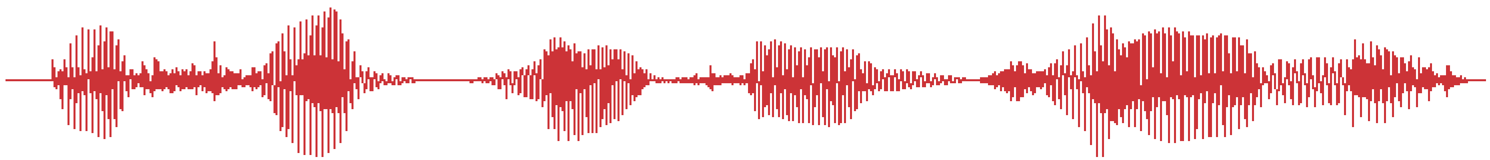
Introduction to Continuous Control Systems

EEME E3601



Week 3

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Vector-Induced L_p Norm of a Matrix

$$\|\mathbf{A}\|_p = \sup_{\vec{x}\neq\vec{0}} \frac{\|\mathbf{A}\vec{x}\|_p}{\|\vec{x}\|_p}$$

Infimum is the Geatest Lower Bound
(does not have to be in the set), eg
The smallest positive real number

Supremum is the Least Upper Bound
(does not have to be in the set), eg
The greatest negative real number

$$\|\mathbf{A}\|_1 = \sup_{\vec{x}\neq\vec{0}} \frac{\|\mathbf{A}\vec{x}\|_1}{\|\vec{x}\|_1}$$

$$= \max_{1\leq j\leq N} \sum_{i=1}^{M} |(\mathbf{A})_{[i][j]}|$$

Maximum of Sum of Absolute
Value of Column Elements

$$\|\mathbf{A}\|_\infty = \sup_{\vec{x}\neq\vec{0}} \frac{\|\mathbf{A}\vec{x}\|_\infty}{\|\vec{x}\|_\infty}$$

$$= \max_{1\leq i\leq M} \sum_{j=1}^{N} |(\mathbf{A})_{[i][j]}|$$

Maximum of Sum of Absolute
Value of Row Elements

Matrix Norm Axioms

In general, all matrix norms satisfy the following four conditions:
For $\mathbf{A}, \mathbf{B} : \mathscr{R}^N \mapsto \mathscr{R}^M$ *and* $\mathbf{C} : \mathscr{R}^M \mapsto \mathscr{R}^N$,

1. $||\mathbf{A}|| \geq 0$ *and* $||A|| = 0$ *iff* $\mathbf{A} = \mathbf{0}$
2. $||k\mathbf{A}|| = |k|\, ||A||$ *where k is any scalar*
3. $||\mathbf{A} + \mathbf{B}|| \leq ||\mathbf{A}|| + ||\mathbf{B}||$ *(Triangular Inequality)*
4. $||\mathbf{AC}|| \leq ||\mathbf{A}||\, ||\mathbf{C}||$ *(Schwarz's Inequality)*

Unitary, Orthogonal Matrices

Definition 23.11 (Unitary / Orthogonal Matrices). *A matrix,* $\mathbf{U} : \mathscr{C}^N \mapsto \mathscr{C}^N$ *is said to be Unitary if,*

$$\mathbf{U}^H\mathbf{U} = \mathbf{U}\mathbf{U}^H = \mathbf{I}_N \tag{23.16}$$

A special case of unitary matrices is $\mathbf{V} : \mathscr{R}^N \mapsto \mathscr{R}^N$ *in which case,*

$$\mathbf{V}^T\mathbf{V} = \mathbf{V}\mathbf{V}^T = \mathbf{I}_N \tag{23.17}$$

Matrices falling under this special case are called orthogonal.

Conjugacy, Orthogonality, Orthonormality

Definition 23.12 (Conjugacy, Orthogonality, and Orthonormality). *Any set of linearly independent vectors,*

$$\mathbf{v}_i : \mathbf{v}_i \in \mathscr{R}^N, i \in \{1,2,...,M\}, M \leq N \tag{23.18}$$

is said to be mutually conjugate about a positive definite, full rank matrix, $\mathbf{Q} : \mathscr{R}^N \mapsto \mathscr{R}^N$ *such that,*

$$\mathbf{v}_i^T \mathbf{Q} \mathbf{v}_j = \begin{cases} a > 0 \ \forall \ i = j \\ \quad 0 \quad \ \forall \ i \neq j \end{cases} \tag{23.19}$$

