Introduction to Continuous Control Systems

EEME E3601

Week 2

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Nonlinear Control Systems

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \\ \mathbf{y}(t) &= \mathbf{h}(\mathbf{x}(t), t)\end{aligned}$$

Linear Time-Invariant (LTI)
Single-Input Single Output (SISO)

$$
\begin{aligned}
\frac{d^{(n)}y(t)}{dt^{(n)}} &+ p_{n-1}\frac{d^{(n-1)}y(t)}{dt^{(n-1)}} + p_{n-2}\frac{d^{(n-2)}y(t)}{dt^{(n-2)}} + \cdots + p_0 y(t) \\
&= q_{n-1}\frac{d^{(n-1)}u(t)}{dt^{(n-1)}} + q_{n-2}\frac{d^{(n-2)}u(t)}{dt^{(n-2)}} + \cdots + q_0 u(t)
\end{aligned}
$$

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Linear Time-Invariant (LTI) State-Space Representation (SISO & MIMO)

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)\end{aligned}$$

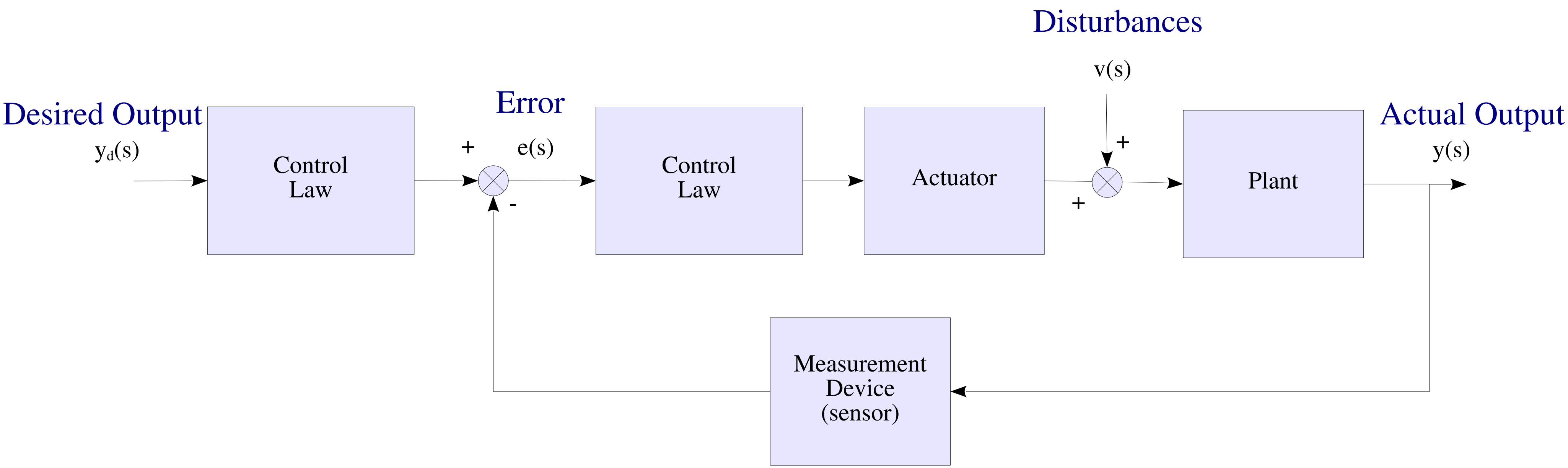
A is the System Matrix
B is the Control Matrix
C is the Observation Control
D is the Direct Input Observation

Time-Variant System

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\
\mathbf{y}(t) &= \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)
\end{aligned}
$$

$\mathbf{A}(t)$ is the time-dependent System Matrix
$\mathbf{B}(t)$ is the time-dependent Control Matrix
$\mathbf{C}(t)$ is the time-dependent Observation Control
$\mathbf{D}(t)$ is the time-dependent Direct Input Observation

