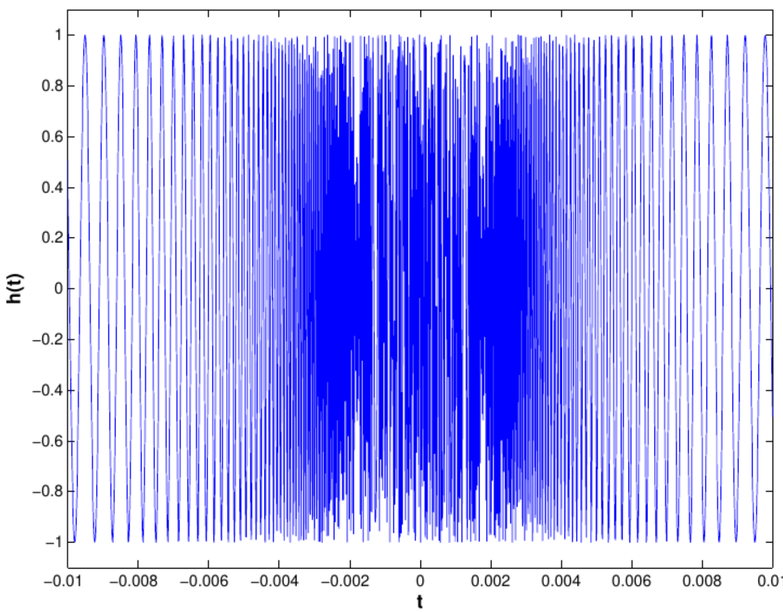
Definition 24.17 (A Point of Oscillatory Discontinuity). *An oscillatory discontinuity s_0 is one where no matter how small the ε neighborhood of s_0 $\{s : |s - s_0| < \varepsilon\}$ is made, the value of s oscillates to different values with function $H(s)$ not being defined at the exact value of s_0, but it may be defined in its neighborhood (it is defined for the Finite Amplitude version – see below).*

The following are the different kinds of oscillatory discontinuities,

1. *Finite Amplitude:* $h(t) = \sin\left(\dfrac{1}{t - t_0}\right)$



Point of Oscillatory Discontinuity at $t = t_0 = 0$ for $h(t) = \sin\left(\frac{1}{t-t_0}\right)$



More Detailed Viewpoint of the Oscillatory Discontinuity at $t = t_0 = 0$

Point of Oscillatory Discontinuity

2. *Infinite Amplitude:*
   In this case, in addition to the oscillatory nature of the discontinuity s_0, the value of the function will approach infinity in the neighborhood of the singularity. An example is,

$$h(t) = \frac{1}{(t - t_0)} \sin\left(\frac{1}{t - t_0}\right)$$

Continuity in an Interval

Definition 24.17 (Continuity of a Function in an Interval). *A function $h(t)$ is said to be continuous in an interval $[a,b]$ if*

[.] means closed interval
(.) means open interval

$$\lim_{t \to t_0} h(t) = h(t_0) \qquad a < t < b$$

$$\lim_{t \to a^+} h(t) = h(a)$$

$$\lim_{t \to b^-} h(t) = h(b)$$

Boundedness

Definition 24.18 (Boundedness). *A function $h(t)$ is bounded in an interval $[a,b]$, if* $\exists\, M \;:\; |h(t)| \;\leq\; M \,\forall\, t \in [a,b]$.

Property 24.6 (Boundedness of a Continuous Function). *A function $h(t)$ which is continuous in an interval $[a,b]$, is bounded.*

Proof.
If a function is continuous in an interval $[a,b]$, then by definition of continuity, a small change, δ, in t can only cause a small change, ε in $h(t)$, therefore, in the finite interval $[a,b]$ where

$$\max_{\substack{a\leq t\leq b\\ a\leq t_0\leq b}} |t-t_0| = b-a$$

$|t-t_0|$ is bounded, so $\exists M : M < \infty$ so that $\displaystyle\max_{\substack{a\leq t\leq b\\ a\leq t_0\leq b}} |h(t)-h(t_0)| < M$

Therefore, based on Definition, $h(t)$ is bounded in interval $[a,b]$. □

Degree of Continuity

Definition 24.19 (Continuity Class (Degree of Continuity)). *A function $h(t)$ is continuous with degree 1 if it is continuous and its first derivative is continuous. First degree continuity is denoted as $\mathfrak{C}^1$.*

If function $h(t)$ is continuous and all its derivatives up to the n^{th} derivative are continuous, then the function is a $\mathfrak{C}^n$ continuous function. N.B., If a function has up to n derivatives, then it is at least of continuity class $\mathfrak{C}^{n-1}$, namely all the derivatives up to and including degree $n-1$ are also continuous.

Sample Application: Fogel's Sampling Theorem

A class $\mathfrak{C}^0$ function is simply continuous.

Degree of Continuity

Definition 24.20 (Smoothness). *A function $h(t)$ is smooth if it is continuous and it has up to order ∞ continuous derivatives, namely it is of class $\mathfrak{C}^\infty$ continuous. N.B. All analytic functions are smooth, but since there is a requirement that analytic functions be determined completely by a power series, not all smooth functions are analytic.*

An Analytic function will be defined soon.

Definition 24.21 (Piecewise Continuity). *A function $h(t)$ is piecewise continuous if it is continuous at all points in an interval except a finite number of discontinuities in that interval.*

Sample Application: Proof of Reconstruction of Sampling Theorem

Definition 24.22 (Piecewise Smoothness). *A function $h(t)$ is piecewise smooth if it is piecewise continuous and its derivatives are piecewise continuous.*

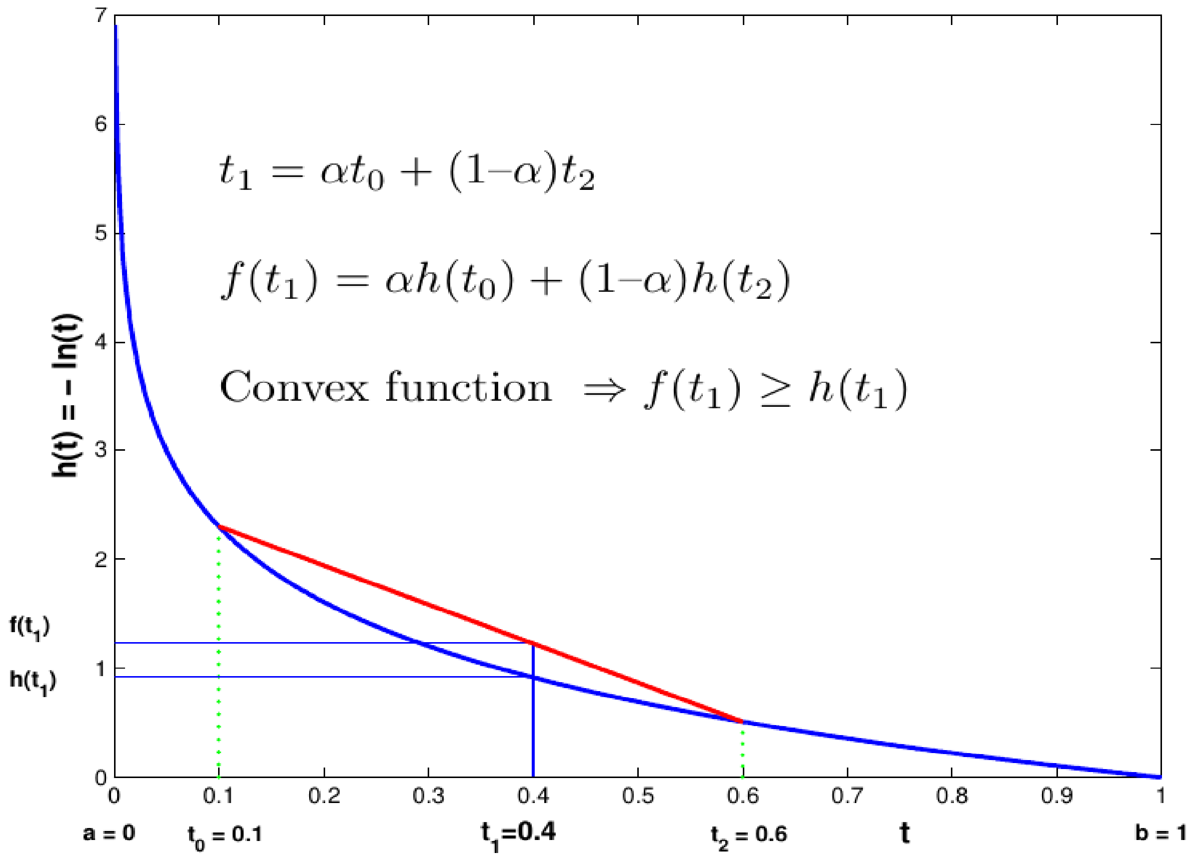
Convexity and Concavity of Functions

Definition 24.23 (Convex Function). *A real-valued function, $h(t)$, which is continuous in the closed interval $\{t : t \in [a,b]\}$, is said to be convex if*

$$h(\alpha t_0 + (1-\alpha)t_2) \le \alpha h(t_0) + (1-\alpha)h(t_2) \;\; \forall t_0, t_2 \in [a,b] \text{ and } \forall \alpha \in [0,1] \quad (24.46)$$