If $\mathbf{Q} = \mathbf{I}_N$*, then the set is a mutually orthogonal set of vectors. If in addition* $a = 1$*, then the set is mutually orthonormal (i.e., for an orthonormal set of vectors,* $\|\mathbf{v}_i\|_{\mathscr{E}} = 1$*).*

Singular Values of a Matrix

Definition 23.13 (Singular Values of a Matrix). *If* $\mathbf{A} : \mathscr{C}^N \mapsto \mathscr{C}^M$*, then the strictly positive square roots* σ_i *of the non-zero eigenvalues of* $\mathbf{A}^H\mathbf{A}$ *(or* $\mathbf{A}\mathbf{A}^H$*) are called the singular values of matrix* $\mathbf{A}$.

Rank of a Matrix

Definition 23.14 (Rank of a Matrix). *Matrix* $\mathbf{A} : \mathscr{C}^N \mapsto \mathscr{C}^M$ *has rank k if it has k singular values.*

Singular Value Decomposition of a Matrix

Definition 23.15 (Singular Value Decomposition). *If* $\mathbf{A} : \mathscr{C}^N \mapsto \mathscr{C}^M$ *has rank* k *and its singular values are denoted by* $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k > 0$*, then there exist two unitary matrices,*

$\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_M] : \mathscr{C}^M \mapsto \mathscr{C}^M$ and $\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N] : \mathscr{C}^N \mapsto \mathscr{C}^N$

Such that

$\mathbf{S} = \mathbf{U}^H \mathbf{A} \mathbf{V} and \mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^H$ where, $\mathbf{S} = \begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} : \mathscr{C}^N \mapsto \mathscr{C}^M$ and $(\mathbf{D})_{[i][j]} = \begin{cases} \sigma_i & \forall \; i = j \\ 0 & \forall \; i \neq j \end{cases}$

Then,

$\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^H$ *is the singular value decomposition of matrix* $\mathbf{A}$*, where, for* $1 \leq i \leq k$, $\mathbf{u}_i = \frac{\mathbf{A}\mathbf{v}_i}{\sigma_i}$ *and* $\mathbf{v}_i = \frac{\mathbf{A}^H \mathbf{u}_i}{\sigma_i}$ *are Eigenvectors of* $\mathbf{A}\mathbf{A}^H$ *and* $\mathbf{A}^H\mathbf{A}$ *respectively, associated with the* k *eigenvalues* $\sigma_i^2 > 0$ *and the vectors* $\mathbf{u}_i, k+1 \leq i \leq M$ *and* $\mathbf{v}_i, k+1 \leq i \leq N$ *are Eigenvectors associated with the zero eigenvalues. If* $\mathbf{A}$ *is real, then* $\mathbf{U}$ *and* $\mathbf{V}$ *will also be real and are therefore orthogonal matrices.*

Vector-Induced L_2 Norm (Spectral Norm) of a Matrix

$$
\begin{aligned}
\|\mathbf{A}\|_2 &= \sup_{\vec{x}\neq\vec{0}} \frac{\|\mathbf{A}\vec{x}\|_2}{\|\vec{x}\|_2} \\
&= \|\mathbf{A}\mathbf{A}^H\|_2 \\
&= \|\mathbf{A}^H\mathbf{A}\|_2 \\
&= \sigma_{max}
\end{aligned}
$$

where σ_{max} is the largest singular value of $\mathbf{A}$

Frobenius Norm of a Matrix

Definition 23.10 (Euclidean (Frobenius) Norm of a Matrix). *The Euclidean (Frobenius) norm of a matrix* $\mathbf{A} : \mathscr{R}^N \mapsto \mathscr{R}^M$ *is denoted by* $\|\mathbf{A}\|_{\mathscr{E}}$ *or* $\|\mathbf{A}\|_{\mathscr{F}}$ *and is defined as,*

$$\begin{aligned}\|\mathbf{A}\|_{\mathscr{E}} &= \|\mathbf{A}\|_{\mathscr{F}} \\ &= \left(\sum_{i=1}^{M}\sum_{j=1}^{N}(\mathbf{A})^2_{[i][j]}\right)^{\frac{1}{2}}\end{aligned}$$

where $(\mathbf{A})_{[i][j]}, (i \in \{1,2,\ldots,M\}; j \in \{1,2,\ldots,N\})$ *is the* $(i,j)^{th}$ *element of matrix* $\mathbf{A}$.

