INTRODUCTION TO CONTINUOUS CONTROL SYSTEMS
COLUMBIA UNIVERSITY MECHANICAL AND ELECTRICAL ENGINEERING DEPARTMENTS: E3601

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Homework 7

Problem 1 (Characteristic Equation and Eigenvalues)**.**

Write the characteristic equations, Eigenvalues, and Eigenvectors of the following matrices.

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 4 & 9 \end{bmatrix} \tag{1}$$

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 5 \\ 0 & -7 & 3 \\ -1 & -6 & -2 \end{bmatrix} \tag{2}$$

Problem 2 (Similarity Transform)**.**

Find the Eigenvalues and Eigenvectors of the following matrix and convert the matrices into diagonal or block diagonal form, whichever is approprate.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} \tag{3}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \tag{4}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 0 \\ 2 & -4 & 3 \end{bmatrix} \tag{5}$$

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Problem 3. *Prove the following Theorem:*

Theorem 1 (Complex Conjugate Eigenvalues)**.** *Suppose* **A** *has the following eigenvalues,*

$$\begin{aligned} \lambda_i &= \sigma_i + i\omega_i && (6)\\ \lambda_{i+1} &= \sigma_i - i\omega_i = \overline{\lambda}_i && (7)\\ & && (8) \end{aligned}$$

for $i = \{1, 3, 5, \cdots, m-1\}$ *and*

$$\lambda_i = \overline{\lambda}_i \tag{9}$$

for $i = \{m+1, m+2, \cdots, n\}$
and a linealy independent set of eigenvectors

$$\begin{aligned} \mathbf{u}_i &= \mathbf{v}_i + i\mathbf{w}_i && (10)\\ \mathbf{u}_{i+1} &= \mathbf{v}_i - i\mathbf{w}_i = \overline{\mathbf{u}}_i && (11)\\ & && (12) \end{aligned}$$

for $i = \{1, 3, 5, \cdots, m-1\}$ *and*

$$\mathbf{u}_i = \overline{\mathbf{u}}_i \tag{13}$$

for $i = \{m+1, m+2, \cdots, n\}$
The, the real-valued matrix,

$$\mathbf{U} = [\mathbf{v}_1 \;\; \mathbf{w}_1 \;\; \mathbf{v}_3 \;\; \mathbf{w}_3 \;\; \cdots \;\; \mathbf{v}_{m-1} \;\; \mathbf{w}_{m-1} \;\; \mathbf{u}_{m+1} \;\; \mathbf{u}_{n-1} \;\; \cdots \;\; \mathbf{u}_n] \tag{14}$$

is nonsingular and may be used to transform **A** *into the block-diagram form,*

$$\mathbf{U}^{-1}\mathbf{A}\mathbf{U} = \begin{bmatrix} \mathbf{\Lambda}_1 & 0 & \cdots & 0 & 0 \\ 0 & \mathbf{\Lambda}_3 & \cdots & 0 & 0 \\ \ldots & \ldots & \ldots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{\Lambda}_{m-1} & 0 \\ 0 & 0 & 0 & \cdots & \mathbf{\Lambda}_{m+1} \end{bmatrix} \tag{15}$$

where,

$$\mathbf{\Lambda}_i = \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix} \tag{16}$$

for $i = \{1, 3, 5, \cdots, m-1\}$ *and*

$$\mathbf{\Lambda}_{m+1} = \begin{bmatrix} \lambda_{m+1} & 0 & \cdots & 0 & 0 \\ 0 & \lambda_{m+2} & \cdots & 0 & 0 \\ \ldots & \ldots & \ldots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \tag{17}$$

This result also holds for non-distinct eigenvalues, provided that the eigenvectors are linearly independent.

Hint:
First show that

$$\begin{aligned} \mathbf{A}\mathbf{v}_i &= \sigma_i\mathbf{v}_i - \omega_i\mathbf{w}_i && (18)\\ \mathbf{A}\mathbf{w}_i &= \omega_i\mathbf{v}_i + \sigma_i\mathbf{w}_i && (19) \end{aligned}$$

for $i = \{1,3,5,\cdots,m-1\}$ and then proceed as if you have all diagonal elements.

Problem 4. *If $\mathbf{A} : \mathscr{R}^n \mapsto \mathscr{R}^n$ and $m \geq n$, show that $\mathbf{A}^m$ may be written as,*

$$\mathbf{A}^m = \lambda_0\mathbf{I} + \lambda_1\mathbf{A} + \lambda_2\mathbf{A}^2 + \cdots + \lambda_{n-1}\mathbf{A}^{n-1} \tag{20}$$

for some coefficients λ_i.

Hint:

Use the Cayley-Hamilton theorem recursively.