Note that a function which has a *non-negative second derivative* over the whole interval, $[a,b]$, is *convex* in that interval.

Applications:
Optimization Theory
Probability Theory



Convexity and Concavity of Functions

Repeated for convenience

Definition 24.23 (Convex Function). *A real-valued function, $h(t)$, which is continuous in the closed interval $\{t : t \in [a,b]\}$, is said to be convex if*

$$h(\alpha t_0 + (1-\alpha)t_2) \leq \alpha h(t_0) + (1-\alpha)h(t_2) \;\; \forall t_0, t_2 \in [a,b] \text{ and } \forall \alpha \in [0,1] \quad (24.46)$$

Definition 24.24 (Strictly Convex Function). *A strictly convex function is defined by Definition 24.23, such that the inequality in Equation 24.46 is changed to a strict inequality, not allowing equality, except when $\{\alpha = 0 \vee \alpha = 1\}$.*

Note that a function which has a *positive second derivative* over the whole interval, $[a,b]$, is *strictly convex* in that interval.

Theorem 24.3 (Convex Function). *A real-valued function, $h(t)$, which is $\mathfrak{C}^1$ continuous in the closed interval $\{t : t \in [a,b]\}$, is said to be convex if it has a non-negative second derivative,*

$$\frac{d^2 h(t)}{dt^2} \geq 0$$

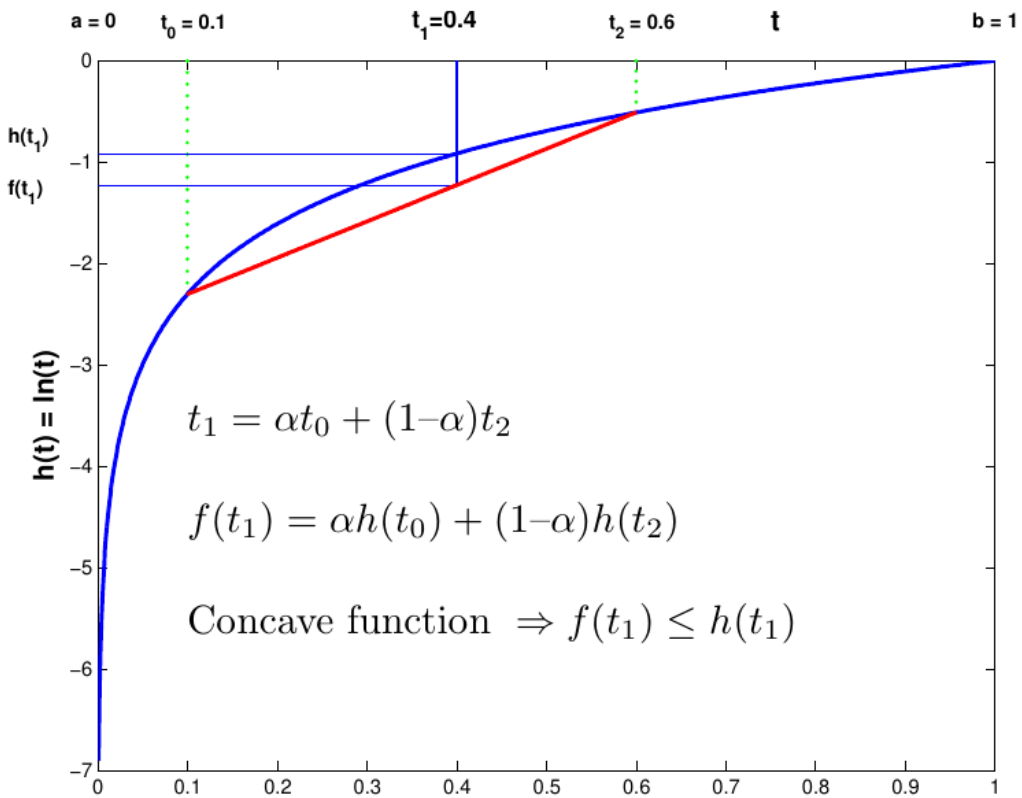
Proof: See the textbook

Convexity and Concavity of Functions

Definition 24.25 (Concave Function). *A real-valued function, $h(t)$, which is continuous in the closed interval $\{t : t \in [a,b]\}$, is said to be concave if*

$$h(\alpha t_0 + (1-\alpha)t_2) \geq \alpha h(t_0) + (1-\alpha)h(t_2) \;\; \forall t_0, t_2 \in [a,b] \text{ and } \forall \alpha \in [0,1] \quad (24.63)$$

Note that a function, which has a *non-positive second derivative* over the whole interval, $[a,b]$, is *concave* in that interval.



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Convexity and Concavity of Functions

Definition 24.25 (Concave Function). *A real-valued function, $h(t)$, which is continuous in the closed interval $\{t : t \in [a,b]\}$, is said to be concave if*

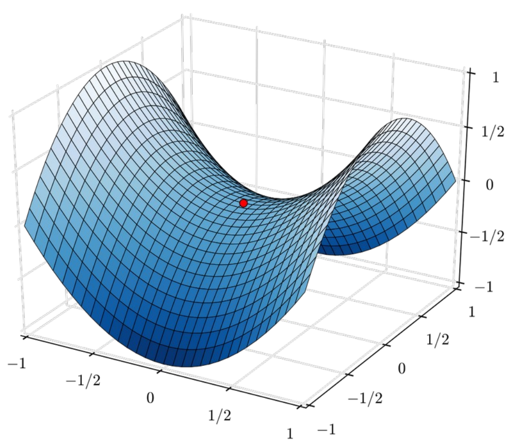
$$h(\alpha t_0 + (1-\alpha)t_2) \geq \alpha h(t_0) + (1-\alpha)h(t_2) \;\; \forall t_0, t_2 \in [a,b] \text{ and } \forall \alpha \in [0,1] \quad (24.63)$$

Definition 24.26 (Strictly Concave Function). *A strictly concave function is defined by Definition 24.25, such that the inequality in Equation 24.63 is changed to a strict inequality, not allowing equality, except when $\{\alpha = 0 \vee \alpha = 1\}$.*

Note that a function, which has a *negative second derivative* over the whole interval, $[a,b]$, is *strictly concave* in that interval.