Generic Control System



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Control Problem Components

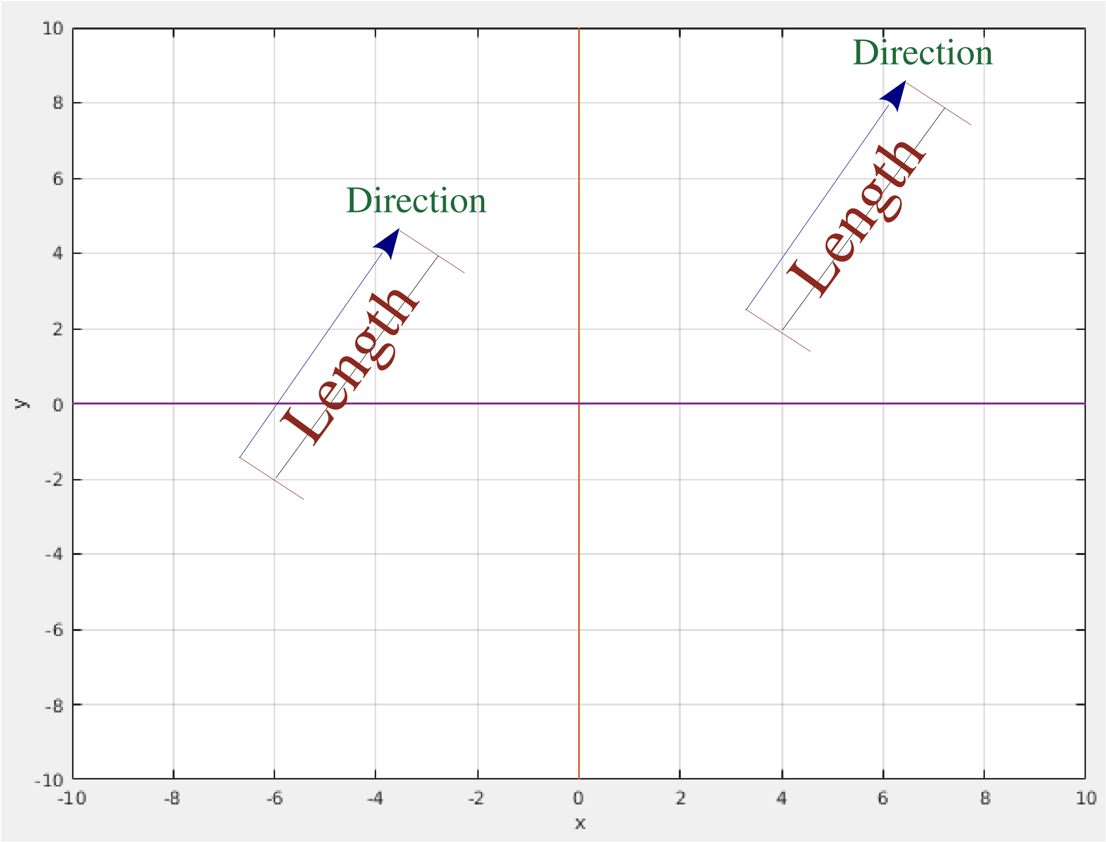
Goals

- Transient Response
- Disturbance Rejection
- Steady-State Error Correction
- Plant Parameter-Change Sensitivity

Approach

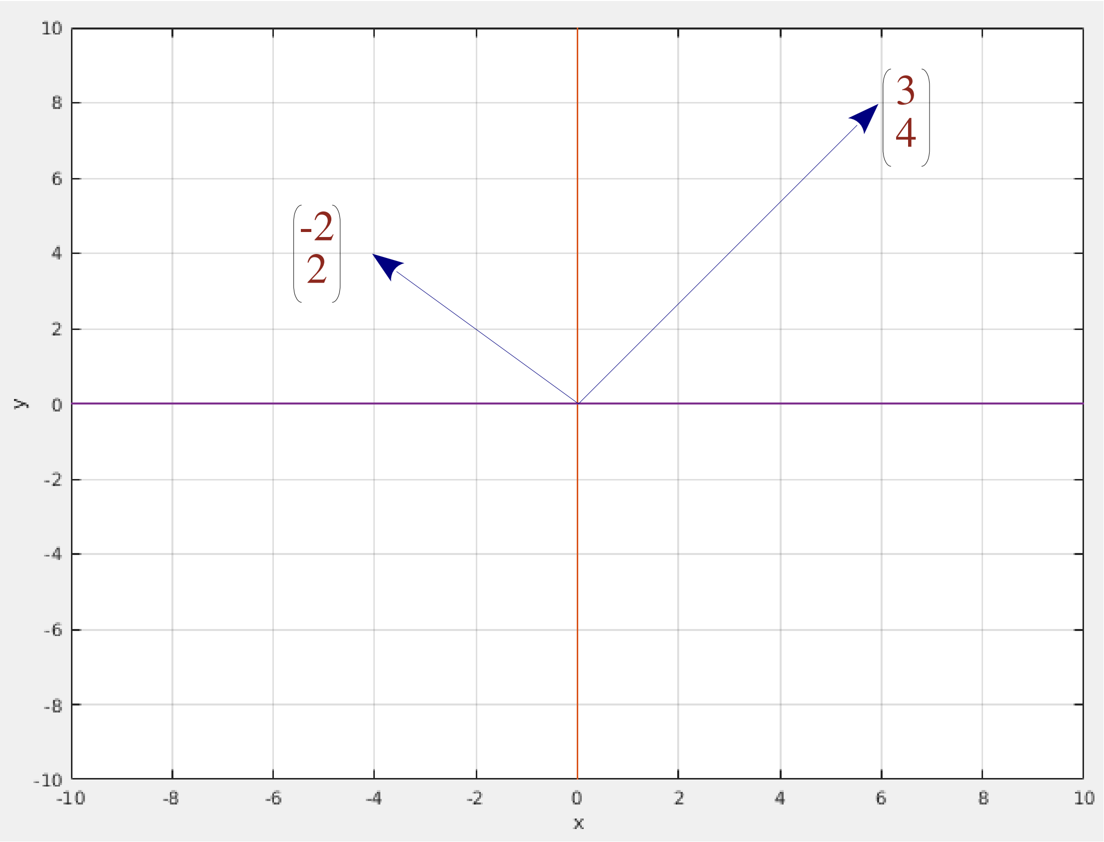
- Sensor Selection
- Actuator Selection
- System Modeling – Developing Equations for the Plant Dynamics, Sensor Response, and Actuator Dynamics
- Controller Design
- Evaluation – analytic evaluation, simulation, hardware test
- Repetition of the tests to achieve repeatable and acceptable results

