INTRODUCTION TO CONTINUOUS CONTROL SYSTEMS
COLUMBIA UNIVERSITY MECHANICAL AND ELECTRICAL ENGINEERING DEPARTMENTS: E3601

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Homework 6

Problem 1 (Laplace Transform). *Show the following:*

$$\int_0^\infty t^2 e^{-at} e^{-st} dt = \frac{2}{(s+a)^3} \tag{1}$$

Problem 2 (Transfer Function). *Consider,*

$$\begin{aligned} X(s) &= \frac{{\omega_n}^2}{s(s^2+2\zeta\omega_n s+{\omega_n}^2)} \\ &= \frac{K_0}{s} + \frac{K_{-\alpha+i\omega}}{s+\alpha-i\omega} + \frac{K_{-\alpha-i\omega}}{s+\alpha+i\omega} \end{aligned} \tag{2}$$

We showed that

$$K_{-\alpha+i\omega} = \left[(s+\alpha-i\omega)X(s)\right]_{s=-\alpha+i\omega} = \frac{{\omega_n}^2}{2i\omega(-\alpha+i\omega)} \tag{3}$$

Please show that

$$K_{-\alpha+i\omega} = \frac{\omega_n}{2\omega} e^{-i(\theta+\frac{\pi}{2})} \tag{4}$$

where $\theta = tan^{-1}(\frac{\omega}{-\alpha})$

Problem 3 (Temporal Response). *Show that equation 5 may be rewritten as 6:*

$$x(t) = u(t) + \frac{\omega_n}{2\omega}\left(e^{-i(\theta+\frac{\pi}{2})}e^{(-\alpha+i\omega)t} + e^{i(\theta+\frac{\pi}{2})}e^{(-\alpha-i\omega)t}\right) \tag{5}$$

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$$x(t) = u(t) + \frac{\omega_n}{\omega} e^{-\alpha t} sin(\omega t - \theta) \tag{6}$$

Problem 4 (Use Laplace Transform and Partial Fractions to solve these ODEs)**.**

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \sin(3t) \tag{7}$$

$$\begin{aligned} y(0) &= -1 \\ \dot{y}(0) &= 0 \end{aligned} \tag{8}$$

Problem 5.

$$\ddot{y}(t) + 4\dot{y}(t) + 4y(t) = \cos(4t) \tag{9}$$

$$\begin{aligned} y(0) &= 1 \\ \dot{y}(0) &= 1 \end{aligned} \tag{10}$$

Problem 6.

$$\ddot{y}(t) + \dot{y}(t) + y(t) = 1 + \cos(6t) \tag{11}$$

$$\begin{aligned} y(0) &= 0 \\ \dot{y}(0) &= 0 \end{aligned} \tag{12}$$

Problem 7.

$$\ddot{y}(t) + y(t) = 1 + e^{(-2t)} \tag{13}$$

$$\begin{aligned} y(0) &= 0 \\ \dot{y}(0) &= 0 \end{aligned} \tag{14}$$

Problem 8.

$$\ddot{y}(t) = e^{-5t} \tag{15}$$

$$\begin{aligned} y(0) &= 0 \\ \dot{y}(0) &= 0 \end{aligned} \tag{16}$$

Problem 9 (Transition Matrix)**.**
Starting with the series definition of $e^{\mathbf{A}t}$*, compute* $e^{\mathbf{A}t}$ *for the following matrices:*

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \tag{17}$$

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \tag{18}$$

Problem 10.

Part A
Solve for $\mathbf{x}(t)$ *in the following:*

$$\dot{\vec{x}}(t) = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \vec{x}(t) \tag{19}$$

$$\vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{20}$$

Part B

Compute $e^{\mathbf{A}t}$ *for*

$$\mathbf{A} = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \tag{21}$$

using the series definition of $e^{\mathbf{A}t}$.