NB: The Euclidean or Frobenius Norm of a Matrix is not the same as its L_2 Norm (Spectral Norm) – in contrast with a vector, where the Euclidean Norm and the L_2 Norm are equivalent.

Frobenius Norm of a Matrix

$$\|\mathbf{A}\|_{\mathscr{E}} = \|\mathbf{A}\|_{\mathscr{F}} = \left(\sum_{i=1}^{M} \|\mathbf{A}\mathbf{u}_i\|_{\mathscr{E}}^2 \right)^{\frac{1}{2}}$$

where $\mathbf{u}_i, i \in \{1, 2, \ldots, M\}$ *is any orthonormal basis*

$$\|\mathbf{A}\|_{\mathscr{E}} = \|\mathbf{A}\|_{\mathscr{F}} = \sqrt{tr\left(\mathbf{A}^T\mathbf{A}\right)}$$

For a complex Matrix A, replace A^T with A^H

where $tr\left(\mathbf{A}^T\mathbf{A}\right)$ *denotes the trace of* $\left(\mathbf{A}^T\mathbf{A}\right)$ *which is equivalent to the sum of all its diagonal elements.*

Therefore the 2-norm (spectral norm) of A is always smaller than its Euclidean (Frobenius) Norm

$$\begin{aligned} \|\mathbf{A}\|_2 &\leq \|\mathbf{A}\|_{\mathscr{F}} \\ \sigma_{max} &\leq \sqrt{tr\left(\mathbf{A}^H\mathbf{A}\right)} \end{aligned}$$

Where equality holds for the case when A is a vector or has rank one.

Pseudo-Inverse (Moore-Penrose Generalized Inverse)

Definition 23.16 (Pseudo-Inverse (Moore-Penrose Generalized Inverse)). *If* $\mathbf{A} : \mathscr{C}^N \mapsto \mathscr{C}^M$ *and* $\mathbf{A}^\dagger : \mathscr{C}^M \mapsto \mathscr{C}^N$, *then* $\mathbf{A}^\dagger$ *is the pseudo-inverse (Moore-Penrose generalized inverse) of A iff,*

1. $\mathbf{A}\mathbf{A}^\dagger\mathbf{A}$ Exists and is A
2. $\mathbf{A}^\dagger\mathbf{A}\mathbf{A}^\dagger$ Exists and is A†
3. $\mathbf{A}$ *and* $\mathbf{A}^\dagger$ *are Hermitian*

Then if $\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^H$ is the SVD of A

the pseudo-inverse of A, $A^\dagger$*, is given by,* $\mathbf{A}^\dagger = \mathbf{V}\mathbf{S}^\dagger\mathbf{U}^H$ where $\mathbf{S}^\dagger = \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} : \mathscr{C}^M \mapsto \mathscr{C}^N$

$\mathbf{E}$ *is the kxk diagonal matrix such that,* $\mathbf{E}_{ij} = \begin{cases} \frac{1}{\sigma_i} & \forall\ i = j \\ 0 & \forall\ i \neq j \end{cases}$ *and k is the rank of A.*

For a real matrix, $\mathbf{A}$, $\mathbf{A}^\dagger$ may be written in terms of the following limit

$$\mathbf{A}^\dagger = \lim_{\varepsilon \to 0} \underbrace{\left(\mathbf{A}^T\mathbf{A} + \varepsilon\mathbf{I}\right)}_{\text{This perturbation will make this term invertible}}{}^{-1} \mathbf{A}^T$$

Positive Definiteness

Definition 23.17 (Positive Definiteness). *Let* $\mathbf{s}$ *be any vector such that* $\mathbf{s} \in \mathscr{R}^N$. *A matrix* $\mathbf{G} : \mathscr{R}^N \mapsto \mathscr{R}^N$ *is said to be positive definite if,*

$$\mathbf{s}^T \mathbf{G} \mathbf{s} > 0 \; \forall \mathbf{s} \neq \mathbf{0}$$

Ordinary Gram-Schmidt Orthogonalization

Suppose, $\mathbf{v}_i : \mathbf{v}_i \in \mathscr{R}^N, i \in \{1,2,...,M\}, M \leq N$ are a set of unit vectors. Then, the following is the Gram-Schmidt procedure [12] which generates the set of vectors $\mathbf{z}_i, i \in \{1,2,...,M\}$ which form an *orthonormal set* spanning the same space as vectors $\mathbf{v}_i$,