Vector



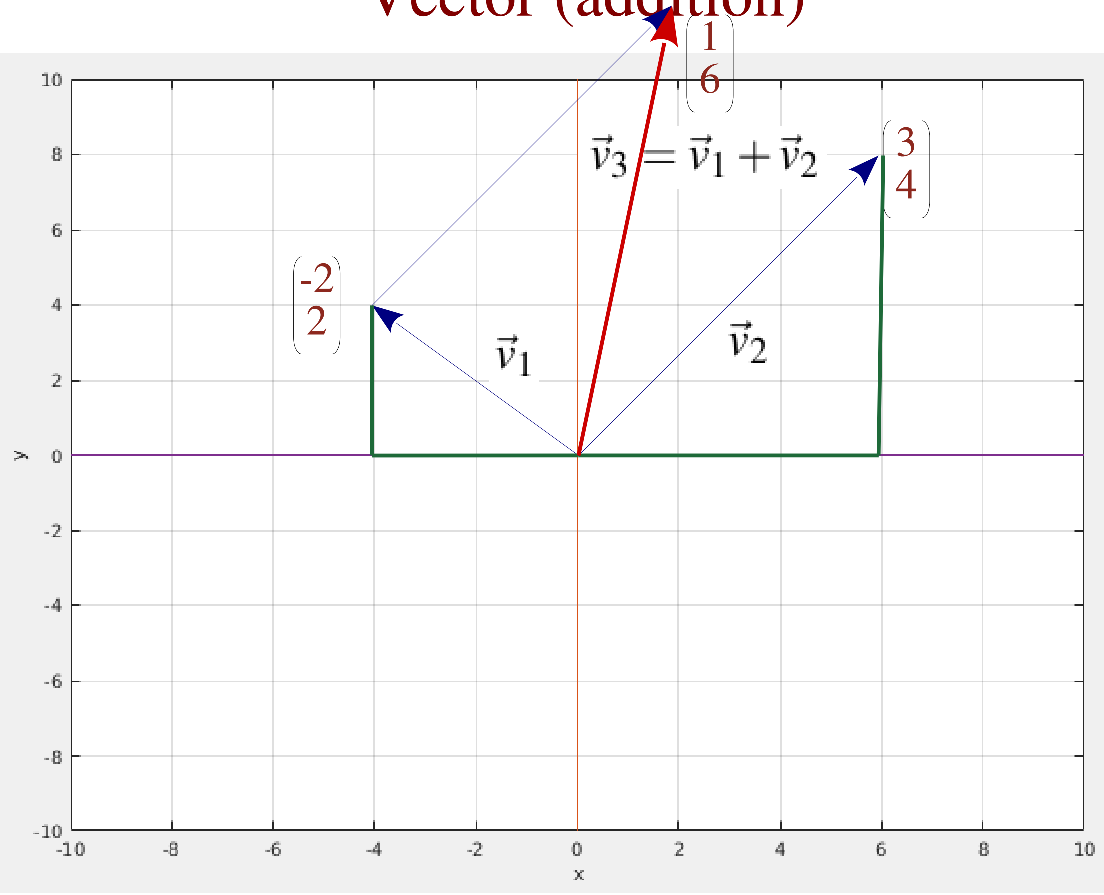
In Engineering and physics
the location does not matter