These definition also work for gradients (vector) and
Hessians (second gradient matrix) in higher dimensions

Hessian (G) can be positive definite, negative definite, and positive/negative semi-definite



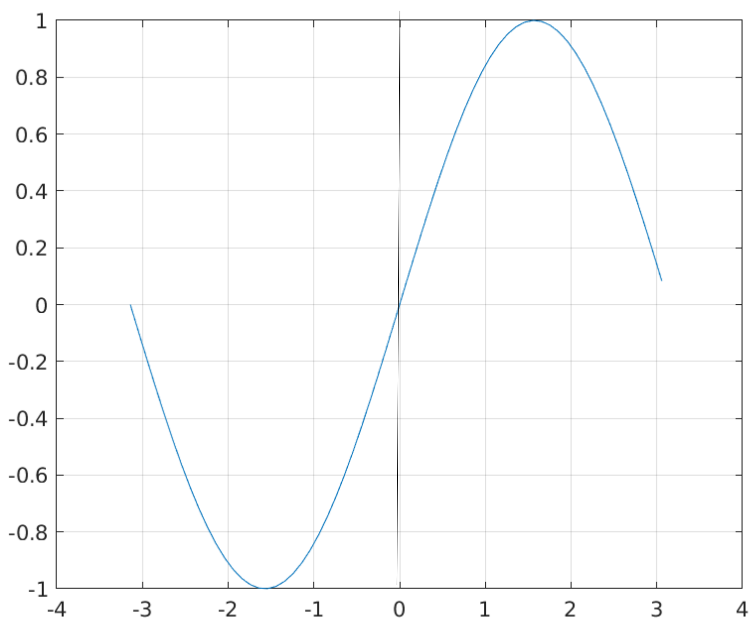
Odd, Even, and Periodic Functions

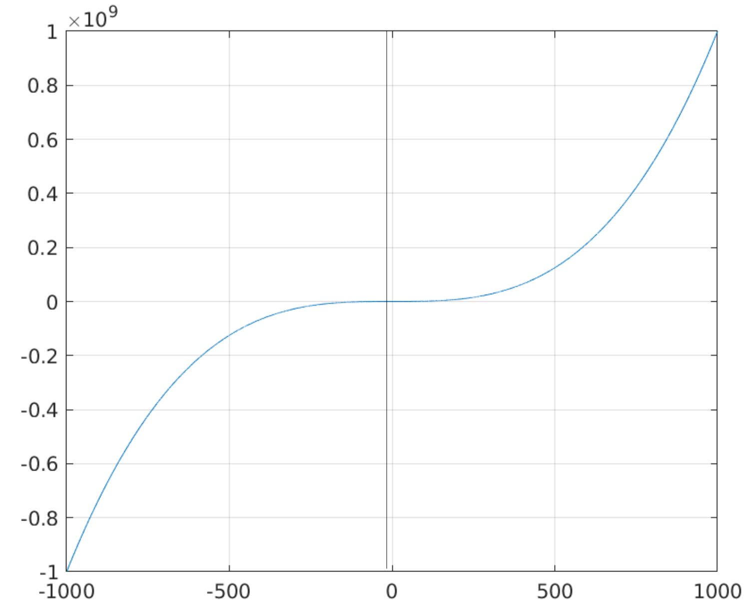
Definition 24.27 (Odd Functions). *A function $h(t)$ is odd if $h(-t) = -h(t) \;\; \forall \;\; t$.*

If $h(t)$ is periodic with period 2π, then oddness implies that,

$$\int_{-\pi}^{\pi} h(t)dt = 0$$

Some examples of odd functions are, $h(t) = \sin(t)$ and $h(t) = t^3$





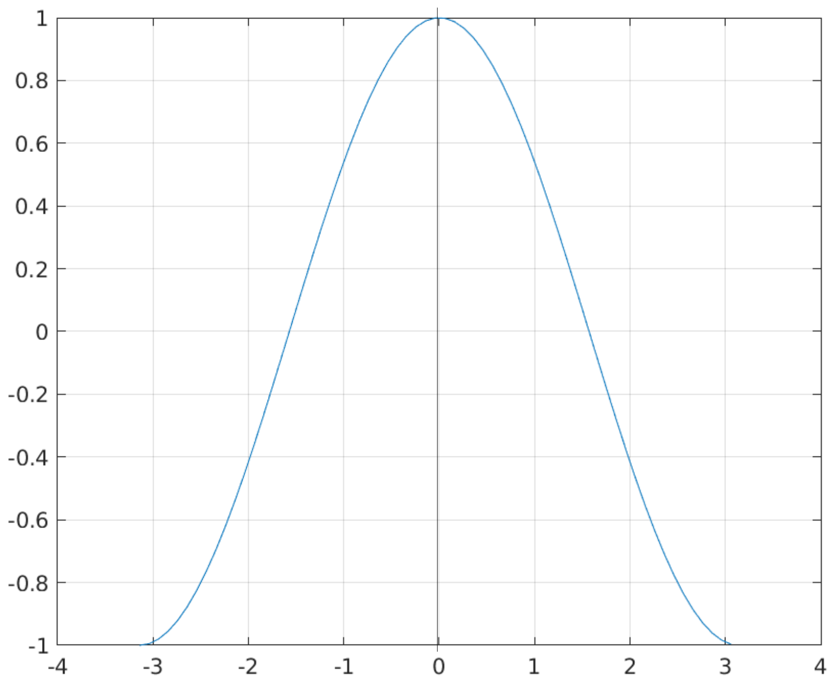
Odd, Even, and Periodic Functions

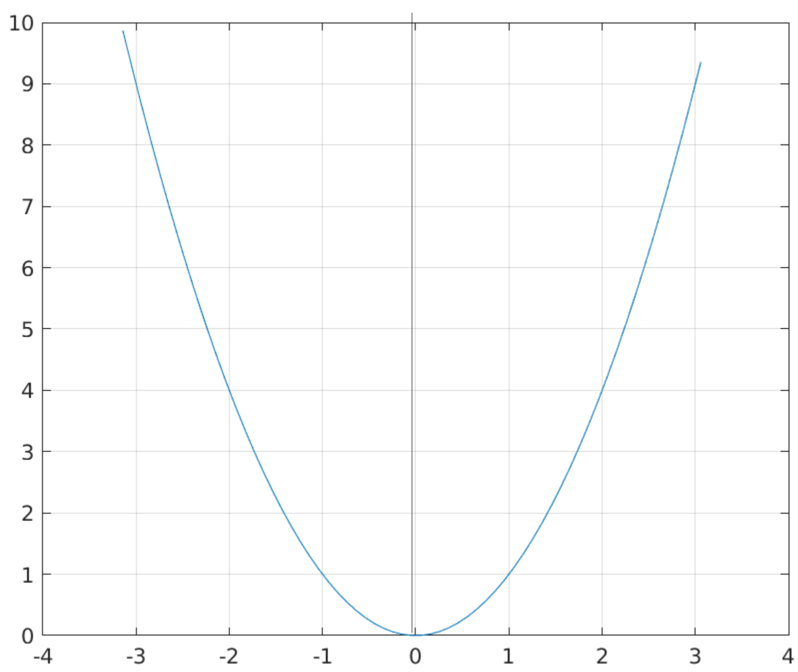
Definition 24.28 (Even Functions). *A function $h(t)$ is even if $h(-t) = h(t) \ \forall \ t$.*

If $h(t)$ is periodic with period 2π, then evenness implies that,

$$\int_{-\pi}^{\pi} h(t)dt = 2\int_{0}^{\pi} h(t)dt$$

Some examples of even functions are, $h(t) = \cos(t)$ and $h(t) = t^2$.

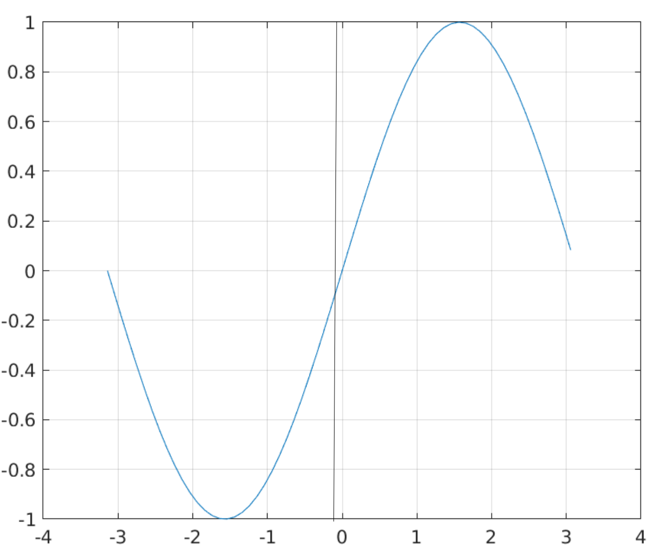




Odd, Even, and Periodic Functions

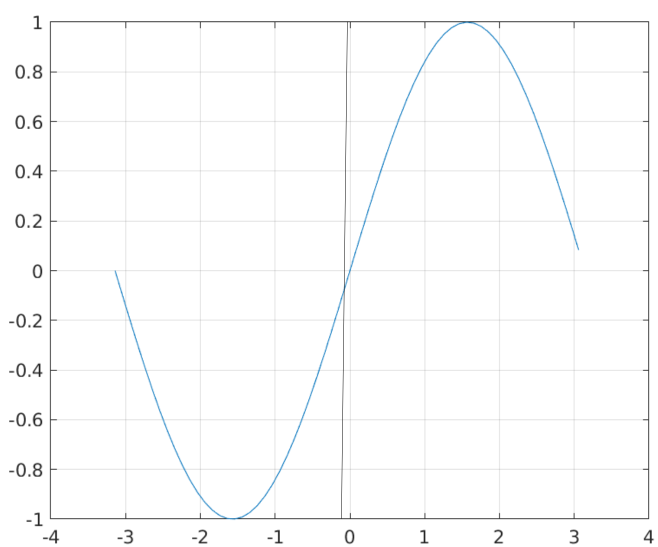
Property 24.7 (Odd and Even Functions). *Here are some properties related to odd and even functions,*