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1 \\
\mathbf{u}_i &= \mathbf{v}_i - \sum_{j=1}^{i-1} (\mathbf{v}_i^T \mathbf{z}_j)\mathbf{z}_j \ \ with \ \ i \in \{1,2,...,M\} \\
\mathbf{z}_i &= \frac{\mathbf{u}_i}{\|\mathbf{u}_i\|_{\mathscr{E}}} \ \ with \ \ i \in \{1,2,...,M\}
\end{aligned}$$

Modified Gram-Schmidt Orthogonalization

The following pseudo-code presents a modified Gram-Schmidt orthogonalization method which, theoretically, gives the same set of vectors as the ordinary procedure (Section 23.3.1), but it is more accurate in its numerical implementation,

1. a. $\mathbf{u}_1 = \mathbf{v}_1$
   b. $\mathbf{z}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|_{\mathscr{E}}}$
2. $\mathbf{v}_i^{(1)} = \mathbf{v}_i - (\mathbf{v}_i^T \mathbf{z}_1)\mathbf{z}_1$ for $i = 2, 3, \ldots, M$
3. a. $\mathbf{u}_j = \mathbf{v}_j^{(j-1)}$
   b. $\mathbf{z}_j = \frac{\mathbf{u}_j}{\|\mathbf{u}_j\|_{\mathscr{E}}}$ for $j = 2, 3, \ldots, M$
   c. $\mathbf{v}_i^{(j)} = \mathbf{v}_i^{(j-1)} - (\mathbf{v}_i^{(j-1)^T} \mathbf{z}_j)\mathbf{z}_j$ for $i = j+1, \ldots, M$

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Sherman-Morrison Inversion Formula

If $\mathbf{G}_{k+1}, \mathbf{G}_k : \mathscr{R}^N \mapsto \mathscr{R}^N$, then the rank $M(M \leq N)$ update to $\mathbf{G}_k$ for obtaining $\mathbf{G}_{k+1}$ is,

$$\mathbf{G}_{k+1} = \mathbf{G}_k + \mathbf{R}\mathbf{S}\mathbf{T}^T \tag{23.35}$$

where $\mathbf{R}, \mathbf{T} : \mathscr{R}^M \mapsto \mathscr{R}^N$ and $\mathbf{S} : \mathscr{R}^M \mapsto \mathscr{R}^M$, then the inverse of $\mathbf{G}_{k+1}$ is given by the following,

$$\mathbf{G}_{k+1}^{-1} = \mathbf{G}_k^{-1} - \mathbf{G}_k^{-1}\mathbf{R}\mathbf{U}^{-1}\mathbf{T}^T\mathbf{G}_k^{-1} \tag{23.36}$$

where,

$$\mathbf{U} = \mathbf{S}^{-1} + \mathbf{T}^T\mathbf{G}_k^{-1}\mathbf{R} \tag{23.37}$$

Equation 23.36 is known as the *Sherman-Morrison* formula [6]. It is used to keep track of the change in the inverse of a matrix as the original matrix is updated through 23.35.

Stochastic Matrices

In *probability theory*, we often run across a special type of matrix called a *stochastic matrix*. Here is a formal definition for such matrices.

Definition 23.18 (Stochastic Matrix). *A Stochastic matrix* $\mathbf{A} : \mathscr{R}^N \mapsto \mathscr{R}^M$ *is a matrix such that* $(\mathbf{A})_{[i][j]} \geq 0$ *and*

Sum of elements of each row

$$\sum_{j=1}^{N} (\mathbf{A})_{[i][j]} = 1 \;\; \forall \{i : 1 \leq i \leq M\} \tag{23.45}$$

For example, *stochastic matrices* are used to denote the *transition probabilities* of *Markov chains*.

Complex Variable Theory

Definition 24.1 (Imaginary Number). *i is the imaginary number and is defined as,*