Vector

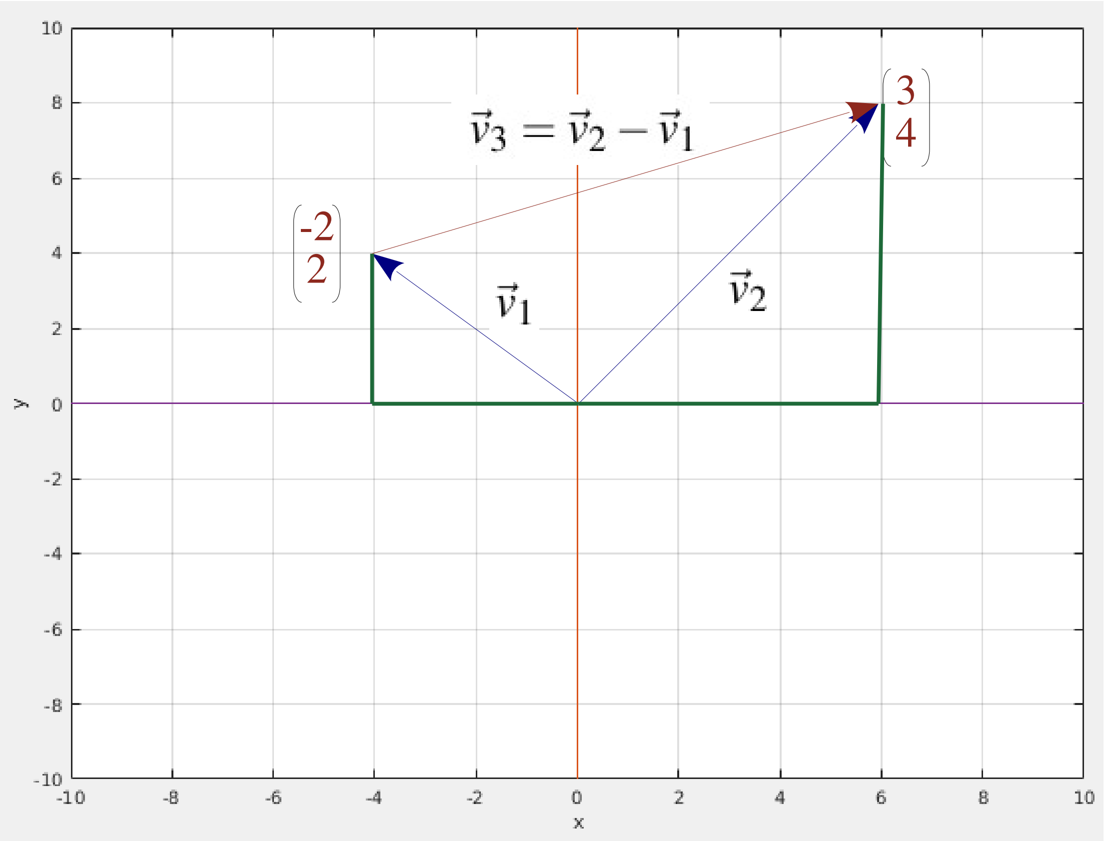


In Mathematics, we define a
vector about the origin

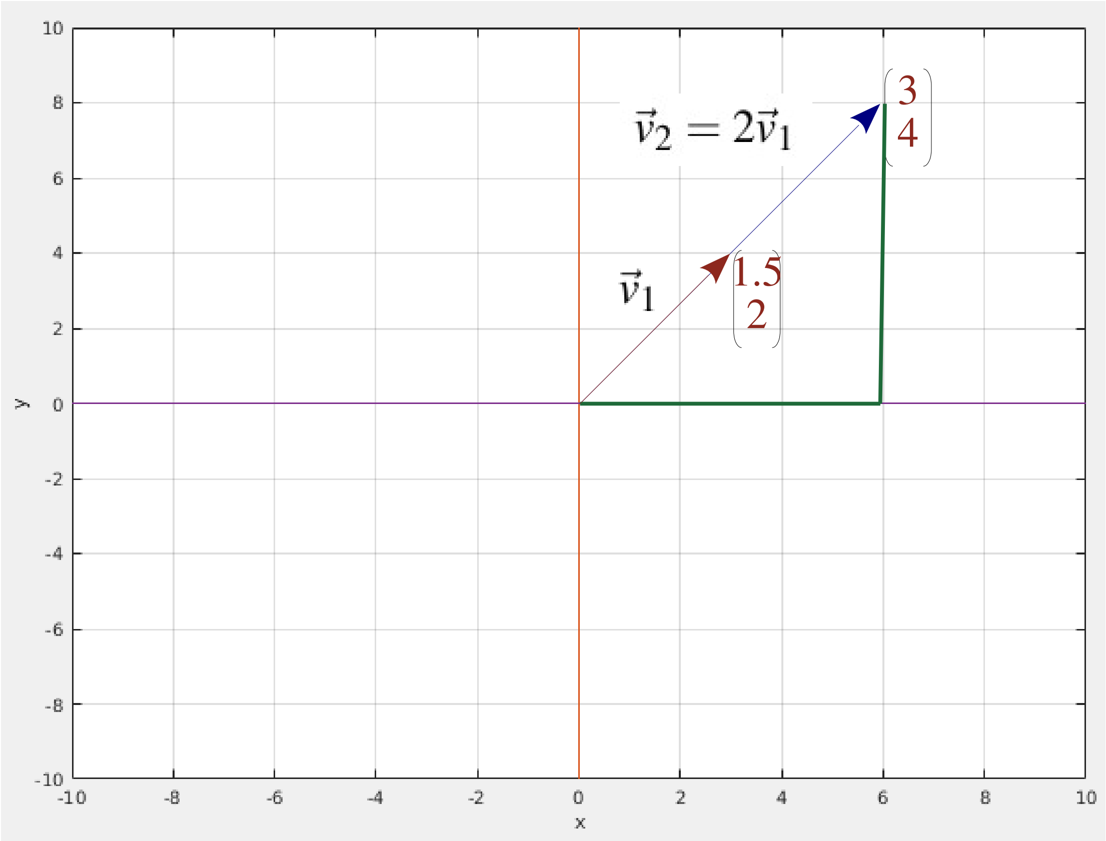
Vector (addition)