- *odd function* $\times$ *odd function* $=$ *even function*
- *odd function* $\times$ *even function* $=$ *odd function*
- *even function* $\times$ *even function* $=$ *even function*



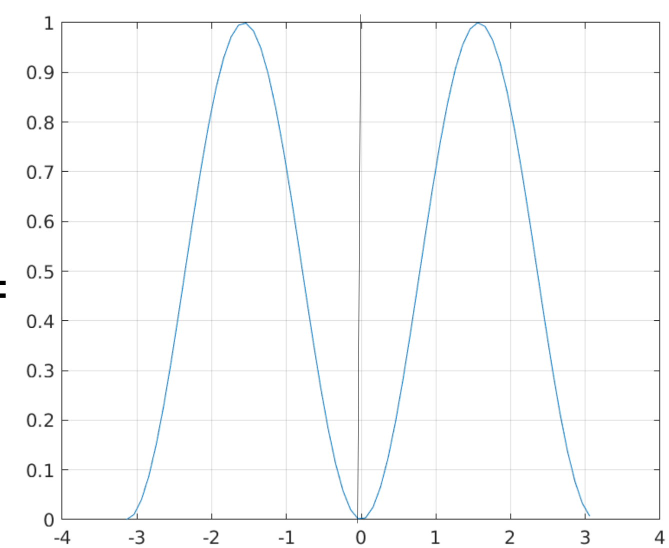
odd

x



odd

=



even

Odd, Even, and Periodic Functions

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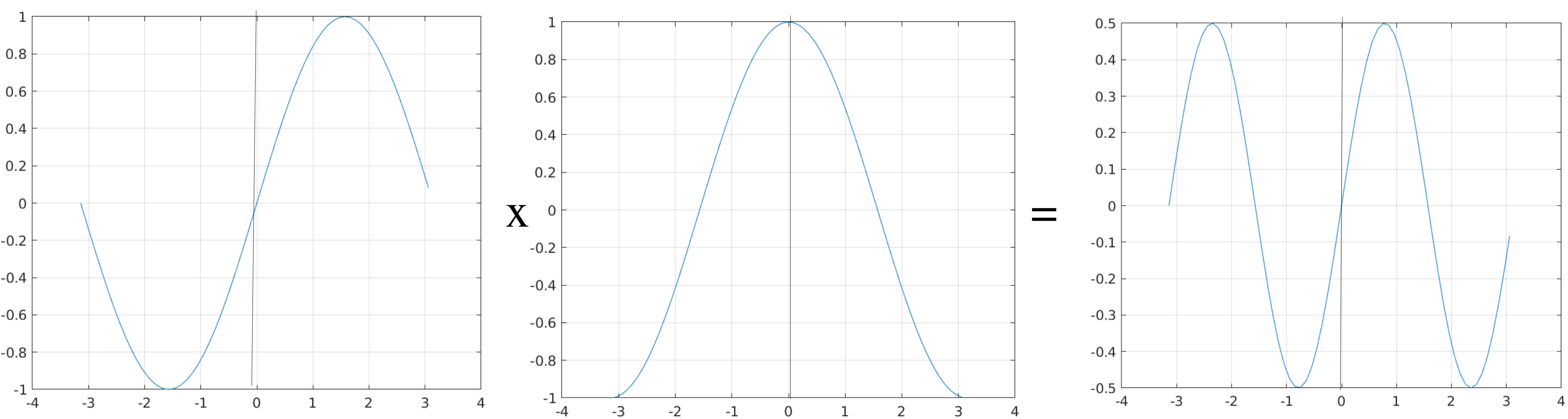

odd　　　even　　　odd

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Odd, Even, and Periodic Functions

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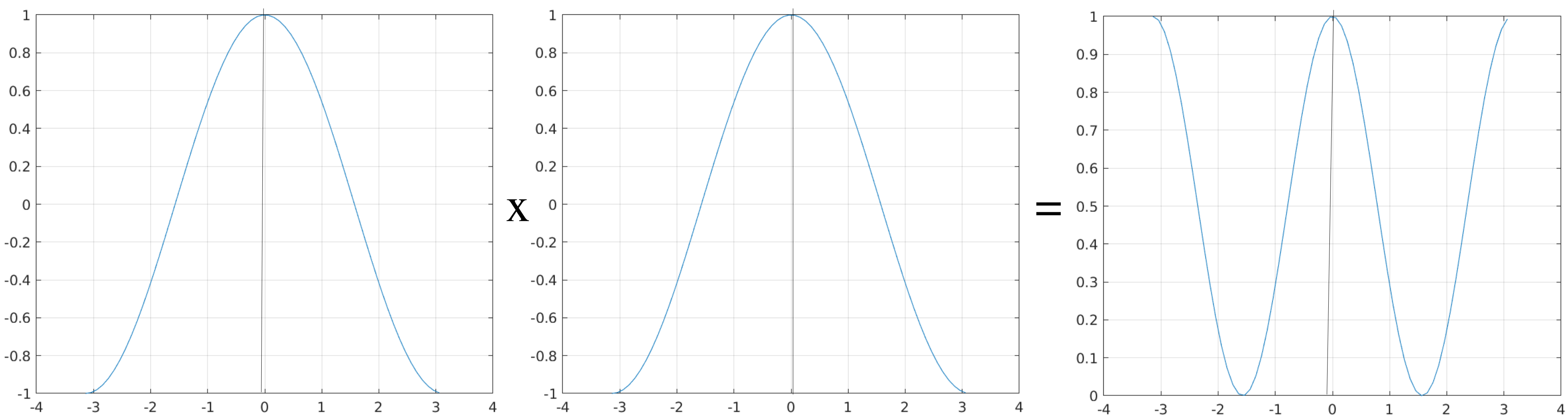

even even even

Odd, Even, and Periodic Functions

Definition 24.29 (Periodic Function). *Let s be a variable in the Domain $\mathscr{D} \subset \mathbb{C}$. Also, let $\tilde{\lambda}$ be a constant where $\tilde{\lambda} \neq 0$ and such that $s + \tilde{\lambda} \in \mathscr{D}$. A function $H(s)$ is said to be periodic with period $\tilde{\lambda}$ if $H(s) = H(s + \tilde{\lambda}) \forall s \in \mathscr{D}$.*

Definition 24.30 (Periodic Extension of a Function). *Let $h(t), t \in \mathbb{R}$ be defined in an interval $t_0 \leq t < t_0 + \tilde{\lambda}$, then the periodic extension of $h(t)$, $\tilde{h}(\tau)$ is defined as a function defined in $-\infty < \tau < \infty$ where $\tilde{h}(t + n\tilde{\lambda}) = h(t), t_0 \leq t < t_0 + \tilde{\lambda}; -\infty < n < \infty$.*

This is essentially the collection of $h(t)$ and its copies which have been shifted by $n\tilde{\lambda}, n = 1, 2, \cdots$ to the right and to the left. The periodic extension is a useful notion for doing manipulations on functions where the function is expected to be periodic, such as the Fourier Series expansion