$$i \stackrel{\Delta}{=} \sqrt{-1}$$

René Descartes called it
imaginary to show its uselessness

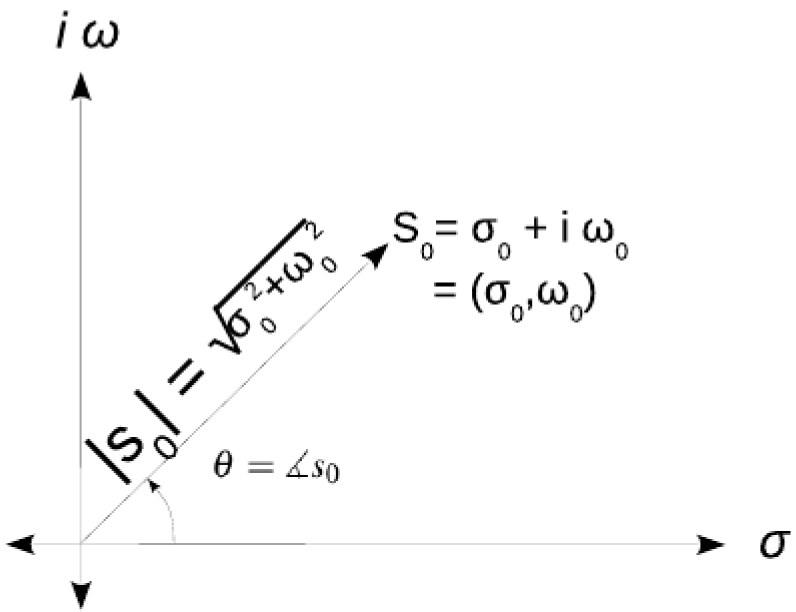

Fig. 24.1: Representation of a Number s_0 in the Complex Plane

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Complex Variable Theory

Definition 24.2 (Modulus or Magnitude of a Complex Number). *The Modulus or Magnitude of the complex number,* $\{s : s = \sigma + i\omega \in \mathbb{C}\}$*, is denoted by* $|s|$ *and is defined as,* $|s| = \sqrt{\sigma^2 + \omega^2}$

N.B., In the complex plain, size is considered using the modulus, e.g., we cannot say $s_1 < s_2$*, but we can say* $|s_1| < |s_2|$.

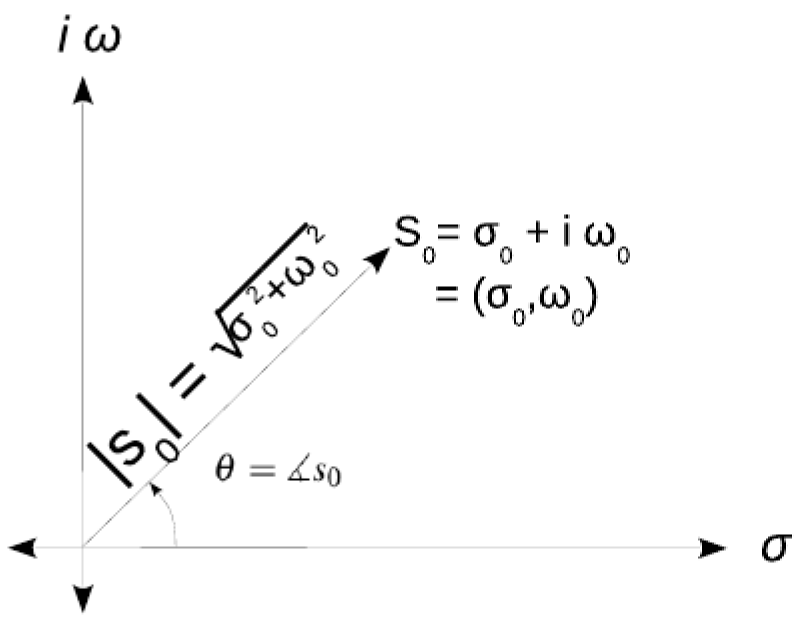

Fig. 24.1: Representation of a Number s_0 in the Complex Plane

Representations of Complex Variables

$\{s_0 : s_0 \in \mathbb{C}\}$ *may be represented in polar coordinates as follows,*

$$s_0 = \rho_0 e^{i\theta_0} \quad \text{where,} \quad \rho_0 = |s_0| \quad \text{and} \quad \theta = \measuredangle s_0$$

$$\begin{aligned} &= \sin^{-1}\left(\frac{\omega_0}{\rho_0}\right) \\ &= \cos^{-1}\left(\frac{\sigma_0}{\rho_0}\right) \\ &= \tan^{-1}\left(\frac{\omega_0}{\sigma_0}\right) \end{aligned}$$