Vector (subtraction)



Vector (scaling)



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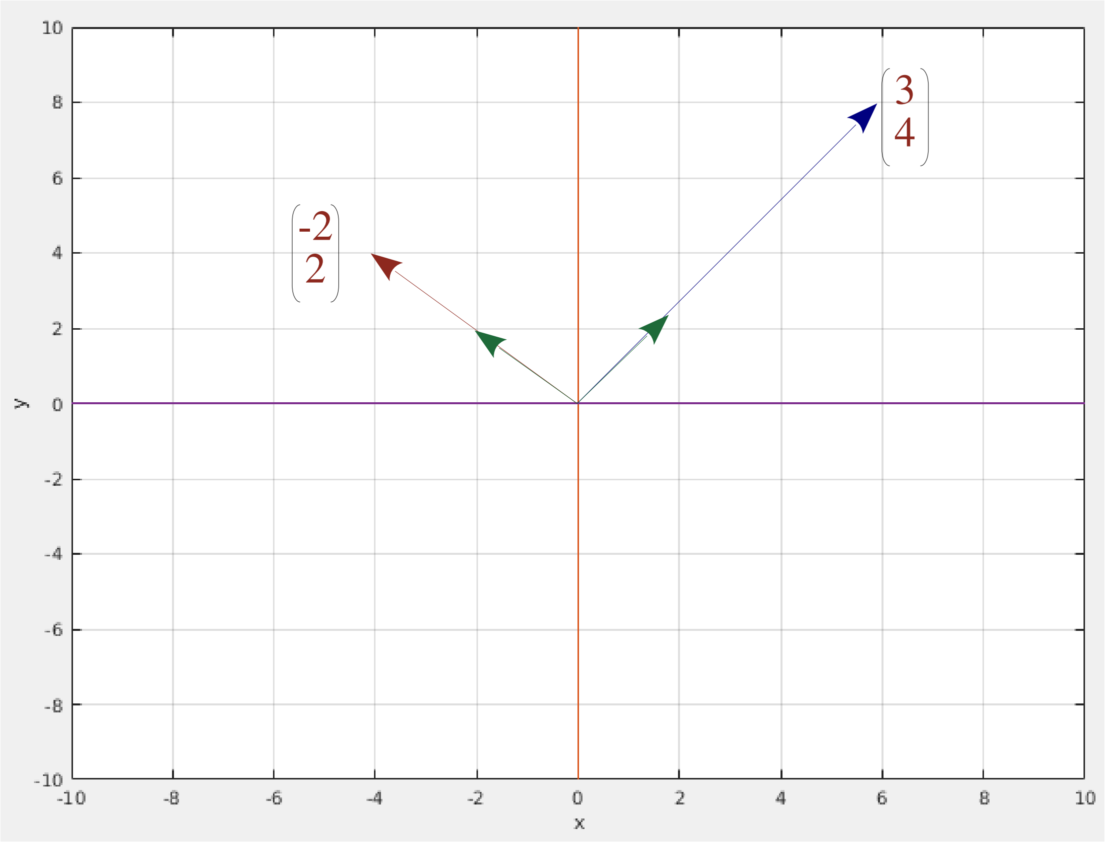
Vector (span of multiple vectors)

Take two 2-D vectors. The Span of two vectors is given by,

$$\vec{s} = \alpha\vec{v}_1 + \beta\vec{v}_2$$

If the two basis vectors are collinear, then only one dimension can be spanned. Otherwise, they will span the whole two-dimensional space. The scaling factors are used to allow access to any point on the spanned plane.