Complex Variables: Differentiation

Definition 24.31 (Differentiation of Functions of Complex Variables). *Let $H(s)$ be a single-valued function of $s \ : \ s \in \mathscr{D} \subset \mathbb{C}$. Let s_0 be any fixed point in domain $\mathscr{D}$. Then, $H(s)$ is said to have a derivative at point s_0 if the limit in Equation 24.70 exists.*

$$\left.\frac{dH(s)}{ds}\right|_{s=s_0} = \lim_{s \to s_0} \frac{H(s) - H(s_0)}{s - s_0} \tag{24.70}$$

Complex Variables: Differentiation

Property 24.8 (Differentiation of Functions of Complex Variables). *The formal rules for the differentiation of functions of complex variables are similar to those for functions of real variables. If $s \in \mathbb{C}$, c is a constant such that $c \in \mathbb{C}$, and $G(s)$ and $H(s)$ are functions of s defined in $\mathbb{C}$, then,*

$$\frac{dc}{ds} = 0 \qquad \frac{ds}{ds} = 1$$

$$\frac{d\left[H(s) \pm G(s)\right]}{ds} = \frac{dH(s)}{ds} \pm \frac{dG(s)}{ds} \qquad \frac{d\left[H(s).G(s)\right]}{ds} = H(s)\frac{dG(s)}{ds} + G(s)\frac{dH(s)}{ds}$$

and assuming $G(s) \neq 0$,

$$\frac{d\left[\frac{H(s)}{G(s)}\right]}{ds} = \frac{G(s)\frac{dH(s)}{ds} - H(s)\frac{dG(s)}{ds}}{G(s)^2}$$

Also, the chain rule still holds in the complex domain, namely,

$$\begin{aligned} w &\stackrel{\Delta}{=} H(\eta) \\ &= H\left(G(s)\right) \end{aligned} \qquad \Longrightarrow \qquad \frac{dw}{ds} = \frac{dw}{d\eta}\frac{d\eta}{ds}$$

Complex Variables: Partial Differentiation

Definition 24.32 (Partial Differentiation Notation). *If $u(\xi_1, \xi_2, \cdots, \xi_n)$ is a function of n variables, then the following shorthand derivative notation is used,*

$$\text{Partial Derivatives, } \quad u_{\xi_i} \triangleq \frac{\partial u(\xi_1, \xi_2, \cdots, \xi_n)}{\partial \xi_i}$$

$$\text{Partial Second Derivatives, } \quad u_{\xi_i \xi_j} \triangleq \frac{\partial^2 u(\xi_1, \xi_2, \cdots, \xi_n)}{\partial \xi_i \partial \xi_j}$$

$$\begin{aligned} \text{Laplacian, } \nabla^2 u &\triangleq \sum_{i=1}^{n} u_{\xi_i \xi_i} \\ &= \sum_{i=1}^{n} \frac{\partial^2 u(\xi_1, \xi_2, \cdots, \xi_n)}{\partial \xi_i^2} \end{aligned}$$

Complex Variables: Laplace's Equation

Definition 24.33 (Laplace's Equation). *Laplace's equation states that*

$$\nabla^2 u(\xi_1, \xi_2, \cdots, \xi_n) = 0$$

It describes many states of nature including steady-state heat conduction and potentials such as gravitation and electric potential.

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Complex Variables: Analyticity

(shows up in the Residue Theorem)

Definition 24.34 (Analytic Function). *A function of a complex variable, $H(s)$ where $s \in \mathscr{D} \subset \mathbb{C}$, is said to be analytic in an interval $[a,b]$ if it is single valued in that domain (only has one value for each point in the domain) and if all its derivatives, $\frac{d^n H(s)}{ds^n} : n \geq 0$, exist at every point of the domain. In addition, an analytic function may be completely described in terms of power series in a Domain $\mathscr{D} \subset \mathbb{C}$.*

If the function satisfies the Cauchy-Riemann conditions at each point in the domain, then the existence of the derivatives may be relaxed to only the existence of the first derivative.

An Analytic function is also known as a Holomorphic or Regular function.

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Complex Variables: Analyticity

Definition 24.35 (Pointwise Analyticity of Functions). *A function $H(s)$ is said to be analytic at point s_0 if $H(s)$ is analytic in neighborhood of s_0.*

Complex Variables: Analyticity

Theorem 24.4 (Relation between existence of derivative and continuity). *If a function of a complex variable, $H(s)$ where $s \in \mathbb{C}$, has a derivative at $s_0 \in \mathbb{C}$, then it is continuous at s_0. All analytic functions are continuous.*

Proof:

$$\begin{aligned}
\lim_{s \to s_0} [H(s) - H(s_0)] &= \lim_{s \to s_0} (s - s_0) \lim_{s \to s_0} \left[\frac{H(s) - H(s_0)}{(s - s_0)} \right] \\
&= 0 \times \left. \frac{dH(s)}{ds} \right|_{s=s_0} \\
&= 0
\end{aligned}$$

Hence,

$$\begin{aligned}
\lim_{s \to s_0} H(s) &= \lim_{s \to s_0} [H(s_0) + (H(s) - H(s_0))] \\
&= H(s_0)
\end{aligned}$$

which is just the definition of continuity *(see Definition 24.12)*

$$\lim_{t \to t_0} h(t) = h(t_0) \qquad \square$$

Continuity does not imply Analyticity

Example: $H(s) = |s|^2 \quad \forall \; s \in \mathbb{C}$

$$\begin{aligned} G(s) &\triangleq \frac{H(s) - H(s_0)}{s - s_0} \\ &= \frac{|s|^2 - |s_0|^2}{s - s_0} \quad \forall \; (s \neq s_0) \\ &= \frac{s\bar{s} - s_0\overline{s_0}}{s - s_0} \\ &= \bar{s} + s_0 \left[\frac{\bar{s} - \overline{s_0}}{s - s_0} \right] \end{aligned}$$

Polar Coordinates

$$\begin{aligned} \rho e^{i\theta} &\equiv s - s_0 \\ &= \rho(\cos(\theta) + i\sin(\theta)) \end{aligned}$$

$$\begin{aligned} G(s) &= \bar{s} + \frac{s_0 \rho e^{-i\theta}}{\rho e^{i\theta}} \\ &= \bar{s} + s_0 e^{-i(2\theta)} \end{aligned}$$

$$\begin{aligned} G(s) &= \frac{H(s) - H(s_0)}{s - s_0} \\ &= \bar{s} + s_0[\cos(2\theta) - i\sin(2\theta)] \end{aligned}$$

Consider two different ways $s \to s_0$ in the complex plane $\mathbb{C}$,

1. $s \to s_0$ along $\theta = 0 \implies G(s) = \overline{s_0} + s_0$
2. $s \to s_0$ along $\theta = \frac{\pi}{4}$ rad. $\implies G(s) = \overline{s_0} - s_0$

Therefore, in general the limit, hence the derivative, does not exist unless $s_0 = 0$ where $G(s) = \overline{s_0} = 0$. This implies that although $H(s) = |s|^2$ exists everywhere and hence is continuous, it is not analytic since its derivative does not exist except at $s = 0$.