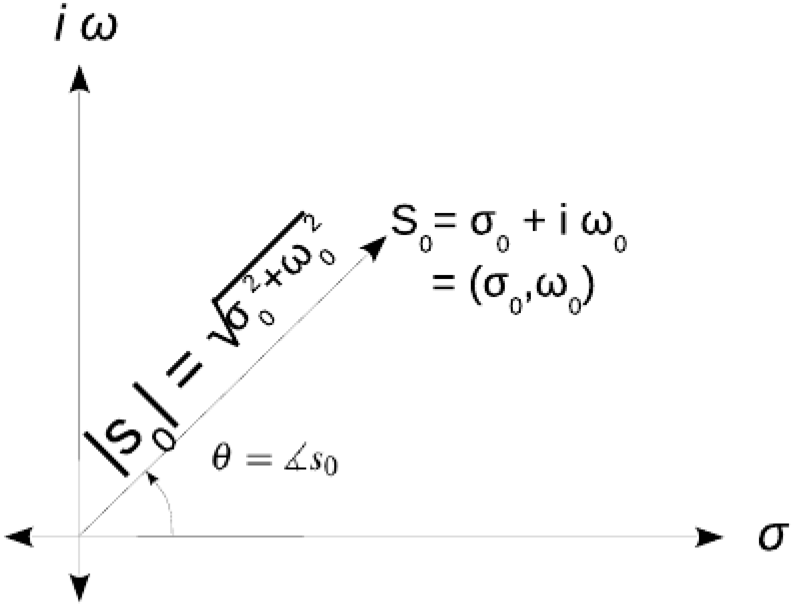

Fig. 24.1: Representation of a Number s_0 in the Complex Plane

Properties of Complex Variables

$$|s| = |\bar{s}|$$

$$|s|^2 = s\bar{s}$$

$$s + \bar{s} = 2\mathscr{R}e\{s\}$$

$$s - \bar{s} = 2\mathscr{I}m\{s\}$$

Property 24.2 (Triangular Inequality in the Complex Plane).

$$|s_1 + s_2| \leq |s_1| + |s_2|$$

Product and Quotient of Complex Variables

Product:

$$
\begin{aligned}
s_1 s_2 &= \rho_1 e^{i\theta_1} \rho_2 e^{i\theta_2} \\
&= \rho_1 \rho_2 e^{i(\theta 1 + \theta_2)} \\
&= \rho_1 \rho_2 e^{i(\theta 1 + \theta_2 + 2n\pi)} \quad where \;\; n = \{0, \pm 1, \pm 2, \cdots\}
\end{aligned}
$$

Periodicity

Quotient:

$$
\begin{aligned}
\frac{s_1}{s_2} &= \frac{\rho_1 e^{i\theta_1}}{\rho_2 e^{i\theta_2}} \\
&= \frac{\rho_1}{\rho_2} e^{i(\theta 1 - \theta_2)} \\
&= \frac{\rho_1}{\rho_2} e^{i(\theta 1 - \theta_2 + 2n\pi)} \quad where \;\; n = \{0, \pm 1, \pm 2, \cdots\}
\end{aligned}
$$

Periodicity of Complex Variables

Problem:

What is the period of the exponential function,

$$H(s) = e^s$$

Solution:

cos and sin have a period of 2π. Also,

$$\cos(2k\pi) + i\sin(2k\pi) = 1 \tag{S.48}$$

where k in any integer.

Writing Equation S.48 in polar coordinates, we have, $e^{i2k\pi} = 1$. Therefore, since we may multiply 1 by e^s without changing its value, we may write,

$$e^s e^{i2k\pi} = e^s = e^{s+i2k\pi} \tag{S.49}$$

Equation S.49 establishes that the period of e^s is $i2\pi$.

Modulus of Product of Complex Variables

Theorem 24.1 (Modulus of the product of two Complex Numbers). *The modulus of the product of two complex numbers, s_1 and s_2, is equal to the product of the moduli of the two numbers, namely,*

$$|s_1 s_2| = |s_1||s_2| \tag{S.29}$$

Proof:

Write the left side of Equation S.29 in terms of the real and imaginary parts of the variables involved,

$$\begin{aligned}
|s_1 s_2| &= |(\sigma_1 + i\omega_1)(\sigma_2 + i\omega_2)| \\
&= |\sigma_1\sigma_2 + i\sigma_1\omega_2 + i\omega_1\sigma_2 - \omega_1\omega_2| \\
&= |(\sigma_1\sigma_2 - \omega_1\omega_2) + i(\sigma_1\omega_2 + \omega_1\sigma_2)| \\
&= \sqrt{(\sigma_1\sigma_2 - \omega_1\omega_2)^2 + (\sigma_1\omega_2 + \omega_1\sigma_2)^2} \\
&= \sqrt{\sigma_1^2\sigma_2^2 + \omega_1^2\omega_2^2 - \cancel{2\sigma_1\sigma_2\omega_1\omega_2} + \sigma_1^2\omega_2^2 + \omega_1^2\sigma_2^2 + \cancel{2\sigma_1\omega_2\omega_1\sigma_2}} \\
&= \sqrt{\sigma_1^2\sigma_2^2 + \omega_1^2\omega_2^2 + \sigma_1^2\omega_2^2 + \omega_1^2\sigma_2^2} \qquad \text{(S.30)}
\end{aligned}$$