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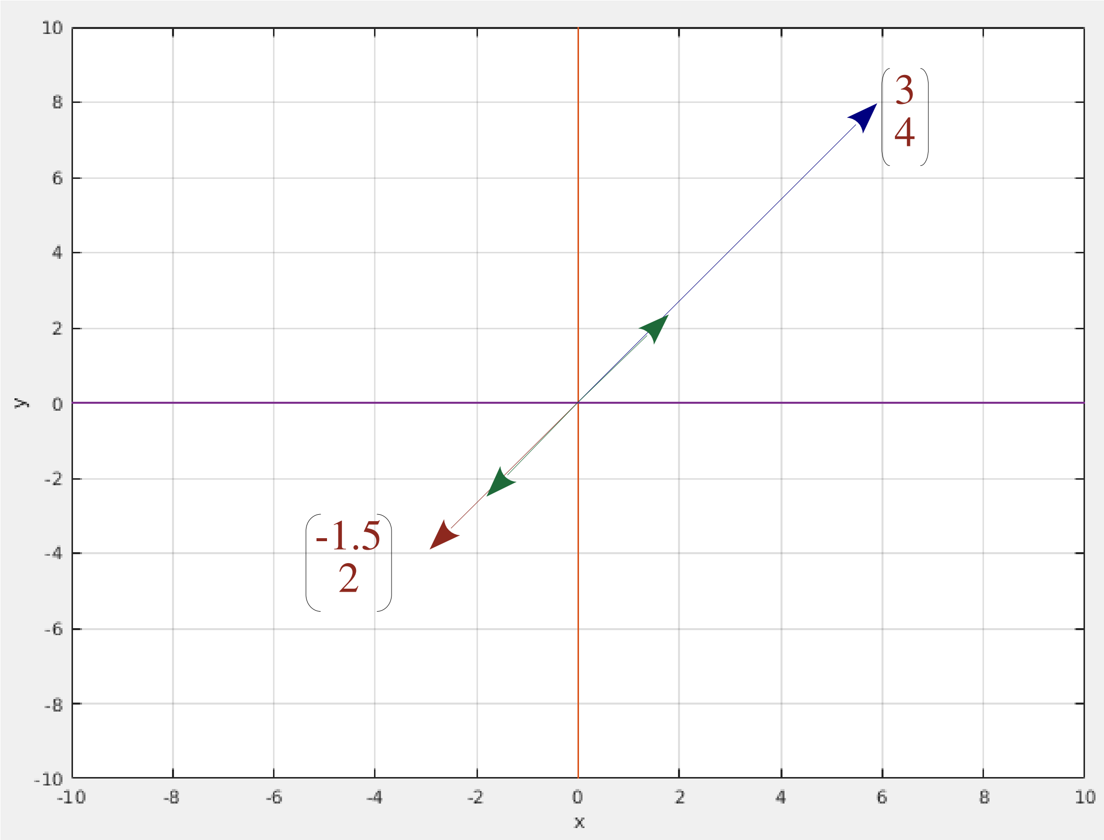

Identity Matrix

Definition 23.1 (Identity Matrix). *The N dimensional identity matrix is denoted by $\mathbf{I}_N$ (or sometimes $\mathbf{I}$) and is defined as follows,*

$\mathbf{I}_N : \mathscr{R}^N \mapsto \mathscr{R}^N$ *is the matrix such that*

$$\mathbf{I}_{ij} = \begin{cases} 1 & \forall \quad i = j \\ 0 & \forall \quad i \neq j \end{cases} \tag{23.1}$$

where $i, j \in \{1, 2, \ldots, N\}$ are the row number and column number of the corresponding element of matrix $\mathbf{I}_N$.

Matrix Transpose

Definition 23.2 (Transpose of a Matrix). *The transpose of a matrix* $\mathbf{A} : \mathscr{R}^N \mapsto \mathscr{R}^M$ *is given by* $\mathbf{A}^T : \mathscr{R}^M \mapsto \mathscr{R}^N$ *such that,*

$$\mathbf{A}^T_{ji} = \mathbf{A}_{ij} \tag{23.2}$$

where indices $i \in \{1, 2, ..., M\}$ *and* $j \in \{1, 2, ..., N\}$ *denote the location of elements of the matrix such that the first index corresponds to the row and the second index corresponds to the column number.*

Hermitian Transpose

Definition 23.3 (Hermitian Transpose). *The Hermitian transpose of a matrix* $\mathbf{A} : \mathscr{C}^N \mapsto \mathscr{C}^M$ *is given by* $\mathbf{A}^H : \mathscr{C}^M \mapsto \mathscr{C}^N$ *such that,*

$$\mathbf{A} = \mathbf{A}_R + i\mathbf{A}_I \tag{23.3}$$

$$\mathbf{A}_R, \mathbf{A}_I : \mathscr{R}^N \mapsto \mathscr{R}^M$$

and

$$\mathbf{A}^H = \mathbf{A}_R^T - i\mathbf{A}_I^T \tag{23.4}$$

Matrix $\mathbf{A}^H$ *is also known as the adjoint matrix of matrix* $\mathbf{A}$.

Hermitian Matrix

Definition 23.4 (Hermitian Matrix). *A Hermitian matrix* $\mathbf{A} : \mathscr{C}^N \mapsto \mathscr{C}^N$ *is the matrix for which,*

$$\mathbf{A} = \mathbf{A}^H \tag{23.5}$$

Determinant of a Square Matrix

$$\det \mathbf{A} = \Delta_A = |\mathbf{A}|$$

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Cofactor of a Determinant Element: *Given any* n*th-order determinant* $|\mathbf{A}|$, *the cofactor* A_{ij} *of any element* a_{ij} *is the determinant obtained by eliminating all elements of the ith row and jth column and then multiplied by* $(-1)^{i+j}$. For example, the cofactor of the element $a11$ of $|\mathbf{A}|$ in Eq. (A-10) is

$$A_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

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Determinant of a Square Matrix