Cauchy-Riemann Conditions

Definition 24.36 (Cauchy-Riemann Conditions). *If $H(s)$ may be written in its real and imaginary components, namely,*

$$H(s) \equiv U(\sigma, \omega) + iV(\sigma, \omega)$$

Then, the Cauchy-Riemann conditions dictate that,

$$U_\sigma = V_\omega$$
$$U_\omega = -V_\sigma$$

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Cauchy-Riemann Theorem

Theorem 24.5 (Cauchy-Riemann Theorem). *A necessary condition for a function, $H(s) = U(\sigma, \omega) + iV(\sigma, \omega)$ to be analytic in a domain $\mathscr{D} \subset \mathbb{C}$ is that the four partial derivatives, $U_\sigma, U_\omega, V_\sigma$, and V_ω exist and satisfy the Cauchy-Riemann conditions at each point in $\mathscr{D}$.*

Proof:

Let $s_0 = \sigma_0 + i\omega_0$ be any fixed point in domain $\mathscr{D}$. Then,

$$\left.\frac{dH(s)}{ds}\right|_{s=s_0} = \lim_{s \to s_0} \frac{H(s) - H(s_0)}{s - s_0}$$

$$= \lim_{s \to s_0} \frac{\Delta H(s)}{\Delta s}$$

Cauchy-Riemann Theorem

Proof (Continued): Consider two paths along which $s \to s_0$,

1. Let $s \to s_0$ along a line parallel to the $\mathbb{R}$-axis, i.e. along $\omega = \omega_0$. Therefore,

$$\begin{aligned} s - s_0 &= \sigma + i\omega_0 - \sigma_0 - i\omega_0 \\ &= \sigma - \sigma_0 \\ &= \Delta\sigma \end{aligned}$$

$$\begin{aligned} \left.\frac{dH(s)}{ds}\right|_{\substack{s \to s_0 \\ \omega = \omega_0}} &= \lim_{\Delta\sigma \to 0} \frac{U(\sigma_0 + \Delta\sigma, \omega_0) - U(\sigma_0, \omega_0)}{\Delta\sigma} + \\ & \quad i \lim_{\Delta\sigma \to 0} \frac{V(\sigma_0 + \Delta\sigma, \omega_0) - V(\sigma_0, \omega_0)}{\Delta\sigma} \\ &= U_\sigma(\sigma_0, \omega_0) + iV_\sigma(\sigma_0, \omega_0) \end{aligned}$$

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Cauchy-Riemann Theorem

Proof (Continued): Consider two paths along which $s \rightarrow s_0$,

2. Let $s \rightarrow s_0$ along a line parallel to the $\mathbb{I}$-axis, i.e. along $\sigma = \sigma_0$. Therefore,

$$\begin{aligned} s - s_0 &= \sigma_0 + i\omega - \sigma_0 - i\omega_0 \\ &= i(\omega - \omega_0) \\ &= i\Delta\omega \end{aligned}$$

$$\begin{aligned} \left.\frac{dH(s)}{ds}\right|_{\substack{s\rightarrow s_0 \\ \sigma=\sigma_0}} &= \lim_{\Delta\omega\rightarrow 0} \frac{U(\sigma_0, \omega_0 + \Delta\omega) - U(\sigma_0, \omega_0)}{i\Delta\omega} + \\ & \quad i \lim_{\Delta\omega\rightarrow 0} \frac{V(\sigma_0, \omega_0 + \Delta\omega) - V(\sigma_0, \omega_0)}{i\Delta\omega} \\ &= \frac{U_\omega(\sigma_0, \omega_0)}{i} + V_\omega(\sigma_0, \omega_0) \\ &= -iU_\omega(\sigma_0, \omega_0) + V_\omega(\sigma_0, \omega_0) \end{aligned}$$

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Cauchy-Riemann Theorem

Proof (Continued): Consider two paths along which $s \rightarrow s_0$,

$$\left.\frac{dH(s)}{ds}\right|_{\substack{s \rightarrow s_0 \\ \omega = \omega_0}} = U_\sigma(\sigma_0, \omega_0) + iV_\sigma(\sigma_0, \omega_0)$$

$$\left.\frac{dH(s)}{ds}\right|_{\substack{s \rightarrow s_0 \\ \sigma = \sigma_0}} = -iU_\omega(\sigma_0, \omega_0) + V_\omega(\sigma_0, \omega_0)$$

If $\frac{dH(s)}{ds}|_{s \rightarrow s_0}$ exists, the expressions should be identical.

$$\begin{aligned} U_\sigma(\sigma_0, \omega_0) &= V_\omega(\sigma_0, \omega_0) \\ U_\omega(\sigma_0, \omega_0) &= -V_\sigma(\sigma_0, \omega_0) \end{aligned}$$

which are the Cauchy-Riemann conditions

□

Alternative Cauchy-Riemann Theorem

Theorem 24.6 (Alternate Cauchy-Riemann Theorem). *Another way of stating Theorem 24.5 is that a necessary condition for a function, $H(s) = U(\sigma,\omega) + iV(\sigma,\omega)$ to be analytic in a domain $\mathscr{D} \subset \mathbb{C}$ is that the Laplace's Equation (see Equation 24.81) be satisfied for both Real and Imaginary parts of $H(s)$, namely,*

$$\nabla^2 U(\sigma,\omega) = 0$$
$$\nabla^2 V(\sigma,\omega) = 0$$

Proof:

Let us consider the Cauchy-Riemann conditions. If we take $\frac{\partial}{\partial\sigma}$ and $\frac{\partial}{\partial\omega}$ of

$$U_\sigma = V_\omega$$
$$U_\omega = -V_\sigma$$

and add the two resulting Equations together we get,

$$U_{\sigma\sigma} + U_{\omega\omega} = \underbrace{V_{\omega\sigma} - V_{\sigma\omega}}_{0} \quad \text{or} \quad \nabla^2 U(\sigma,\omega) = 0$$

Alternative Cauchy-Riemann Theorem

Proof (Continued):

Similarly, if we take $\frac{\partial}{\partial \omega}$ and $\frac{\partial}{\partial \sigma}$ of

$$U_\sigma = V_\omega$$

$$U_\omega = -V_\sigma$$

and add the two resulting Equations together we get,

$$\underbrace{U_{\sigma\omega} - U_{\omega\sigma}}_{0} = V_{\sigma\sigma} + V_{\omega\omega} \qquad \text{or} \qquad \nabla^2 V(\sigma,\omega) = 0$$

Cauchy-Riemann Theorem

Theorem 24.7 (Necessary and Sufficient Cauchy-Riemann Theorem (General Analyticity)). *A necessary and sufficient condition for a function,* $H(s) = U(\sigma, \omega) + iV(\sigma, \omega)$ *to be analytic in a domain* $\mathscr{D} \subset \mathbb{C}$ *is that the four partial derivatives,* $U_\sigma, U_\omega, V_\sigma$*, and* V_ω *exist, be continuous in domain* $\mathscr{D}$*, and satisfy the Cauchy-Riemann conditions (see Definition 24.36) at each point in* $\mathscr{D}$*.*

See the book for the proof.

Analyticity of the Exponential Function

Theorem 24.8 (Analyticity of the Exponential Function). *The Exponential function, e^s is analytic.*

Proof:

$$H(s) = e^s = \underbrace{e^{\sigma}\cos(\omega)}_{U} + i\,\underbrace{e^{\sigma}\sin(\omega)}_{V}$$

Let us write the four partial derivatives of $H(s)$,

$$U_{\sigma} = e^{\sigma}\cos(\omega)$$
$$U_{\omega} = -e^{\sigma}\sin(\omega)$$
$$V_{\sigma} = e^{\sigma}\sin(\omega)$$
$$V_{\omega} = e^{\sigma}\cos(\omega)$$

All four partial derivatives are continuous and are defined in the $\mathbb{C}$ plane. Also, the Cauchy-Riemann conditions are satisfied. Therefore, $H(s) = e^s$ is analytic everywhere. $\square$

Analyticity of the Trigonometric Functions

$$\sin(s) \triangleq \frac{e^{is} - e^{-is}}{2i} \qquad \csc(s) \triangleq \frac{1}{\sin(s)} \qquad \tan(s) \triangleq \frac{\sin(s)}{\cos(s)}$$

$$\cos(s) \triangleq \frac{e^{is} + e^{-is}}{2} \qquad \sec(s) \triangleq \frac{1}{\cos(s)} \qquad \cot(s) \triangleq \frac{\cos(s)}{\sin(s)}$$

All these functions are analytic everywhere in the $\mathbb{C}$ plane. $\sin(s)$ *and* $\cos(s)$ *are periodic with period* 2π. Except for a finite point in the domain in some cases.

Prove for Homework

Cauchy Integral Theorem

Theorem 24.11 (Cauchy Integral Theorem).