Modulus of Product of Complex Variables

Theorem 24.1 (Modulus of the product of two Complex Numbers). *The modulus of the product of two complex numbers, s_1 and s_2, is equal to the product of the moduli of the two numbers, namely,*

$$|s_1 s_2| = |s_1||s_2| \tag{S.29}$$

Proof (continued):

Now do the same for the right hand side of Equation S.29,

$$\begin{aligned} |s_1||s_2| &= |\sigma_1 + i\omega_1||\sigma_2 + i\omega_2| \\ &= \sqrt{(\sigma_1^2 + \omega_1^2)}\sqrt{(\sigma_2^2 + \omega_2^2)} \\ &= \sqrt{\sigma_1^2\sigma_2^2 + \sigma_1^2\omega_2^2 + \omega_1^2\sigma_2^2 + \omega_1^2\omega_2^2} \end{aligned} \tag{S.31}$$

We have arrived at the same expression in Equations S.30 and S.31, proving Equation S.29, hence proving Theorem 24.1.

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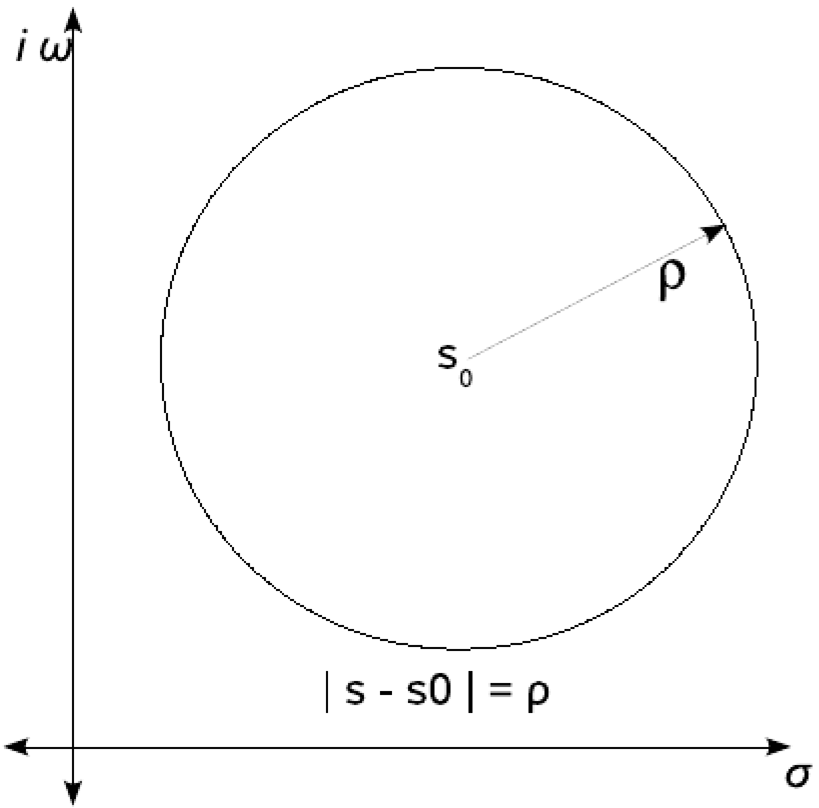
A Circle in the Complex Plane

Definition 24.3 (A Circle in the Complex Plane). *A circle is defined by its center, s_0 and its radius, ρ. In the complex place, such a circle is defined by,*

$$|s - s_0| = \rho \tag{24.16}$$



Sample Application: the Cauchy Residue Theorem

Euler's Formula and de Moivre's Theorem

Euler's Formula:

$$\begin{aligned} e^{i\theta} &= \cos(\theta) + i\ \sin(\theta) \\ e^{-i\theta} &= \cos(\theta) - i\ \sin(\theta) \end{aligned} \quad \Longrightarrow \quad \textit{Euler's Identity:} \quad \begin{aligned} e^{i\pi} + 1 &= 0 \\ e^{i\pi} &= -1 \end{aligned}$$