In general, the value of a determinant can be written in terms of the cofactors. Let **A** be an $n \times n$ matrix, then the determinant of **A** can be written in terms of the cofactor of any row or the cofactor of any column. That is,

$$\det A = \sum_{j=1}^{n} a_{ij} A_{ij} \quad (i = 1, \text{ or } 2, \ldots, \text{ or } n)$$

or

$$\det A = \sum_{i=1}^{n} a_{ij} A_{ij} \quad (j = 1, \text{ or } 2, \ldots, \text{ or } n)$$

Determinant of a Square Matrix

Adjoint of a Matrix: *Let* **A** *be a square matrix of order n. The adjoint matrix of* **A**, *denoted by* adj **A**, *is defined as*

$$\text{adj}\,\mathbf{A} = [A_{ij} \text{ of } \det \mathbf{A}]'_{n,n}$$

where A_{ij} *denotes the cofactor of* a_{ij}.

Inverse of a Square Matrix

Definition 23.5 (Inverse of a Square Matrix). *The Inverse of a Square Matrix* $\mathbf{A} : \mathscr{R}^N \mapsto \mathscr{R}^N$ *(if it exists) is denoted by* $\mathbf{A}^{-1} : \mathscr{R}^N \mapsto \mathscr{R}^N$ *and is that unique matrix such that,*

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_N \tag{23.6}$$

$$(\mathbf{A}^{-1})^{-1} = \mathbf{A}$$

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Inverse of a Square Matrix

$Ax = y$

$x = A^{-1}y$

A^{-1} denotes the matrix inverse of A

A is a square matrix

A must be nonsingular

If A^{-1} exists, it is given by

$$\mathbf{A}^{-1} = \frac{\operatorname{adj}\mathbf{A}}{|\mathbf{A}|}$$

Some Properties of Matrix Inverse

If $\mathbf{AB} = \mathbf{AC} \nRightarrow \mathbf{B} = \mathbf{C}$.

But if A^{-1} exists, then $\mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}\mathbf{AC} \Rightarrow \mathbf{B} = \mathbf{C}$.

If A and B are square matrices and are nonsingular, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

Inverse of a Square Matrix (Example)

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{\mathrm{adj}\mathbf{A}}{|\mathbf{A}|} = \frac{\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}{a_{11}a_{22} - a_{12}a_{21}}$$

$|\mathbf{A}| \neq 0$ or $a_{11}\, a_{22} - a_{12}\, a_{21} \neq 0$

Inverse of a Square Matrix (Example)

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\mathrm{Cof}(\mathbf{A}) = \begin{bmatrix} a_{22}a_{33} - a_{23}a_{32} & -(a_{21}a_{33} - a_{23}a_{31}) & a_{21}a_{32} - a_{22}a_{31} \\ -(a_{12}a_{33} - a_{13}a_{32}) & a_{11}a_{33} - a_{13}a_{31} & -(a_{11}a_{32} - a_{12}a_{31}) \\ (a_{12}a_{23} - a_{13}a_{22}) & -(a_{11}a_{23} - a_{13}a_{21}) & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix}$$

$$\mathrm{Adj}(\mathbf{A}) = \begin{bmatrix} a_{22}a_{33} - a_{23}a_{32} & -(a_{12}a_{33} - a_{13}a_{32}) & (a_{12}a_{23} - a_{13}a_{22}) \\ -(a_{21}a_{33} - a_{23}a_{31}) & a_{11}a_{33} - a_{13}a_{31} & -(a_{11}a_{23} - a_{13}a_{21}) \\ a_{21}a_{32} - a_{22}a_{31} & -(a_{11}a_{32} - a_{12}a_{31}) & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix}$$

$$|\mathbf{A}| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

Kronecker Product

Definition 23.6 (Kronecker Product). *The Kronecker product of two matrices,* $\mathbf{A} : \mathscr{R}^N \mapsto \mathscr{R}^M$ *and* $\mathbf{B}$ *of arbitrary dimension is denoted by* $\mathbf{A} \otimes \mathbf{B}$ *and is defined as follows,*

We will see this symbol again in Abstract Algebra used for Finite State Transducers – We'll see that it is a user-defined product, used here for the Kronecker Product

$$\mathbf{A} \otimes \mathbf{B} \stackrel{\Delta}{=} \begin{bmatrix} (\mathbf{A})_{[1][1]}\mathbf{B} & (\mathbf{A})_{[1][2]}\mathbf{B} & \cdots & (\mathbf{A})_{[1][N]}\mathbf{B} \\ (\mathbf{A})_{[2][1]}\mathbf{B} & (\mathbf{A})_{[2][2]}\mathbf{B} & \cdots & (\mathbf{A})_{[2][N]}\mathbf{B} \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \ddots \\ (\mathbf{A})_{[M][1]}\mathbf{B} & (\mathbf{A})_{[M][2]}\mathbf{B} & \cdots & (\mathbf{A})_{[M][N]}\mathbf{B} \end{bmatrix} \tag{23.7}$$