- *Simply Connected Domains:*
  Let $H(s)$ be analytic in a simply connected Domain $\mathscr{D} \subset \mathbb{C}$ and let Γ be any closed contour in $\mathscr{D}$. Then,

$$\oint_{\Gamma} H(s) ds = 0 \tag{24.134}$$

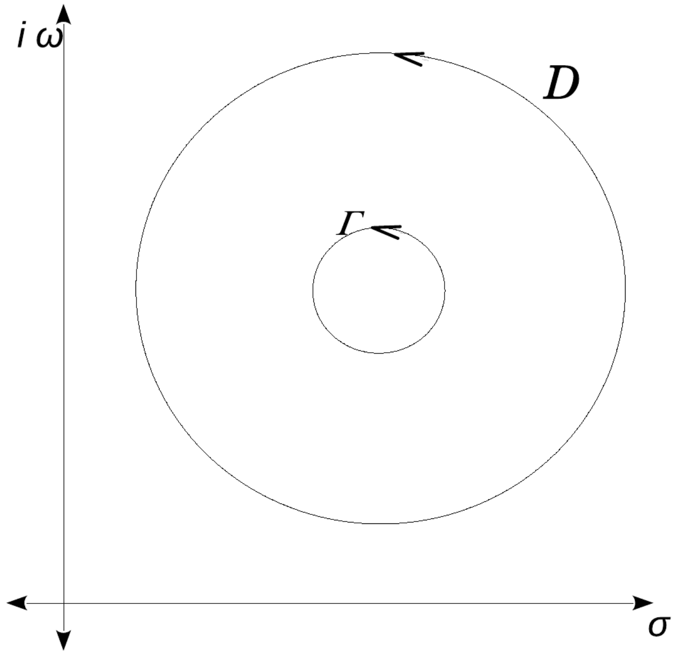

Fig. 24.10: Simply Connected Domain

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Cauchy Integral Theorem

Theorem 24.11 (Cauchy Integral Theorem).

- *Multiply Connected Domains:*
  Let $\Gamma, \Gamma_1, \Gamma_2, \cdots, \Gamma_n$ be simple closed contours, each described in the positive (counter clockwise) direction and such that each Γ_j is inside Γ and outside $\Gamma_k \forall j \neq k; j,k = \{1,2,\cdots,n\}$. See Figure 24.12.

  Let $H(s)$ be analytic on each of the contours Γ and $\Gamma_j, j = \{1,2,\cdots,n\}$ and at each point interior to Γ and exterior to all the $\Gamma_j, j = \{1,2,\cdots,n\}$. Then, the contour integral which contains all the said analytic points of $H(s)$ is zero, namely,

$$\oint_{\Gamma} H(s)ds + \oint_{\Gamma_1} H(s)ds + \oint_{\Gamma_2} H(s)ds + \oint_{\Gamma_3} H(s)ds + \cdots + \oint_{\Gamma_n} H(s)ds = 0$$

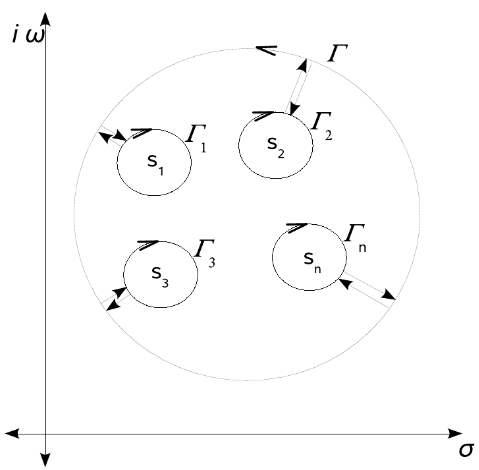


Fig. 24.11: Integration Path of Multiply Connected Contours

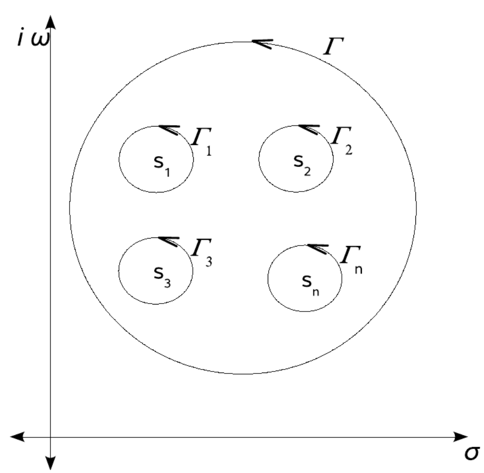


Fig. 24.12: Individual Contour Paths Used by the Cauchy Integral Theorem

Cauchy Integral Theorem

Theorem 24.11 (Cauchy Integral Theorem).

$$\oint_{\Gamma} H(s)ds + \oint_{\Gamma_1} H(s)ds + \oint_{\Gamma_2} H(s)ds + \oint_{\Gamma_3} H(s)ds + \cdots + \oint_{\Gamma_n} H(s)ds = 0$$

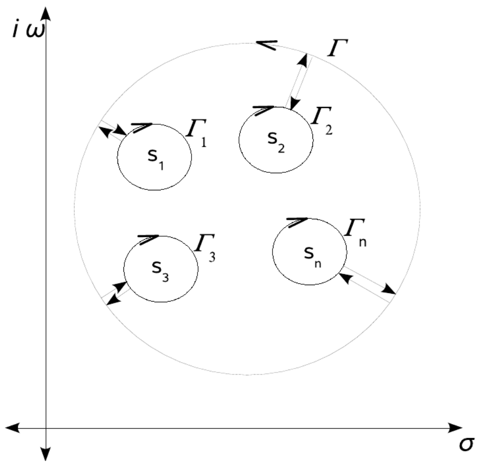


Fig. 24.11: Integration Path of Multiply Connected Contours

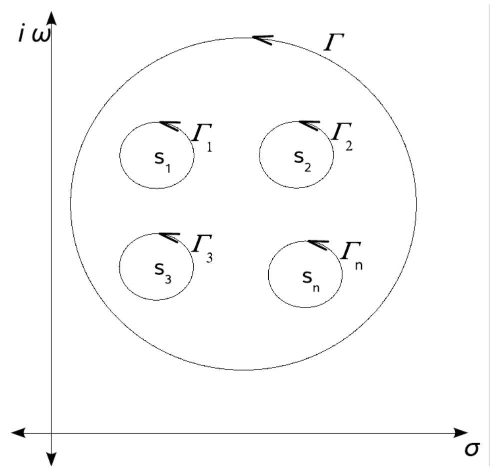


Fig. 24.12: Individual Contour Paths Used by the Cauchy Integral Theorem

Changing the direction of integration for contours $\Gamma_j, j = \{1, 2, \cdots, n\}$,

$$\oint_{\Gamma} H(s)ds = \oint_{\Gamma_1} H(s)ds + \oint_{\Gamma_2} H(s)ds + \oint_{\Gamma_3} H(s)ds + \cdots + \oint_{\Gamma_n} H(s)ds$$

$$= \sum_{j=1}^{n} \oint_{\Gamma_j} H(s)ds$$

Morera's Theorem

Theorem 24.14 (Morera's Theorem (Converse of Cauchy's Integral Theorem)). *If* $H(s)$ *is continuous in a Domain* $\mathscr{D} \subset \mathbb{C}$ *and if,* $\oint_{\Gamma} H(s)ds$ *is zero for every closed contour,* Γ*, then* $H(s)$ *is analytic.*

Power Series Expansion of Functions

Definition 24.42 (Taylor Series (Expansion of an analytic function into a Power Series)). *Let $H(s)$ be analytic within the interior of a circular Domain $\mathscr{D} \subset \mathbb{C}$ with center s_0 and radius ρ, i.e. $\mathscr{D} = \{s : |s - s_0| \leq \rho\}$. Then, at each point s interior to $\mathscr{D}$, the function $H(s)$ may be written in terms of the following power series,*

$$H(s) = \sum_{n=0}^{\infty} a_n (s - s_0)^n \quad \textit{where,} \quad a_n = \frac{1}{2\pi i} \oint_C \frac{H(\zeta)}{(\zeta - s_0)^{n+1}} d\zeta \quad n = \{0, 1, 2, \cdots\}$$

Recall,

$$\underbrace{H(s) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n H(s)}{ds^n} \right|_{s=s_0} (s - s_0)^n}_{\text{Taylor series expansion of } H(s) \text{ about } s=s_0}$$