De Moivre's Theorem:

$$(\cos\theta + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta) \qquad n = \{0, \pm 1, \pm 2, \cdots\}$$

Proof:

Using the Euler's formula

$$\begin{aligned} s &= \rho\,[\cos\theta + i\sin\theta] \\ &= \rho e^{i\theta} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} s^n &= \rho^n\,[\cos\theta + i\sin\theta]^n \\ &= \rho^n e^{in\theta} \end{aligned}$$

$\square$

Sample Application: Solution to Differential Equations and Integral Transforms

A Hermitian Function

Definition 24.5 (A Hermitian Function). *A function* $H(s) = U(\sigma,\omega) + iV(\sigma,\omega)$, $\{s \in \mathbb{C}, \sigma \in \mathbb{R}, \omega \in \mathbb{R}, s = \sigma + i\omega\}$, *is called a Hermitian function if*

$$
\begin{aligned}
H(-s) &= \overline{H(s)} \\
&= U(\sigma,\omega) - iV(\sigma,\omega)
\end{aligned}
$$

Sample Application: Cauchy-Riemann Conditions
and the Cauchy-Riemann Theorem

Limits – Limit of a Sequence

Definition 24.6 (Limit of a Sequence of Numbers).

$$\lim_{n \to \infty} S_n = A$$

A is the limit of sequence S_n as $n \to \infty$. Examples are,

$$\lim_{n \to \infty} \frac{2n}{n+1} = 2$$

and

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

Sample Application: Financial Interest, music (tonal theory)

Sample Application: Convergence of Fourier and Power Series, as well as in the Probability Theory

Limits – One-Sided Limit of a Function

Definition 24.7 (One Sided Limit of a Function – Right Hand Limit).

$$\lim_{t \to t_{0+}} h(t) = A$$

The limit exists if

Implies that given $\varepsilon > 0, \;\exists\; \delta(\varepsilon) > 0 \;:\; |h(t) - A| < \varepsilon$ *if* $t_0 < t < t_0 + \delta$

Definition 24.8 (One Sided Limit of a Function – Left Hand Limit).

$$\lim_{t \to t_{0-}} h(t) = A$$

The limit exists if

Implies that given $\varepsilon > 0, \;\exists\; \delta(\varepsilon) > 0 \;:\; |h(t) - A| < \varepsilon$ *if* $t_0 - \delta < t < t_0$

Limits – Limit of a Function of a Continuous Variable

Definition 24.9 (Limit of a Function of a Continuous Variable).

$$\lim_{t \to t_0} h(t) = A$$

Implies that given $\varepsilon > 0, \; \exists \; \delta(\varepsilon) > 0 \; : \; |h(t) - A| < \varepsilon$ *when* $0 < |t - t_0| < \delta$
Note that all the points lying inside a circle of radius ε *and center* t_0 *are in an* ε *neighborhood of* t_0.

$$\exists \lim_{t \to t_0} h(t) = A \iff \lim_{t \to t_{0+}} h(t) = \lim_{t \to t_{0-}} h(t) = A.$$

If and Only If

Limits – Existence of a Limit

Example 24.1 (Existence of the Limit).
Take the following question: If

$$h(t) = 1 + \frac{|t|}{t} \tag{24.28}$$

does the limit, $\lim_{t \to 0} h(t)$ *exist?*

The right hand limit of 24.28 is,

$$\lim_{t \to 0^+} 1 + \frac{|t|}{t} = 1 + 1 = 2 \;\; (|t| = t) \tag{24.29}$$

The left hand limit of 24.28 is,

$$\lim_{t \to 0^-} 1 + \frac{|t|}{t} = 1 - 1 = 0 \;\; (|t| = -t) \tag{24.30}$$

Therefore, the limit, $\lim_{t \to 0} h(t)$ *does not exist since the left hand and right hand limits are not equal.*

Limits – Infinite Limits

Definition 24.10 (Positive Infinite Limit). *If* $\forall\, M > 0\; \exists \delta > 0 : h(t) > M$ *when* $0 < |t - t_0| < \delta$, *then,*

$$\lim_{t \to t_0} h(t) = \infty$$

Definition 24.11 (Negative Infinite Limit). *If* $\forall\, M > 0\; \exists \delta > 0 : h(t) < -M$ *when* $0 < |t - t_0| < \delta$, *then,*

$$\lim_{t \to t_0} h(t) = -\infty$$