Used in Information Theory

Norms: L_p Norm of a Vector

Euclidean norm is a special case
of the L_p norm (L_2 norm)

Definition 23.8 (L_p-norm of a vector). *The L_p-norm of a vector $\mathbf{x} \in \mathscr{R}^N$, where $\{p : p \in \mathbb{R}, p \geq 1\}$, is denoted by $\|\mathbf{x}\|_p$ and is defined as,*

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{N} |\, (\mathbf{x})_{[i]} \,|^p \right)^{\frac{1}{p}} \tag{23.9}$$

where, $(\mathbf{x})_{[i]}, i \in \{1, 2, \ldots, N\}$ is the i^{th} element of vector $\mathbf{x}$.

Since p is only on the real-line (does not include
infinity), the infinite norm is computed in the limit

Norms: Some Special L_p Norms

L_1 Norm

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{N} |\,(\mathbf{x})_{[i]}\,|$$

L_∞ Norm

$$\|\mathbf{x}\|_\infty = \lim_{p\to\infty} \|\mathbf{x}\|_p$$

As we said, since p is only the real-line (does not include infinity), the infinite norm is computed in the limit

$$= \max_{i=1}^{N} |\,(\mathbf{x})_{[i]}\,|$$

As p goes to infinity, only the maximum will matter

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Norms: Euclidean Norm

Defined for a vector – Coincides with L2 norm of a vector and its Frobenius Norm – Not the same with Matrices

Definition 23.7 (Euclidean Norm of a Vector). *The Euclidean norm of a vector* $\mathbf{x} \in \mathscr{R}^N$ *is denoted by* $\|\mathbf{x}\|_{\mathscr{E}}$ *and is defined as,*

$$\|\mathbf{x}\|_{\mathscr{E}} = \left(\sum_{i=1}^{N} (\mathbf{x})_{[i]}^2 \right)^{\frac{1}{2}} \tag{23.8}$$

where, $(\mathbf{x})_{[i]}, i \in \{1, 2, \ldots, N\}$ *is the* i^{th} *element of vector* $\mathbf{x}$.

Euclidean Norm = L_2 Norm = Frobenius norm for a vector

Euclidean Norm is smooth and has derivatives – desirable

Linear Dependence

Definition 23.9 (Linear Dependence / Independence). *A set of vectors* $\mathbf{s}_i \in \mathscr{R}^N, i \in \{1, 2, \ldots, N\}$ *is said to be a linearly dependent set if there exist numbers* $\lambda_i, i \in \{1, 2, \ldots, N\}$, *not all zero, such that,*

If all are zero, it will be always true – trivial case

$$\sum_{i=1}^{N} \lambda_i \mathbf{s}_i = 0$$

Linearly dependent

Sometimes I use bold lower case for vectors and sometimes use an arrow above

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Vector-Induced L_p Norm of a Matrix

$$\|\mathbf{A}\|_p = \sup_{\vec{x}\neq\vec{0}} \frac{\|\mathbf{A}\vec{x}\|_p}{\|\vec{x}\|_p}$$

Infimum is the Geatest Lower Bound
(does not have to be in the set), eg
The smallest positive real number

Supremum is the Least Upper Bound
(does not have to be in the set), eg
The greatest negative real number

$$\begin{aligned}\|\mathbf{A}\|_1 &= \sup_{\vec{x}\neq\vec{0}} \frac{\|\mathbf{A}\vec{x}\|_1}{\|\vec{x}\|_1} \\ &= \max_{1\leq j\leq N} \sum_{i=1}^{M} |\,(\mathbf{A})_{[i][j]}\,|\end{aligned}$$

Maximum of Sum of Absolute
Value of Column Elements

$$\begin{aligned}\|\mathbf{A}\|_\infty &= \sup_{\vec{x}\neq\vec{0}} \frac{\|\mathbf{A}\vec{x}\|_\infty}{\|\vec{x}\|_\infty} \\ &= \max_{1\leq i\leq M} \sum_{j=1}^{N} |\,(\mathbf{A})_{[i][j]}\,|\end{aligned}$$

Maximum of Sum of Absolute
Value of Row Elements

Homework 2

- Read Chapter 23 and start reading Chapter 24 of the textbook
- Find the rank of these matrices using row echelon normal form. If matrix is square and has full rank, compute the inverse using the adjucate (adjoint) method. Show all the work. Double check your results with matlab's rank and inverse functions.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} -2 & 1 \\ 3 & -2 \\ 0 & 5 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 2 & -6 & 4 \\ 1 & 7 & -2 \\ 3 & -3 & 0 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} 2 & 1 \\ 4 & 9 \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} 1 & -1 & 5 \\ 0 & -7 & 3 \\ -1 & -6 & -2 \end{bmatrix}$$