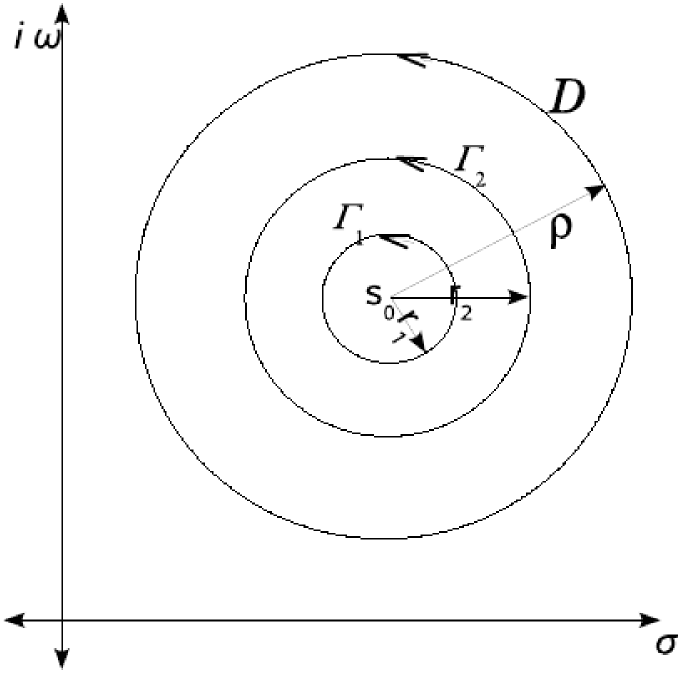
Consequence of Cauchy Integral Formula for the n^{th} derivative

$$\overbrace{\boxed{\left. \frac{d^n H(s)}{ds^n} \right|_{s=s_0} = \frac{n!}{2\pi i} \oint_\Gamma \frac{H(\zeta)}{(\zeta - s_0)^{n+1}} d\zeta}}^{\text{Consequence of Cauchy Integral Formula for the } n^{th} \text{ derivative}}$$

Taylor Series Expansion – Continued

$$H(s) = \sum_{n=0}^{\infty} a_n (s - s_0)^n \quad \text{where,} \quad a_n = \frac{1}{2\pi i} \oint_C \frac{H(\zeta)}{(\zeta - s_0)^{n+1}} d\zeta \quad n = \{0, 1, 2, \cdots\}$$

Moreover, the series converges uniformly (Definition 24.69) for points within and on any circle Γ with center at s_0 and radius $r < \rho$ where ρ is the radius of convergence. Given any center s_0, the radius of convergence, ρ, is the distance from s_0 to the nearest singularity of the function.



N.B. When $s_0 = 0$, the Taylor Series is known as the McLauren series.

Radius of Convergence of Taylor Series
(Examples)

Infinite Radius of Convergence

The radius of convergence of $H(s) = e^s$ *is* $\rho = \infty$ *for any center* s_0.

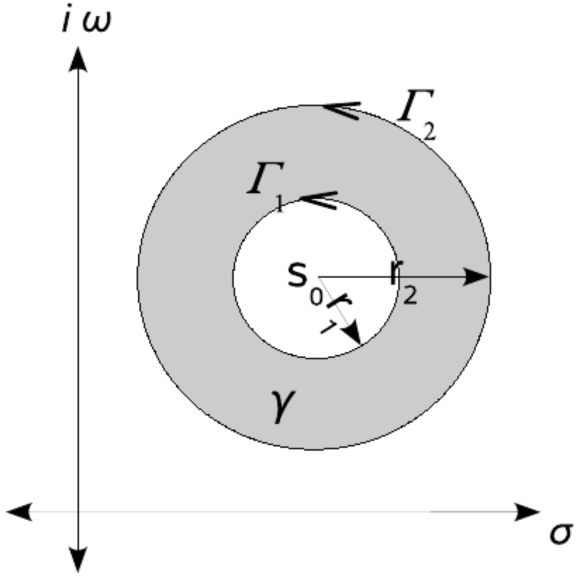
Radius of Convergence

If the center of the domain Γ *is taken to be* $s_0 = 0$, *then the radius of convergence of*

$$H(s) = \frac{e^s}{s-1} \quad \textit{will be } \rho = 1.$$

Laurent Series

Definition 24.43 (Laurent Series (Expansion of analytic functions in an Annular Region)). *Let Υ be an annular region bounded by two concentric circles, Γ_1 and Γ_2 (see Figure 24.14) with centers at s_0 and radii r_1 and r_2 where $r_1 < r_2$. Let $H(s)$ be analytic within Υ and on Γ_1 and Γ_2. Then at each point in the interior of Υ, H(s) can be represented by a convergent power series consisting of both positive and negative powers of $(s - s_0)$ as follows,*



$$H(s) = \sum_{n=0}^{\infty} a_n (s - s_0)^n + \sum_{n=1}^{\infty} b_n (s - s_0)^{-n}$$

$$a_n = \frac{1}{2\pi i} \oint_{\Gamma_2} \frac{H(\zeta)}{(\zeta - s_0)^{n+1}} d\zeta \qquad n = \{0, 1, 2, \cdots\}$$

$$b_n = \frac{1}{2\pi i} \oint_{\Gamma_1} \frac{H(\zeta)}{(\zeta - s_0)^{-n+1}} d\zeta \qquad n = \{1, 2, \cdots\}$$

N.B. $b_1 = \frac{1}{2\pi i} \oint_{\Gamma_1} H(\zeta) d\zeta$ *is the residue of* $H(s)$ *at* $s = s_0$.

Taylor Series expansion is a special case of the Laurent Series where $r_1 \rightarrow 0$.

Uniqueness of Power Series

Property 24.10 (Uniqueness of Power Series). *If two power series,* $\sum_{n=0}^{\infty} a_n(s-s_0)^n$ *and* $\sum_{n=0}^{\infty} b_n(s-s_0)^n$ *both converge to the same function* $H(s)$*, in the same neighborhood of* s_0*,* $|s-s_0| < \rho$*, then the two series are identical. i.e.* $a_n = b_n \ \forall\, n = \{0,1,2,\cdots\}$.

Addition and Multiplication of Power Series

Property 24.11 (Addition and Multiplication of Power Series). *If two power series,* $G(s) = \sum_{n=0}^{\infty} a_n (s - s_0)^n$ *and* $H(s) = \sum_{n=0}^{\infty} b_n (s - s_0)^n$ *both converge with nonzero convergence radii* r_1 *and* r_2 *respectively, such that,* $r_1 \leq r_2$*, then,*

$$G(s) \pm H(s) = \sum_{n=0}^{\infty} (a_n \pm b_n)(s - s_0)^n \quad \textit{where}\ |s - s_0| < r1$$

and

$$G(s) \cdot H(s) \quad = \quad \sum_{n=0}^{\infty} (c_n)(s - s_0)^n \quad \textit{where}\ c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

$$|s - s_0| < r1$$

$$n = \{0, 1, 2, \cdots\}$$

Division of Power Series

Property 24.12 (Division of Power Series). *Consider the two power series $G(s)$ and $H(s)$ of Property 24.11. If $H(s) \neq 0$, then there exists a power series $\sum_{n=0}^{\infty} c_n (s - s_0)^n$ and a number $\zeta > 0$ such that*

$$\frac{G(s)}{H(s)} = \sum_{n=0}^{\infty} c_n (s - s_0)^n \qquad \forall \quad |s - s_0| < \zeta$$

where the coefficients c_n satisfy the following equations

$$a_n = \sum_{k=0}^{n} c_k b_{n-k} \qquad\qquad b_n = \sum_{k=0}^{n} c_k a_{n-k}$$

Zeros and Poles of a Function

Definition 24.44 (Zeros of a Function). *A point s_0 is called a zero of order r of $H(s)$ if*

$$\lim_{s \to s_0} \left[(s - s_0)^{-r} H(s) \right] = M \quad \textit{where } M \neq 0 \wedge M < \infty$$

Definition 24.45 (Isolated Singularities and Poles of a Function). *A point s_0 is called an isolated singularity or an isolated singular point of $H(s)$ if $H(s)$ is not analytic at s_0, but it is analytic in a deleted neighborhood of s_0. s_0 is also called a pole of order r of function $H(s)$ if,*

$$\lim_{s \to s_0} [(s - s_0)^{r} H(s)] = M \quad \textit{where } M \neq 0 \wedge M < \infty$$